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A New Reduction of the Self-Dual Yang–Mills Equations and its Applications

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Abstract: Through imposing on space–time symmetries, a new reduction of the self-dual Yang–Mills equations is obtained for which a Lax pair is established. By a proper exponent transformation, we transform the Lax pair to get a new Lax pair whose compatibility condition gives rise to a set of partial differential equations (PDEs). We solve such PDEs by taking different Lax matrices; we develop a new modified Burgers equation, a generalised type of Kadomtsev–Petviasgvili equation, and the Davey–Stewartson equation, which also generalise some results given by Ablowitz, Chakravarty, Kent, and Newman.

Keywords: (2+1)-Dimensional Integrable System; Lie Algebra; Self-Dual Yang–Mills Equation.

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1 Introduction

The self-dual Yang–Mills (SDYM) equations are of great importance in their own right and have found a remarkable number of applications in both physics and mathematics. These equations arise in the context of gauge theory, in the classical general relativity, and can be used as a powerful tool in the analysis of four-manifolds [1]. SDYM equations are originated from the non-perturbative approach to the quantum theory of gauge fields [2]. Actually, the SDYM equations have given rise to the self-dual Einstein equations and many other equations of physical interest, including the (1+1)-dimensional Korteweg de Vries (KdV) equation which governs a long one-dimensional, small-amplitude, and surface gravity wave propagating in a shallow channel of water, and the (1+1)-dimensional nonlinear Schrödinger equation which is a partial differential equation in quantum mechanics that describes how the quantum state of a quantum system changes with time. The SDYM equations have also led to (2+1)-dimensional nonlinear equations, such as the Kadomtsev–Petviasgvili (KP) equation which can be thought of as a two-spatial-dimensional analogue of the KdV equation, and it is one of the classical prototype problems in the field of exactly solvable equations and arises generically in physical contexts such as the plasma physics and the surface water wave, and the (2+1)-dimensional Davey–Stewartson (DS) equation which is used to describe the evolution of a three-dimensional wave-packet on water of finite depth in fluid dynamics [3]. In the aspect of mathematical applications, by symmetric reductions of the SDYM equations, some different Lax pairs have been introduced, whose compatibility conditions not only generate some known (1+1)- and (2+1)-dimensional physical equations, but also help us study their some properties, such as Hamiltonian structures, Darboux transformations, and symmetries. Ablowitz et al. [1] have shown us some reductions of the SDYM equations, derived the KdV equation and the (2+1)-dimensional KP equation, and studied their Lax pairs and Painlevé properties. Based on this, Chakravarty et al. [2] introduced a reduction of the SDYM equations in the aspect of mathematics whose compatibility condition leads to a set of integrable equations. By choosing an operator Lie algebra, here the operator is ∂_y, whose coefficients of different powers are 2×2 matrices, a forced Burgers equation, the KP equation, and an mKP equation were re-obtained. Zhang and Hon [4] developed an approach for constructing Lie algebras by the reduction of the SDYM equations for which a (2+1)-dimensional expanding integrable model of the Giachetti Johnson (GJ) hierarchy was obtained; furthermore, a few Hamiltonian equations were generated in the linear space R^3. In [5], we introduced a reduction of the SDYM equations whose compatibility condition admits a variable-coefficient Burgers equation and a (2+1)-dimensional integrable coupling system. In this paper, we shall construct a new reduction of the SDYM equations by imposing a space–time symmetry whose compatibility condition gives rise to a set of (2+1)-dimensional integrable systems containing multi-potential functions which are similar to those in [2, 5], but they are different from each other. It is remarkable that the variables presented in this paper are different from those

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in [2]. Therefore, the differential equations obtained by us are not the same as those in [2]. In [2] and [5], the chosen Lie algebras are 2×2 matrix coefficients; however, in this paper we choose 3×3 coefficients of the various powers of the operator ∂_y so that some new (2+1)-dimensional integrable systems with mathematical structures or physical backgrounds could be generated. Hence, when the central matrix is degenerated, some new (2+1)-dimensional integrable systems containing six potential functions are obtained, which can be reduced to a (2+1)-dimensional integrable coupling of a generalised KP equation which enriches the theory on integrable couplings, where such (2+1)-dimensional integrable couplings could describe models of atmospheric turbulence. In particular, we obtain a (2+1)-dimensional Burgers equation which could be the equation of motion governing the surface perturbations of shallow viscous fluid. When the central matrix is nonsingular, a (2+1)-dimensional integrable system containing six potential functions is obtained, which can be reduced to a (2+1)-dimensional generalised DS system; in particular, it is again reduced to the DS integrable system in [2]. Because the KP equation has the known physical applications stated as above, the (2+1)-dimensional integrable system obtained in this paper also describes some physical phenomena just like the surface water waves.

2 A New Reduction of the SDYM Equations

Set M to be a complex four-dimensional manifold whose complex coordinates are denoted by $(\omega, \bar{\omega}, z, \bar{z})$. Then the SDYM equations are the compatibility condition of the following isospectral problem [1, 2]:

$$D_1\psi=A_1\psi, \quad D_2\psi=A_2\psi, \tag{1}$$

where

$$D_1=\partial_\omega+\lambda\partial_{\bar z}, \quad D_2=\partial_z-\lambda\partial_{\bar\omega}, \quad A_1=A_\omega+\lambda A_{\bar z}, \quad A_2=A_z-\lambda A_{\bar\omega},$$

and $A_a(a=\omega, \bar\omega, z, \bar z)$ stand for Yang–Mills potentials; the variable $\lambda\in\mathbf{C}$ is referred to as the spectral parameter. The compatibility condition of (1) possesses the form

$$D_2A_1-D_1A_2+[A_1, A_2]=0. \tag{2}$$

In [1], it is shown that the components $A_{\bar\omega}$ and $A_{\bar z}$ can be represented by two commuting elements in a Lie algebra g through a suitable gauge. Hence, we assume that

$$[A_{\bar\omega}, A_{\bar z}]=0. \tag{3}$$

By following the way presented in [2], we also impose space–time symmetry on the components A_1 and A_2 in (2), but the variables chosen by us are different from those in [2], which lead to the similar differential equations. Suppose the Yang–Mills potentials are independent of the variables $\omega+\bar\omega$ and $\bar z$. We redefine the remaining coordinates as $\omega-\bar\omega=x$ and $z=t$; then one gets that

$$D_1=\partial_x, \quad D_2=\partial_t+\lambda\partial_x, \tag{4}$$

which is slightly different from that in [2]. In terms of (4), (1) becomes

$$\psi_x=(A_\omega+\lambda A_{\bar z})\psi, \quad \psi_t=[A_z-(A_\omega+A_{\bar\omega})\lambda-\lambda^2A_{\bar z}]\psi. \tag{5}$$

We assume that the potentials in (5) are elements of an infinite-dimensional Lie algebra,

$$g=\text{span}\{A=a_0+a_1\partial_y+a_2\partial_{yy}\}, \tag{6}$$

where a_0, a_1, a_2 are $n\times n$ matrices, which belong to a ring of matrix functions of y, x, and t, and the Lie bracket on g is closed, that is, $[A, B]\in g$, $A, B\in g$. Reference [2] takes the 2×2 matrices in (6) to generate some (2+1)-dimensional integrable systems. In this paper, we want to take 3×3 matrices in (6) to deduce more richer (2+1)-dimensional integrable dynamical systems to further supplement the results in [2, 4, 5].

Assume that

$$A_\omega=U_0+U_1\partial_y+U_2\partial_y^2, \quad A_{\bar\omega}=B_0+B_1\partial_y+B_2\partial_y^2, \quad A_{\bar z}=A,$$
$$A_z=V_0+V_1\partial_y+V_2\partial_y^2,$$

and substitute them into (2); we have

$$\begin{cases}\psi_x=(U_0+U_1\partial_y+U_2\partial_y^2+\lambda A)\psi,\\ \psi_t=[V_0+V_1\partial_y+V_2\partial_y^2-\lambda(U_0+B_0+(U_1+B_1)\partial_y\\ \quad+(U_2+B_2)\partial_y^2)-\lambda^2A]\psi.\end{cases} \tag{7}$$

Equation (3) is equivalent to

$$[A, B_0]=B_1A_y+B_2A_{yy}, \quad [A, B_1]=2B_2A_y, \quad [A, B_2]=0. \tag{8}$$

To eliminate the spectral parameter λ in (7), we make a transformation

$$\psi=\varphi e^{\lambda y}$$

and substitute it into (7); we get

$$\varphi_x=(U_0+U_1\partial_y+U_2\partial_y^2)\varphi+\lambda(U_1+2U_2+A)\varphi+\lambda^2U_2\varphi,$$
$$\varphi_t=(V_0+V_1\partial_y+V_2\partial_y^2)\varphi+\lambda[V_1+2V_2\partial_y-(U_0+B_0)$$
$$-(U_1+B_1)\partial_y-(U_2+B_2)\partial_y^2]\varphi+\lambda^2[V_2-U_1-B_1-A$$
$$-2(U_2+B_2)\partial_y-\lambda^3(U_2+B_2)\varphi,$$

which give the following equations by setting the coefficients of λ^j $(j=3, 2, 1)$ to be zero that

$$\begin{cases} U_1+2U_2+A=0, \\ U_2=0, \\ V_1+2V_2\partial_y-U_0-B_0-(U_1+B_1)\partial_y-(U_2+B_2)\partial_y^2=0, \\ V_2-U_1-B_1-A-2(U_2+B_2)\partial_y=0, \\ U_2+B_2=0, \end{cases} \tag{9}$$

and

$$\begin{cases} \varphi_x=(U_0+U_1\partial_y)\varphi, \\ \varphi_t=(V_0+V_1\partial_y+V_2\partial_y^2)\varphi, \end{cases} \tag{10}$$

$$[A, B_0]=0. \tag{11}$$

Solving (9) yields

$$U_1=-A, \quad U_2=B_2=0, \quad V_1=U_0+B_0, \quad V_2=-A. \tag{12}$$

Substituting (12) into (10) and dropping the subscripts of U_0, V_0, and B_0, one gets that

$$\begin{cases} \varphi_x=(U-A\partial_y)\varphi, \\ \varphi_t=[V+(U+B)\partial_y-A\partial_y^2]\varphi. \end{cases} \tag{13}$$

The compatibility condition of (13) reads

$$\begin{cases} U_t+[U, V]-AV_y-V_x+AU_{yy}-(U+B)U_y=0, \\ -A_t+[U, B]-[A, V]-A(U+B)_y-(U+B)_x \\ +(U+B)A_y+2AU_y-AA_{yy}=0, \\ AA_y-A_x=0. \end{cases} \tag{14}$$

Assume that A is independent of x, y, and t; then (14) reduces to

$$\begin{cases} [U, B]-[A, V]+A(U-B)_y-(U+B)_x=0, \\ U_t+[U, V]-AV_y-V_x+AU_{yy}-(U+B)U_y=0. \end{cases} \tag{15}$$

Equation (11) can be written as

$$[A, B]=0. \tag{16}$$

We call the matrix A in (15) and (16) a central matrix.

3 A Few (2+1)-Dimensional Integrable Systems

In the section, we shall deduce a few (2+1)-dimensional integrable systems according to (15) and (16) as well as the operator Lie algebra g with 3×3 matrix coefficients of the powers of the operator ∂_y. A simple case is that U, V, A in (15) and (16) are scalar functions. With no loss of generality, we take $A=1$; (15) becomes

$$(U+B)_x=(U-B)_y, \tag{17}$$

$$U_t-V_y-V_x+U_{yy}-(U+B)U_y=0. \tag{18}$$

A special solution to (17) is given by

$$U=\frac{1}{2}(f_x+f_y), \quad B=\frac{1}{2}(f_y-f_x), \tag{19}$$

where f is a derivative function in x, y, and t. Substituting (19) into (18) yields that

$$(f_x+f_y)_t-2(V_x+V_y)+f_{xyy}+f_{yyy}-f_y(f_{xy}+f_{yy})=0. \tag{20}$$

Set $Y=y-x$, $X=x$; then (20) becomes

$$f_{X,t}-2V_X+f_{XYY}-f_Yf_{XY}=0. \tag{21}$$

Integrating (21) on the variable X gives

$$f_t-2V+f_{YY}-\frac{1}{2}(f_Y)^2=0, \tag{22}$$

which is the same form except for negative sign as that in [2]. When $V=0$, (22) is a form of Burgers' equation.

Next, we take some explicit matrices in (14) to deduce a type of new KP equation. Set

$$U=\begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix}, \quad A=\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad B=-\frac{u}{2}A,$$
$$V=\frac{1}{2}\begin{pmatrix} f(x, y)+u_x+\gamma u_y & -3u \\ u_{xx}+3u^2+u_y & f(x, y)-u_x-\gamma y_y \end{pmatrix},$$

where $f(x, y)$ is a derivative function and γ is an arbitrary parameter. Then (14) admits the following equations:

$$\begin{aligned} & u_x=2\gamma u_y, \\ & 6u^2+u_y-f_x-\gamma u_{xy}-2u_y=0, \\ & u_t+\frac{1}{2}(2uu_x+2\gamma uu_y)+\frac{1}{2}(f_y+\gamma u_{yy}-u_{xxx}-6uu_x)=0, \\ & 6u^2+u_y-f_x-\gamma u_{xy}-2u_y=0. \end{aligned}$$

Solving the above four equations yields that

$$u_t-uu_x+6\partial_x^{-1}uu_x-\frac{1}{4}u_{xx}-\frac{1}{2}\partial_x^{-1}u_{yy}-\frac{1}{2}u_{xxx}=0,$$

which could describe the surface water waves in fluid dynamics. It is a generalised form given by Ablowtiz et al. in [1]. In what follows, we consider only two cases of the matrix A; the first case is degenerative, while another case is non-degenerate.

3.1 The Central Matrix is $A=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

We take

$$U=\begin{pmatrix} u_1 & u_2 & u_3 \\ u_4 & u_5 & u_6 \\ u_7 & u_8 & u_9 \end{pmatrix}, \quad V=\begin{pmatrix} v_1 & v_2 & v_3 \\ v_4 & v_5 & v_6 \\ v_7 & v_8 & v_9 \end{pmatrix}. \tag{23}$$

Since (16) holds, we set

$$B=\begin{pmatrix} b_1 & 0 & 0 \\ b_4 & b_5 & 0 \\ b_7 & b_8 & b_1 \end{pmatrix}. \tag{24}$$

Substituting (23) and (24) into the first equation in (15) yields

$$\begin{cases} u_{3x}=0, \\ u_2b_4-(u_1+b_1)_x+v_3+u_3b_7=0, \\ u_2b_5+u_3b_8-b_1u_2-u_{2x}=0, \\ u_4b_1+u_5b_4+u_6b_7-u_1b_4-u_4b_5+v_6-(u_4+b_4)_x=0, \\ u_6b_8-u_2b_4-(u_5+b_5)_x=0, \\ u_6b_1-u_3b_4-u_6b_5-u_{6x}=0, \\ u_8b_4+u_9b_7-u_1b_7-u_4b_8+v_9-v_1+(u_1-b_1)_y-(u_7+b_7)_x=0, \\ u_8b_5+u_9b_8-u_2b_7-u_5b_8-b_1u_8-v_2+u_{2y}-(u_8+b_8)_x=0, \\ -u_3b_7-u_6b_8-v_3+u_{3y}-(u_9+b_1)_x=0. \end{cases} \tag{25}$$

Taking $u_3=1$, one can get from (25) that

$$b_1=-\frac{1}{2}(u_1+u_9+u_2u_6),$$

$$b_4=-\frac{3}{2}u_2u_6^2-\frac{1}{2}(u_1u_6+u_6u_9)+u_5u_6-u_{6x},$$

$$b_5=u_2u_6-u_5,$$

$$b_8=-\frac{3}{2}u_2^2u_6-\frac{1}{2}(u_1u_2+u_2u_9)+u_2u_5+u_{2x},$$

$$v_3+b_7=\frac{3}{2}(u_2u_6)^2+\frac{1}{2}(u_1u_2u_6+u_2u_6u_9)-u_2u_5u_6+u_2u_{6x} \\ +\frac{1}{2}(u_1-u_9-u_2u_6)_x,$$

$$v_2=-\frac{3}{2}(u_9-u_5)u_2^2u_6-\frac{1}{2}(u_9-u_5)(u_1u_2+u_2u_9)+(u_9-u_5)(u_2u_5+u_{2x}) \\ +u_2u_6u_8-u_5u_8-u_2u_7+\frac{1}{2}(u_1u_8+u_8u_9+u_2u_6u_8)+u_{2y}-u_{8x} \\ +\frac{3}{2}(u_2^2u_6)_x+\frac{1}{2}(u_1u_2+u_2u_9)_x-(u_2u_5)_x-u_{2xx},$$

$$v_6=\frac{1}{2}u_1u_4+\frac{1}{2}u_4u_9+\frac{1}{2}u_2u_4u_6 \\ +(u_1-u_5)\left[-\frac{3}{2}u_2u_6^2-\frac{1}{2}(u_1u_6+u_6u_9)+u_5u_6-u_{6x}\right]+u_2u_4u_6 \\ -u_4u_5+u_{4x}-\frac{3}{2}(u_2u_6^2)_x-\frac{1}{2}(u_1u_6+u_6u_9)_x+(u_5u_6)_x-u_{6xx},$$

$$v_1-v_9=-\frac{3}{2}u_2u_8u_6^2-\frac{1}{2}u_8(u_1u_6+u_5u_9)+u_5u_6u_8-u_8u_{6x} \\ +(u_9-u_1)b_7+\frac{3}{2}u_2^2u_4u_6+\frac{1}{2}u_4(u_1u_2+u_2u_9)-u_2u_4u_5-u_4u_{2x} \\ +\frac{1}{2}(u_1-u_9-u_2u_6)_y-u_{7x}-b_{7x},$$

$$-\frac{1}{2}u_1u_6-\frac{1}{2}u_6u_9-\frac{3}{2}u_2u_6^2+\frac{3}{2}u_2u_3u_6^2+\frac{1}{2}u_1u_3u_6 \\ +\frac{1}{2}u_3u_6u_9-u_3u_5u_6+u_3u_{6x}+u_5u_6-u_{6x}=0, \tag{26}$$

$$-u_2u_5-\frac{3}{2}u_2^2u_3u_6-\frac{1}{2}(u_1u_2u_3+u_2u_3u_9)+u_2u_3u_5+u_3u_{2x} \\ +\frac{1}{2}u_1u_2+\frac{1}{2}u_2u_9+\frac{3}{2}u_2^2u_6-u_{2x}=0. \tag{27}$$

Based on the above results, we take

$$U=\begin{pmatrix} u_1 & u_2 & 1 \\ u_4 & u_5 & u_6 \\ u_7 & u_8 & u_9 \end{pmatrix}, \quad V=\begin{pmatrix} v_1 & v_2 & v_3 \\ 0 & 0 & v_6 \\ 0 & 0 & v_9 \end{pmatrix}$$

and substitute them into the second equation in (15); one infers that

$$\begin{cases} u_{1t}-v_2u_4-v_3u_7-v_{1x}-(u_1+b_1)u_{1y}-u_2u_{4y}-u_{7y}=0, \\ u_{2t}+u_1v_2-u_2v_1-u_5v_2-u_8v_3-v_{2x}-(u_1+b_1)u_{2y}-u_2u_{5y}-u_{8y}=0, \\ u_1v_3+u_2v_6+v_9-v_1-v_2u_6-v_3u_9-v_{3x}-u_2u_{6y}-u_{9y}=0, \\ u_{4t}+u_4v_1-u_7v_6-(u_4+b_4)u_{1y}-(u_5+b_5)u_{4y}-u_6u_{7y}=0, \\ u_{5t}+u_4v_2-u_8v_6-(u_4+b_4)u_{2y}-(u_5+b_5)u_{5y}-u_6u_{8y}=0, \\ u_{6t}+u_4v_3+u_5v_6+u_6v_9-u_9v_6-v_{6x}-(u_5+b_5)u_{6y}-u_6u_{9y}=0, \\ u_{7t}+u_7(v_1-v_9)-v_{1y}+u_{1y}-(u_7+b_7)u_{1y}-(u_8+b_8)u_{4y} \\ \quad -(u_9+b_1)u_{7y}=0, \\ u_{8t}+u_7v_2-u_8v_9-v_{2y}+u_{2y}-(u_7+b_7)u_{2y}-(u_8+b_8)u_{5y} \\ \quad -(u_9+b_1)u_{8y}=0, \\ u_{9t}+u_7v_3+u_8v_6-v_{3y}-v_{9x}-(u_8+b_8)u_{6y}-(u_9+b_1)u_{9y}=0. \end{cases} \tag{28}$$

In what follows, we discuss some reductions on (28). Suppose $u_2=u_6=0$; then we get that

$$b_1=-\frac{1}{2}(u_1+u_9), \quad b_4=b_8=0, \quad b_5=-u_5, \quad v_3+b_7=\frac{1}{2}u_{1x},$$

$$v_2=-u_5u_8+\frac{1}{2}(u_1u_8+u_8u_9)-u_{8x},$$

$$v_6=\frac{1}{2}u_1u_4+\frac{1}{2}u_4u_9-u_4u_5+u_{4x},$$

$$v_1-v_9=(u_9-u_1)b_7+\frac{1}{2}(u_1-u_9)_y-u_{7x}-b_{7x}.$$

Equations (26) and (27) identically hold, and (28) reduces to

$$\begin{cases} u_{1t}+u_4u_5u_8-\frac{1}{2}u_1u_4u_8-\frac{1}{2}u_4u_8u_9+u_4u_{8x}-v_3u_7-v_{1x}-\frac{1}{2}u_1u_{1y} \\ \quad +\frac{1}{2}u_9u_{1y}-u_{7y}=0, \\ u_1v_2-u_5v_2-u_8v_3-v_{2x}-u_{8y}=0, \\ u_1v_3+v_9-v_1-v_3u_9-v_{3x}-u_{9y}=0, \\ u_{4t}+u_4v_1-u_7v_6-u_4u_{1y}=0, \\ u_{5t}+u_4v_2-u_8v_6=0, \\ u_4v_3+u_5v_6-u_9v_6-v_{6x}=0, \\ u_{7t}+u_7(v_1-v_9)-v_{1y}+u_{1y}-(u_1+b_7)u_{1y}-u_8u_{4y} \\ \quad -\frac{1}{2}(u_9-u_1)u_{7y}=0, \\ u_{8t}+u_7v_2-u_8v_9-v_{2y}-u_8u_{5y}-\frac{1}{2}(u_9-u_1)u_{8y}=0, \\ u_{9t}+u_7v_2+u_8v_6-v_{3y}-v_{9x}-\frac{1}{2}(u_9-u_1)u_{9y}=0. \end{cases} \tag{29}$$

Case 1: Assume $u_4=u_8=b_7=v_9=0$, $u_5=1$. Equation (29) reduces to

$$\begin{cases} u_{1t}-\frac{1}{2}u_{1x}u_7-\frac{1}{2}(u_1-u_9)_{xx}+u_{7xx}+\frac{1}{2}(u_9-u_1)u_{1y}-u_{7y}=0, \\ \frac{1}{2}(u_9-u_1)_y+u_{7x}-\frac{1}{2}u_{1xx}-u_{9y}+\frac{1}{2}(u_1-u_9)u_{1x}=0, \\ u_{9t}+\frac{1}{2}u_{1x}u_7-\frac{1}{2}u_{1xy}-\frac{1}{2}(u_9-u_1)u_{9y}=0, \\ u_{7t}+\frac{1}{2}u_7(u_1-u_9)_y-u_7u_{7x}-\frac{1}{2}(u_1-u_9)_{yy}+u_{7xy}+u_{1y}-u_1u_{1y} \\ \quad -\frac{1}{2}(u_9-u_1)u_{7y}=0. \end{cases} \tag{30}$$

Substituting the second equation into the fourth equation yields

$$u_{7t}-\frac{1}{2}u_7u_{1xx}-u_7u_{9y}+\frac{1}{2}u_7(u_1-u_9)u_{1x}+\frac{1}{2}u_{1xxy}+u_{9yy}$$
$$-\frac{1}{2}(u_1-u_9)u_{1xy}-u_{1y}-u_1u_{1y}+\frac{1}{2}(u_1-u_9)u_{7y}-u_{1x}u_{7x}$$
$$+\frac{1}{2}u_{1x}u_{1xx}+u_{1x}u_{9y}-\frac{1}{2}(u_1-u_9)u_{1x}^2=0. \tag{31}$$

Assuming $u_9=0$, (30) and (31) reduce to a new (2+1)-dimensional integrable coupling of a generalised KP equation which is different from that in [4]:

$$\begin{cases} u_{1t}-\frac{1}{2}u_{1xxx}+\frac{1}{2}\partial_x^{-1}u_{1yy}+\frac{1}{4}u_{1x}\partial_x^{-1}u_{1y}+\frac{3}{4}u_{1x}^2-\frac{1}{8}u_1^2u_{1x}-\frac{1}{2}u_{1xy} \\ \quad +\frac{1}{2}u_1u_{1xx}-u_1u_{1y}=0, \\ u_{7t}-\frac{1}{2}u_{1xx}u_7+\frac{1}{2}u_{1xy}+\frac{1}{2}u_{1xxy}-\frac{1}{2}u_1u_{1xy}-u_{1y}-u_1u_{1y}+\frac{1}{2}u_1u_{7y} \\ \quad -u_{1x}u_{7x}+\frac{1}{2}u_{1x}u_{1xx}-\frac{1}{2}u_1u_{1x}^2=0, \end{cases}$$

along with a constraint as follows:

$$2u_{1x}u_{1xxy}-u_{1y}u_{1x}^2-2u_{1xx}u_{1xy}-u_{1xx}u_{1x}^2+u_1u_{1x}^3=0.$$

Case 2: Assume $u_1=u_9=0$. Then we have

$$b_1=-\frac{1}{2}u_1,\quad b_5=-u_5,\quad v_3=-b_7,\quad v_2=-u_5u_8-u_{8x},$$
$$v_6=-u_4u_5+u_{4x}.$$

Substituting the above computations into (29) yields the following (2+1)-dimensional integrable systems along with four potential functions u_4, u_5, u_7, u_8:

$$\begin{cases} u_{4t}+u_4v_1-u_7u_{4x}+u_4u_5u_7=0, \\ u_{5t}=(u_4u_8)_x, \\ u_{7t}+u_7b_{7x}-v_{1y}-u_8u_{4y}=0, \\ u_{8t}-u_5u_7u_8-u_7u_{8x}-u_8v_9+(u_5u_8)_y+u_{8xy}-u_8u_{5y}=0, \end{cases} \tag{32}$$

and

$$\begin{cases} u_4u_5u_8+u_4u_{8x}+b_7u_7-v_{1x}-u_{7y}=0, \\ u_5^2u_8+u_5u_{8x}+u_8b_7+(u_5u_8)_x+u_{8xx}-u_{8y}=0, \\ v_1-v_9=b_{7x}, \\ -u_7b_7-u_4u_5u_8+u_8u_{4x}+b_{7y}-v_{9x}=0. \end{cases} \tag{33}$$

Specially, let $u_4=u_5=u_8=0$. Equations (32) and (33) reduce to a new (2+1)-dimensional integrable system with an arbitrary parameter b_7:

$$u_{7t}+u_7b_{7x}+\frac{1}{2}\partial_x^{-1}(u_7-b_7)_y-\frac{1}{2}b_{7x}=0, \tag{34}$$

where the parameter b_7 satisfies

$$2u_7b_7-(u_7+b_7)_y-b_{7xx}=0. \tag{35}$$

Setting $w=u_7-\frac{1}{2}$, (34) reduces to a generalised (2+1)-dimensional Burgers equation

$$w_t+ww_x-ww_{xy}+\frac{1}{2}\partial_x^{-1}w_{yy}=0, \tag{36}$$

which could be the equation of motion governing the surface perturbations of a shallow viscous fluid heated from below.

3.2 The Central Matrix is $A=\begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{pmatrix}$

We take

$$B=\begin{pmatrix}b_1 & 0 & 0\\ 0 & b_2 & 0\\ 0 & 0 & b_3\end{pmatrix},\quad U=\begin{pmatrix}0 & u_2 & u_3\\ u_4 & 0 & u_6\\ u_7 & u_8 & 0\end{pmatrix},\quad V=\begin{pmatrix}v_1 & v_2 & v_3\\ v_4 & v_5 & v_6\\ v_7 & v_8 & v_9\end{pmatrix}. \tag{37}$$

Substituting (41) into the first equation in (15) gives

$$\begin{cases}\lambda_1 b_{1y}+b_{1x}=0,\\ \lambda_2 b_{2y}+b_{2x}=0,\\ \lambda_3 b_{3y}+b_{3x}=0,\\ u_2b_2-b_1u_2-(\lambda_1-\lambda_2)v_2+\lambda_1u_{2y}-u_{2x}=0,\\ u_3(b_3-b_1)-(\lambda_1-\lambda_3)v_3+\lambda_1u_{3y}-u_{3x}=0,\\ u_4(b_1-b_2)-(\lambda_2-\lambda_1)v_4+\lambda_2u_{4y}-u_{4x}=0,\\ u_6(b_3-b_2)-(\lambda_2-\lambda_3)v_6+\lambda_2u_{6y}-u_{6x}=0,\\ u_7(b_1-b_3)-(\lambda_3-\lambda_1)v_7+\lambda_3u_{7y}-u_{7x}=0,\\ u_8(b_2-b_3)-(\lambda_3-\lambda_2)v_8+\lambda_3u_{8y}-u_{8x}=0.\end{cases} \tag{38}$$

Assuming $B=0$, $\partial_i^{\pm}=\partial_x\pm\lambda_i\partial_y$, we get from (38) that

$$\begin{cases}v_2=\frac{1}{\lambda_2-\lambda_1}\partial_2^{-}u_2,\quad v_3=\frac{1}{\lambda_3-\lambda_1}\partial_1^{-}u_3,\\ v_4=\frac{1}{\lambda_1-\lambda_2}\partial_2^{-1}u_4,\quad v_6=\frac{1}{\lambda_3-\lambda_2}\partial_2^{-}u_6,\\ v_7=\frac{1}{\lambda_1-\lambda_3}\partial_3^{-}u_7,\quad v_8=\frac{1}{\lambda_2-\lambda_3}\partial_3^{-}u_8.\end{cases} \tag{39}$$

Substituting (37) into the second equation in (15) yields

$$\begin{cases}u_2v_4+u_3v_7-v_2u_4-v_3u_7-\lambda_1v_{1y}-v_{1x}-u_2u_{4y}-u_3u_{7y}=0,\\ u_4v_2+u_6v_8-v_4u_2-v_6u_8-\lambda_2v_{5y}-v_{5x}-u_4u_{2y}-u_6u_{8y}=0,\\ u_7v_3+u_8v_6-v_7u_3-v_8u_6-\lambda_3v_{9y}-v_{9x}-u_7u_{3y}-u_8u_{6y}=0,\end{cases} \tag{40}$$

and

$$\begin{cases}u_{2t}+u_2(v_5-v_1)+u_3v_8-v_3u_8-\lambda_1v_{2y}-v_{2x}+\lambda_1u_{2yy}-b_1u_{2y}\\ \quad -u_3u_{8y}=0,\\ u_{3t}+u_2v_3+u_3v_9-v_1u_3-v_2u_6-\lambda_1v_{3y}-v_{3x}+\lambda_1u_{3yy}-b_1u_{3y}\\ \quad -u_2u_{6y}=0,\\ u_{4t}+u_4v_1+u_6v_7-v_5u_4-v_6u_7-\lambda_2v_{4y}-v_{4x}+\lambda_2v_{4yy}-b_2u_{4y}\\ \quad -u_6u_{7y}=0,\\ u_{6t}+u_4v_3+u_6(v_9-v_5)-v_4u_3-\lambda_2v_{6y}-v_{6x}+\lambda_2u_{6yy}-u_4u_{3y}\\ \quad -b_2u_{6y}=0,\\ u_{7t}+u_7(v_1-v_9)+u_8v_4-v_8u_4-\lambda_3v_{7y}-v_{7x}+\lambda_3u_{7xx}-u_8u_{4y}\\ \quad -b_3u_{7y}=0,\\ u_{8t}+u_7v_2+u_8(v_5-v_9)-u_2v_7-\lambda_3v_{8y}-v_{8x}+\lambda_3u_{8yy}-u_7u_{2y}\\ \quad -b_3u_{8y}=0.\end{cases} \tag{41}$$

When $B=0$, one infers from (40) that

$$\begin{aligned}\partial_1^{+}v_1&=\frac{1}{\lambda_1-\lambda_2}[u_2(\partial_2^{-}u_4)+u_4(\partial_1^{-}u_2)]\\ &\quad+\frac{1}{\lambda_1-\lambda_3}[u_3(\partial_3^{-}u_7)+u_7(\partial_1^{-}u_3)]-u_2u_{4y}-u_3u_{7y},\\ \partial_2^{+}u_5&=\frac{1}{\lambda_1-\lambda_2}(u_4\partial_2^{-}u_2-u_2\partial_2^{-}u_4)\\ &\quad+\frac{1}{\lambda_2-\lambda_3}(u_6\partial_3^{-}u_8+u_8\partial_2^{-}u_6)-u_4u_{2y}-u_6u_{8y},\\ \partial_3^{+}v_9&=-\frac{1}{\lambda_1-\lambda_3}(u_7\partial_1^{-}u_3+u_3\partial_3^{-}u_7)\\ &\quad-\frac{1}{\lambda_2-\lambda_3}(u_8\partial_2^{-}u_6+u_6\partial_3^{-}u_8)-u_7u_{3y}-u_8u_{6y},\\ \partial_1^{+}v_2&=\frac{1}{\lambda_1-\lambda_2}(\lambda_1^2u_{2yy}-u_{2xx}).\end{aligned}$$

Substituting the above results into (41) reads

$$\begin{aligned}u_{2t}&=u_2(v_1-v_5)+\frac{1}{\lambda_2-\lambda_3}(\lambda_2u_3u_{8y}-2\lambda_3u_3u_{8y}+u_3u_{8x})\\ &\quad+\frac{1}{\lambda_1-\lambda_3}(u_8u_{3x}-\lambda_1u_8u_{3y})+\frac{1}{\lambda_1-\lambda_2}(\lambda_1\lambda_2u_{2yy}-u_{2xx}),\end{aligned} \tag{42}$$

$$\begin{aligned}u_{3t}&=\frac{1}{\lambda_1-\lambda_3}(u_2u_{3x}-\lambda_1u_2u_{3y}+\lambda_1\lambda_3u_{3yy}-u_{3xx})\\ &\quad+\frac{1}{\lambda_1-\lambda_2}(\lambda_1u_6u_{2y}+\lambda_1u_2u_{6y}-\lambda_2u_2u_{6y}-u_6u_{2x})+u_3(v_1-v_9),\end{aligned} \tag{43}$$

$$\begin{aligned}u_{4t}&=u_4(v_5-v_1)+\frac{1}{\lambda_1-\lambda_3}(u_6u_{7x}+\lambda_1u_6u_{7y}-2\lambda_3u_6u_{7y})\\ &\quad+\frac{1}{\lambda_2-\lambda_3}(\lambda_2u_7u_{6y}-u_7u_{6x})+\frac{1}{\lambda_1-\lambda_2}(u_{4xx}-\lambda_1\lambda_2u_{4yy}),\end{aligned} \tag{44}$$

$$u_{6t}=u_6(v_5-v_9)+\frac{1}{\lambda_1-\lambda_3}(u_4u_{3x}-\lambda_3u_4u_{3y})$$
$$+\frac{1}{\lambda_1-\lambda_2}(u_3u_{4x}-\lambda_2u_3u_{4y})+\frac{1}{\lambda_2-\lambda_3}(\lambda_2\lambda_3u_{6yy}-u_{6xx}), \quad (45)$$

$$u_{7t}=u_7(v_9-v_1)+\frac{1}{\lambda_1-\lambda_2}(u_8u_{4x}-2\lambda_2u_8u_{4y}+\lambda_1u_8u_{4y})$$
$$+\frac{1}{\lambda_2-\lambda_3}(u_4u_{8x}-\lambda_3u_4u_{8y})+\frac{1}{\lambda_1-\lambda_3}(\lambda_3u_{7xx}-\lambda_1\lambda_3u_{7yy}), \quad (46)$$

$$u_{8t}=u_8(v_9-v_5)+\frac{1}{\lambda_1-\lambda_2}(u_7u_{2x}-\lambda_2u_7u_{2y})$$
$$+\frac{1}{\lambda_1-\lambda_3}(u_2u_{7x}-\lambda_3u_2u_{7y})+\frac{1}{\lambda_2-\lambda_3}(u_{8xx}-\lambda_2\lambda_3u_{8yy}). \quad (47)$$

Taking $u_6=u_7=u_8=0$, (42)–(47) reduce to the following (2+1)-dimensional generalised DS system:

$$\begin{cases} u_{2t}=u_2(v_1-v_5)+\frac{1}{\lambda_1-\lambda_2}(\lambda_1\lambda_2u_{2yy}-u_{2xx}), \\ u_{3t}=u_3v_1+\frac{1}{\lambda_1-\lambda_3}(\lambda_1\lambda_3u_{3yy}-u_{3xx}+u_2u_{3x}-\lambda_1u_2u_{3y}), \\ u_{4t}=u_5(v_5-v_1)+\frac{1}{\lambda_2-\lambda_1}(\lambda_1\lambda_2u_{4yy}-u_{4xx}). \end{cases} \quad (48)$$

If we again take $u_3=0$, (48) further reduces to

$$\begin{cases} u_{2t}=u_2(v_1-v_5)+\frac{1}{\lambda_1-\lambda_2}(\lambda_1\lambda_2u_{2yy}-u_{2xx}), \\ u_{4t}=u_4(v_5-v_1)+\frac{1}{\lambda_2-\lambda_1}(\lambda_1\lambda_2u_{4yy}-u_{4xx}), \end{cases} \quad (49)$$

which is the generalised DS system once presented in [2], which can be transformed to the standard DS equation by a suitable transformation in [3].

Remark: Some (2+1)-dimensional equations were obtained in the paper. It is an interesting and important work to investigate their properties, such as Hamiltonian structures, exact solutions, symmetries, and so on, in terms of ideas in [6–15].

4 Conclusion

We have derived some different (2+1)-dimensional integrable equations starting from the SDYM equations and Lie algebras. We found that these high-dimensional equations have different physical applications and backgrounds. Therefore, an important work is how to generate them from the viewpoint of mathematics. For the reason, in this paper, we extended the reductions of the SDYM equations to higher order matrices and derived some (2+1)-dimensional integrable systems. Except for the way of the SDYM equations, the approach of the Hamiltonian operators is also an efficient method for generating (2+1)-dimensional integrable systems. For this, Dorfman and Fokas [16] built a Hamiltonian theory over a noncommutative ring and applied the generalised Adler–Gel'fand–Dikii scheme to have generated (2+1)-dimensional bi-Hamiltonian integrable systems. Based on this, Athorne and Dorfman [17] employed the Hamiltonian theory and the modified Lenard scheme to have derived a generalised KP hierarchy and the well-known Novikov–Veselov equation. In addition, Tu et al. [18] introduced a residue operator over an associative algebra to propose a scheme for generating (2+1)-dimensional nonlinear equations which was called the TAH scheme [19, 20]. By using the scheme, the KP equation and the DS equation were also obtained by different approaches. Hence, different mathematical methods really could generate (2+1)-dimensional equations which have applications in physics.

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References

[1] M. J. Ablowitz, S. Chkravarty, and L. A. Takhtajan, Commun. Math. Phys. **158**, 289 (1993).
[2] S. Chakravarty, S. L. Kent, and E. T. Newman, J. Math. Phys. **36**, 763 (1995).
[3] M. J. Ablowtitz and P. A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering, Cambridge University Press, England, 1992.
[4] Y. F. Zhang and Y. C. Hon, Commun. Theor. Phys. **56**, 856 (2011).
[5] Y. F. Zhang and W. H. Tam, Commun. Theor. Phys. **61**, 203 (2014).
[6] Y. F. Zhang, L. X. Wu, and W. J. Rui, Commun. Theor. Phys. **63**, 535 (2015).
[7] S. Y. Lou, Phys. Lett. A **151**, 133 (1990).
[8] S. Y. Lou, Phys. Rev. Lett. **71**, 4099 (1993).
[9] S. Y. Lou, J. Phys. A **26**, 4387 (1993).
[10] E. G. Fan, Stud. Appl. Math. **127**, 284 (2011).
[11] E. G. Fan, J. Math. Phys. **53**, 013503 (2012).
[12] Q. P. Liu, Lett. Math. Phys. **35**, 115 (1995).

[13] Q. P. Liu and X. B. Hu, J. Phys. A **38**, 6371 (2005).
[14] M. Wadati, Stud. Appl. Math. **59**, 153 (1978).
[15] M. Wadati, K. Konno, and Y. H. Ichikawa, J. Phys. Soc. Japan **47**, 1698 (1979).
[16] I. Y. Dorfman and A. S. Fokas J. Math. Phys. **33**, 2504 (1992).
[17] C. Athorne and I. Y. Dorfman, J. Math. Phys. **34**, 3507 (1993).
[18] G. Z. Tu, R. I. Andrushkiw, and X. C. Huang, J. Math. Phys. **32**, 1900 (1991).
[19] Y. F. Zhang, J. Gao, and G. M. Wang, Appl. Math. Comput. **244**, 672 (2014).
[20] Y. F. Zhang, W. J. Rui, and H. W. Tam, Discontin. Nonlinearity Complex. **3**, 427 (2014).