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Exact Solutions for *N*-Coupled Nonlinear Schrödinger Equations With Variable Coefficients

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Abstract: In this paper, based on similarity transformation and auxiliary equation method, we construct many exact solutions of *N*-coupled nonlinear Schrödinger equations with variable coefficients, which include soliton solutions, combined soliton solutions, triangular periodic solutions, Jacobi elliptic function solutions and combined Jacobi elliptic function solutions. These solutions may give insight into many considerable physical processes.

Keywords: Auxiliary Equation Method; Exact Solutions; *N*-Coupled Nonlinear Schrödinger Equations with Variable Coefficients; Similarity Transformation.

1 Introduction

In recent years, the *N*-coupled nonlinear Schrödinger (NLS) equations which are used to describe the simultaneous propagation of *N* nonlinear waves in a uniform medium have received considerable interest due to their numerous applications in the areas of plasma physics [1], nonlinear optics and quantum electronics [2], Bose–Einstein condensates [3], and hydrodynamics [4, 5]. For better understanding the complicated nonlinear physical phenomena, the solution is much involved. In the past, various methods have been used to handle nonlinear partial differential nonlinear equations, such as the variational iteration method [6, 7], homotopy perturbation method [8, 9], the Bäcklund transformation method [10], the subsidiary ordinary differential equation method (sub-ODE for short) [11, 12], F-expansion method [13–15], sine-cosine method [16, 17], sech-tanh method [18, 19], Exp-function method [20, 21], Jacobi elliptic function method [22–24], residual power series method (RPSM for short) [25, 26], and homotopy analysis method [27, 28].

It is well known that the nonlinear partial differential equations with variable-coefficients are more realistic in various physical situations than their constant coefficients counterparts. So the aim of this paper is to construct exact solutions of the following *N*-coupled NLS equations with variable coefficients using similarity reduction and auxiliary equation method:

$$i\frac{\partial\Psi_k}{\partial t}=-\frac{\partial^2\Psi_k}{\partial x^2}+v_k(x,\,t)\Psi_k+\sum_{j=1}^{N}g_{kj}(t)|\Psi_j|^2\Psi_k+i\gamma(t)\Psi_k, \quad (1)$$

where the physical field $\Psi_k\equiv\Psi_k(x,\,t)(k=1,\,2,\,\dots,\,N)$, the external potential $v_k(x,\,t)(k=1,\,2,\,\dots,\,N)$ is a real-valued function of time and spatial coordinates, and the nonlinear coefficient $g_{kj}(t)(j,\,k=1,\,2,\,\dots,\,N)$ and gain or loss coefficient $\gamma(t)$ are real valued functions of time.

The rest of this paper is organised as follows: In Section 2, we study the similarity solutions by reducing (1) to the (1+1)-dimensional standard NLS equations with constant coefficients via the similarity transformation and auxiliary equation method. In Section 3, some conclusions are given.

2 Similarity Transformation and Solutions

In general, it is difficult to seek directly analytical solutions of (1). Here, we search for a similarity transformation connecting solutions of (1) with those of the (1+1)-dimensional standard NLS equations with constant coefficients

$$i\frac{\partial\Phi_k(\eta,\,\tau)}{\partial\tau}=-\frac{\partial^2\Phi_k(\eta,\,\tau)}{\partial\eta^2}+\sum_{j=1}^{N}G_{kj}|\Phi_j(\eta,\,\tau)|^2\Phi_k(\eta,\,\tau), \quad (2)$$

where the physical field $\Phi_k(\eta,\,\tau)(k=1,\,2,\,\dots,\,N)$ are functions of two variables $\eta\equiv\eta(x,\,t)$ and $\tau\equiv\tau(t)$, which are to be determined later, and $G_{kj}(j,\,k=1,\,2,\,\dots,\,N)$ are constants.

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To do this, we write the solution of (1) as

$$\Psi_k(x,t)=\rho(t)e^{i\varphi_k(x,t)}\Phi_k[\eta(x,t),\tau(t)] \tag{3}$$

Substituting transformation (3) into (1) and after relatively simple algebra obtain the system of partial differential equations

$$\eta_t+2\varphi_{kx}\eta_x=0,\quad \eta_{xx}=0,\quad \tau_t-\eta_x^2=0, \tag{4}$$

$$\rho_t+\rho\varphi_{kxx}-\gamma(t)\rho=0, \tag{5}$$

$$g_{kj}(t)\rho^2-G_{kj}\eta_x^2=0, \tag{6}$$

$$v_k(x,t)+\varphi_{kx}^2+\varphi_{kt}=0. \tag{7}$$

Solving (4), we can write the similarity variables $\eta(x, t)$, $\tau(t)$, and the phase $\varphi(x, t)$ in the following form

$$\eta(x,t)=k(t)x+c(t), \tag{8}$$

$$\tau(t)= =\int_0^t k^2(s)\mathrm{d}s, \tag{9}$$

$$\varphi_k(x,t)=-\frac{\dot{k}(t)}{4k(t)}x^2-\frac{\dot{c}(t)}{2k(t)}x+w_k(t), \tag{10}$$

where $k(t)$, $c(t)$, $w(t)$ are time-dependent functions and an overdot stands for the derivative with respect to time. Now, from (5) to (7), we obtain the functions $\rho(t)$, $v_k(x, t)$, and $g_{jk}(t)$ as follows:

$$\rho(t)=\rho_0\sqrt{k(t)}\exp\left[\int_0^t\gamma(s)\mathrm{d}s\right], \tag{11}$$

$$g_{kj}=\frac{k(t)G_{kj}}{\rho_0^2\exp\left[2\int_0^t\gamma(s)\mathrm{d}s\right]}, \tag{12}$$

$$v_k(x,t)=\frac{\ddot{k}(t)k(t)-2\dot{k}^2(t)}{4k^2(t)}x^2+\frac{\ddot{c}(t)k(t)-2\dot{k}(t)\dot{c}(t)}{2k^2(t)}x-\dot{w}_k(t)-\frac{\dot{c}^2(t)}{4k^2(t)}, \tag{13}$$

where ρ_0 is a constant.

To obtain the solutions of (2), we make the complex transformation

$$\Phi_k(\eta,\tau)=A_k(\eta,\tau)e^{iB_k(\eta,\tau)}, \tag{14}$$

where $A_k(\eta, \tau)$ and $B_k(\eta, \tau)(k=1, 2, \ldots, N)$ are real functions of η and τ.

Substituting $\Phi_k(\eta, \tau)(k=1, 2, \ldots, N)$ into (2) and setting the real and imaginary parts of the resulting equations to zero lead to the following set of PDEs:

$$\frac{\partial A_k}{\partial\tau}+2\frac{\partial A_k}{\partial\eta}\frac{\partial B_k}{\partial\eta}+A_k\frac{\partial^2 B_k}{\partial\eta^2}=0, \tag{15}$$

$$A_k\frac{\partial B_k}{\partial\tau}-\frac{\partial^2 A_k}{\partial\eta^2}+A_k\left(\frac{\partial B_k}{\partial\eta}\right)^2+\sum_{j=1}^N G_{kj}A_j^2A_k=0. \tag{16}$$

The solution of (15) and (16) is chosen in the following form:

$$A_k(\eta,\tau)=a_k+b_kF(\theta)+d_kF^{-1}(\theta), \tag{17}$$

$$\theta=p\eta+q(\tau), \tag{18}$$

$$B_k(\eta,\tau)=m_k\eta+l_k(\tau), \tag{19}$$

where a_k, b_k, $d_k(k=1, 2, \ldots, N)$, p, m_k, and $q(\tau)$, $h(\tau)$ to be determined later, and $F'(\theta)=\frac{\mathrm{d}F}{\mathrm{d}\theta}$ satisfies

$$F'^2(\theta)=h_0+h_2F^2(\theta)+h_4F^4(\theta). \tag{20}$$

With the aid of Mathematica, substituting (17)–(19) along with (20) into (15) and (16), setting the coefficients of monomials of $F(\theta)$ and η of the resulting system numerator to zero, we obtain a set of ODEs with respect to unknowns a_k, b_k, d_k, p, m_k, and $q(\tau)$, $l_k(\tau)$.

$$a_k(l_k'(\tau)+m_k^2)=0,\quad q'(\tau)+2m_kp=0,$$
$$\sum_{j=1}^N G_{kj}b_kb_j^2=2p^2h_4b_k,\quad \sum_{j=1}^N G_{kj}d_kd_j^2=2p^2h_0d_k,$$
$$[l_k'(\tau)+(m_k^2-p^2h_2)]b_k+\sum_{j=1}^N G_{kj}(b_j^2d_k+2b_kb_jd_j)=0,$$
$$[l_k'(\tau)+(m_k^2-p^2h_2)]d_k+\sum_{j=1}^N G_{kj}(d_j^2b_k+2d_kd_jb_j)=0.$$

Solving the system, we get the following solution set:

Case 1:

$$a_k=0,\quad q(\tau)=-2mp\tau+\xi_0,$$
$$l_k(\tau)=p^2(h_2-2(2\pm1)\sqrt{h_0h_4})\tau-m^2\tau+\xi_k,$$
$$b_k=\pm p\sqrt{\frac{2h_4D_k}{D}},\quad d_k=\pm\epsilon p\sqrt{\frac{2h_0D_k}{D}},\quad p=p,\quad m_k=m, \tag{21}$$

where ξ_0, ξ_k, p, and m are arbitrary constants, and the parameter ϵ can have the values $\epsilon=0, \pm1$.

Case 2:

$$a_k=a_k,\quad q(\tau)=-2mp\tau+\zeta_0,\quad l_k(\tau)=-m^2\tau+\zeta_k,$$
$$h_2=2(2\pm1)\sqrt{h_0h_4},\quad b_k=\pm p\sqrt{\frac{2h_4D_k}{D}},\quad d_k=\pm\epsilon p\sqrt{\frac{2h_0D_k}{D}},$$
$$p=p,\quad m_k=m, \tag{22}$$

where a_k, ζ_0, ζ_k, p, and m are arbitrary constants, and the parameter ϵ can have the values $\epsilon=0$, ±1, the expression of D as following:

$$D=\begin{vmatrix} G_{11} & G_{12} & \cdots & G_{1N} \\ G_{21} & G_{22} & \cdots & G_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ G_{N1} & G_{N2} & \cdots & G_{NN} \end{vmatrix}$$

and D_k is the determinant formed by replacing the k-th column of D by the unit column vector.

Therefore, we obtain exact traveling wave solutions of (2)

$$\Phi_k=\pm p\sqrt{\frac{2D_k}{D}}\left[\sqrt{h_4}F(\theta)+\epsilon\sqrt{h_0}\frac{1}{F(\theta)}\right]e^{iB_k}, \tag{23}$$

where $B_k=mk(t)x+p^2(h_2-2(2\pm1)\sqrt{h_0h_4})\int_0^t k^2(s)\mathrm{d}s-m^2\int_0^t k^2(s)\mathrm{d}s+mc(t)+\xi_k$.

$$\Phi_k=a_k\pm p\sqrt{\frac{2D_k}{D}}\left[\sqrt{h_4}F(\theta)+\epsilon\sqrt{h_0}\frac{1}{F(\theta)}\right]e^{iB_k}, \tag{24}$$

where $B_k=mk(t)x-m^2\int_0^t k^2(s)\mathrm{d}s+mc(t)+\zeta_k$.

Substituting (21) and (24) along with (8)–(11) into (3), we get exact traveling wave solutions of (1) in the following form:

$$\Psi_k=\pm\rho_0\sqrt{k(t)}p\sqrt{\frac{2D_k}{D}}\left[\sqrt{h_4}F\left(pk(t)x+pc(t)-2mp\int_0^t k^2(s)\mathrm{d}s+\xi_0\right)+\frac{\epsilon\sqrt{h_0}}{F(pk(t)x+pc(t)-2mp\int_0^t k^2(s)\mathrm{d}s+\xi_0)}\right]\exp\left[\int_0^t\gamma(s)\mathrm{d}s+i(\varphi_k(x,t)+B_k)\right], \tag{25}$$

where $B_k=mk(t)x+p^2(h_2-2(2\pm1)\sqrt{h_0h_4})\int_0^t k^2(s)\mathrm{d}s-m^2\int_0^t k^2(s)\mathrm{d}s+mc(t)+\xi_k$.

$$\Psi_k=\rho_0\sqrt{k(t)}\left\{a_k\pm p\sqrt{\frac{2D_k}{D}}\left[\sqrt{h_4}F\left(pk(t)x+pc(t)-2mp\int_0^t k^2(s)\mathrm{d}s+\zeta_0\right)+\frac{\epsilon\sqrt{h_0}}{F(pk(t)x+pc(t)-2mp\int_0^t k^2(s)\mathrm{d}s+\zeta_0)}\right]\right\}\exp\left[\int_0^t\gamma(s)\mathrm{d}s+i(\varphi_k(x,t)+B_k)\right], \tag{26}$$

where $B_k=mk(t)x-m^2\int_0^t k^2(s)\mathrm{d}s+mc(t)+\zeta_k$.

Remark 2.1. As we know the more solutions of (20) we find, the more exact solutions of (1) may be obtained. However, the general solutions are difficult to be listed because of the complexity of (20). Some special solutions [29–33] are listed in Tables 1–3.

For example, if we choose $\epsilon=0$, we can obtain the following solutions using (25) and Table 1:

Table 1: Solutions of (20) m ($0<m<1$) denotes the modulus of the Jacobi elliptic function.

h_0	h_2	h_4	$F(\xi)$
0	>0	<0	$\sqrt{-\frac{h_2}{h_4}}\operatorname{sech}(\sqrt{h_2}\xi)$
0	>0	>0	$\sqrt{\frac{h_2}{h_4}}\operatorname{csch}(\sqrt{h_2}\xi)$
$\frac{h_2^2}{4h_4}$	<0	>0	$\sqrt{-\frac{h_2}{2h_4}}\tanh\left(\frac{\sqrt{-h_2}\xi}{2}\right)$
0	<0	>0	$\sqrt{-\frac{h_2}{h_4}}\sec(\sqrt{-h_2}\xi)$
0	<0	>0	$\sqrt{-\frac{h_2}{h_4}}\csc(\sqrt{-h_2}\xi)$
$\frac{h_2^2}{4h_4}$	>0	>0	$\sqrt{\frac{h_2}{2h_4}}\tan\left(\frac{\sqrt{h_2}\xi}{2}\right)$
1	$-(1+m^2)$	m^2	$\mathrm{sn}\xi$, $\mathrm{cd}\xi=\frac{\mathrm{cn}\xi}{\mathrm{dn}\xi}$
$(1-m^2)$	$2m^2-1$	$-m^2$	$\mathrm{cn}\xi$
m^2-1	$2-m^2$	-1	$\mathrm{dn}\xi$
m^2	$(-1+m^2)$	1	$\mathrm{ns}\xi=\frac{1}{\mathrm{sn}\xi}$, $\mathrm{dc}\xi=\frac{\mathrm{dn}\xi}{\mathrm{cn}\xi}$
$\frac{1}{4}$	$\frac{1-2m^2}{2}$	$\frac{1}{4}$	$\mathrm{ns}\xi\pm\mathrm{cs}\xi$
$\frac{1-m^2}{4}$	$\frac{1+m^2}{2}$	$\frac{1-m^2}{4}$	$\mathrm{nc}\xi\pm\mathrm{sc}\xi$

$k_1=\sqrt{1-m^2}$, A, B, C($ABC\neq0$), and D are arbitrary constants.

Table 2: Solutions of (20) (continued) m $(0<m<1)$ denotes the modulus of the Jacobi elliptic function.

h_0	h_2	h_4	$F(\xi)$
$-m^2$	$2m^2-1$	$1-m^2$	$\mathrm{nc}\xi$
-1	$2-m^2$	m^2-1	$\mathrm{nd}\xi$
1	$2-m^2$	$1-m^2$	$\mathrm{sc}\xi$
1	$2m^2-1$	$-m^2(1-m^2)$	$\mathrm{sd}\xi$
$1-m^2$	$2-m^2$	1	$\mathrm{cs}\xi$
$-m^2(1-m^2)$	$2m^2-1$	1	$\mathrm{ds}\xi$
$\frac{m^2}{4}$	$\frac{m^2-2}{2}$	$\frac{1}{4}$	$\mathrm{ns}\xi\pm\mathrm{ds}\xi$
$\frac{m^2}{4}$	$\frac{m^2-2}{2}$	$\frac{m^2}{4}$	$\mathrm{sn}\xi\pm i\mathrm{cn}\xi,\ \frac{\mathrm{dn}(\xi)}{\sqrt{1-m^2}\mathrm{sn}(\xi)\pm\mathrm{cn}(\xi)}$
$\frac{1}{4}$	$\frac{1-2m^2}{2}$	$\frac{1}{4}$	$m\mathrm{sn}\xi\pm i\mathrm{dn}\xi,\ \frac{\mathrm{sn}(\xi)}{1\pm\mathrm{cn}(\xi)}$
$\frac{1}{4}$	$\frac{m^2-2}{2}$	$\frac{m^2}{4}$	$\frac{\mathrm{sn}(\xi)}{1\pm\mathrm{dn}(\xi)}$
$\frac{m^2-1}{4}$	$\frac{m^2+1}{2}$	$\frac{m^2-1}{4}$	$\frac{\mathrm{dn}(\xi)}{1\pm m\mathrm{sn}(\xi)},\ m\mathrm{sd}(\xi)\pm\mathrm{nd}(\xi)$
$\frac{1-m^2}{4}$	$\frac{m^2+1}{2}$	$\frac{1-m^2}{4}$	$\frac{\mathrm{cn}(\xi)}{1\pm\mathrm{sn}(\xi)},\ \mathrm{nc}(\xi)\pm\mathrm{sc}(\xi)$
$\frac{1}{4}$	$\frac{m^2+1}{2}$	$\frac{(1-m^2)^2}{4}$	$\frac{\mathrm{sn}(\xi)}{\mathrm{dn}(\xi)\pm\mathrm{cn}(\xi)}$
$\frac{1}{4}$	$\frac{m^2-2}{2}$	$\frac{m^4}{4}$	$\frac{\mathrm{cn}(\xi)}{\sqrt{1-m^2}\pm\mathrm{dn}(\xi)}$
$\frac{-(1-m^2)^2}{4}$	$\frac{m^2+1}{2}$	$\frac{-1}{4}$	$m\mathrm{cn}(\xi)\pm\mathrm{dn}(\xi)$

$k_1=\sqrt{1-m^2}$, A, B, $C(ABC\neq 0)$, and D are arbitrary constants.

Case 2.1 Soliton and soliton-like solutions

$$\Psi_k=\pm\rho_0(t)\sqrt{k(t)}p\sqrt{\frac{-2h_2D_k}{D}}\mathrm{sech}(\sqrt{h_2}\theta)\\ \exp\left[\int_0^t\gamma(s)\mathrm{d}s+i(\varphi_k(x,\ t)+B_k)\right],\tag{27}$$

where $B_k=mk(t)x+p^2h_2\int_0^t k^2(s)\mathrm{d}s-m^2\int_0^t k^2(s)\mathrm{d}s+mc(t)+\xi_k$.

$$\Psi_k=\pm\rho_0(t)\sqrt{k(t)}p\sqrt{\frac{2h_2D_k}{D}}\mathrm{csch}(\sqrt{h_2}\theta)\\ \exp\left[\int_0^t\gamma(s)\mathrm{d}s+i(\varphi_k(x,\ t)+B_k)\right],\tag{28}$$

where $B_k=mk(t)x+p^2h_2\int_0^t k^2(s)\mathrm{d}s-m^2\int_0^t k^2(s)\mathrm{d}s+mc(t)+\xi_k$.

$$\Psi_k=\pm\rho_0(t)\sqrt{k(t)}p\sqrt{\frac{h_2D_k}{-2D}}\tanh\left(\frac{\sqrt{-h_2}\theta}{2}\right)\\ \exp\left[\int_0^t\gamma(s)\mathrm{d}s+i(\varphi_k(x,\ t)+B_k)\right],\tag{29}$$

where $B_k=mk(t)x-m^2\int_0^t k^2(s)\mathrm{d}s+mc(t)+\xi_k$.

$$\Psi_k=\pm\rho_0(t)\sqrt{k(t)}p\sqrt{\frac{h_2D_k}{-2D}}\tanh\left(\frac{\sqrt{-h_2}\theta}{2}\right)\\ \exp\left[\int_0^t\gamma(s)\mathrm{d}s+i(\varphi_k(x,\ t)+B_k)\right],\tag{30}$$

where $B_k=mk(t)x+2p^2h_2\int_0^t k^2(s)\mathrm{d}s-m^2\int_0^t k^2(s)\mathrm{d}s+mc(t)+\xi_k$.

Case 2.2 Triangular periodic solutions

$$\Psi_k=\pm\rho_0(t)\sqrt{k(t)}p\sqrt{\frac{-2h_2D_k}{D}}\sec(\sqrt{-h_2}\theta)\\ \exp\left[\int_0^t\gamma(s)\mathrm{d}s+i(\varphi_k(x,\ t)+B_k)\right],\tag{31}$$

where $B_k=mk(t)x+p^2h_2\int_0^t k^2(s)\mathrm{d}s-m^2\int_0^t k^2(s)\mathrm{d}s+mc(t)+\xi_k$.

$$\Psi_k=\pm\rho_0(t)\sqrt{k(t)}p\sqrt{\frac{-2h_2D_k}{D}}\csc(\sqrt{-h_2}\theta)\\ \exp\left[\int_0^t\gamma(s)\mathrm{d}s+i(\varphi_k(x,\ t)+B_k)\right],\tag{32}$$

where $B_k=mk(t)x+p^2h_2\int_0^t k^2(s)\mathrm{d}s-m^2\int_0^t k^2(s)\mathrm{d}s+mc(t)+\xi_k$.

$$\Psi_k=\pm\rho_0(t)\sqrt{k(t)}p\sqrt{\frac{h_2D_k}{2D}}\tan\left(\frac{\sqrt{h_2}\theta}{2}\right)\\ \exp\left[\int_0^t\gamma(s)\mathrm{d}s+i(\varphi_k(x,\ t)+B_k)\right],\tag{33}$$

where $B_k=mk(t)x-m^2\int_0^t k^2(s)\mathrm{d}s+mc(t)+\xi_k$.

$$\Psi_k=\pm\rho_0(t)\sqrt{k(t)}p\sqrt{\frac{h_2D_k}{2D}}\tan\left(\frac{\sqrt{h_2}\theta}{2}\right)\\ \exp\left[\int_0^t\gamma(s)\mathrm{d}s+i(\varphi_k(x,\ t)+B_k)\right],\tag{34}$$

where $B_k=mk(t)x+2p^2h_2\int_0^t k^2(s)\mathrm{d}s-m^2\int_0^t k^2(s)\mathrm{d}s+mc(t)+\xi_k$.

Case 2.3 Jacobi elliptic function solutions and combined Jacobi elliptic function solutions

$$\Psi_k=\pm mp\rho_0(t)\sqrt{k(t)}\sqrt{\frac{2D_k}{D}}\mathrm{sn}(\theta)\\ \exp\left[\int_0^t\gamma(s)\mathrm{d}s+i(\varphi_k(x,\ t)+B_k)\right],\tag{35}$$

Table 3: Solutions of (20) (continued) m $(0<m<1)$ denotes the modulus of the Jacobi elliptic function.

h_0	h_2	h_4	$F(\xi)$
$\frac{1}{4}$		$\frac{1}{4}$	$\frac{\mathrm{dn}(\xi)}{m\,\mathrm{cn}(\xi)\pm i\sqrt{1-m^2}}$
	$\frac{1-2m^2}{2}$		$\frac{\mathrm{sn}(\xi)}{1\pm \mathrm{cn}(\xi)}$
	$\frac{1-2m^2}{2}$		$\frac{\mathrm{cn}(\xi)}{\sqrt{1-m^2}\,\mathrm{sn}(\xi)\pm \mathrm{dn}(\xi)}$
1	$2-4m^2$	1	$\frac{\mathrm{sn}(\xi)\mathrm{dn}(\xi)}{\mathrm{cn}(\xi)}$
$\frac{(m-1)^2}{4A^2}$	$\frac{1+m^2+6m}{2}$	$\frac{A^2(m-1)^2}{4}$	$\frac{\mathrm{dn}(\xi)\mathrm{cn}(\xi)}{A(1+\mathrm{sn}(\xi))(1+m\,\mathrm{sn}(\xi))}$
$\frac{(m+1)^2}{4A^2}$	$\frac{1+m^2-6m}{2}$	$\frac{A^2(m+1)^2}{4}$	$\frac{\mathrm{dn}(\xi)\mathrm{cn}(\xi)}{A(1+\mathrm{sn}(\xi))(1-m\,\mathrm{sn}(\xi))}$
$-2m^3+m^4+m^2$	$6m-m^2-1$	$\frac{-4}{m}$	$\frac{m\,\mathrm{dn}(\xi)\mathrm{cn}(\xi)}{1+m\,\mathrm{sn}^2(\xi)}$
$2m^3+m^4+m^2$	$-6m-m^2-1$	$\frac{4}{m}$	$\frac{m\,\mathrm{dn}(\xi)\mathrm{cn}(\xi)}{-1+m\,\mathrm{sn}^2(\xi)}$
$2+2k_1-m^2$	$6k_1-m^2+2$	$4k_1$	$\frac{m^2\mathrm{sn}(\xi)\mathrm{cn}(\xi)}{k_1-\mathrm{dn}^2(\xi)}$
$2-2k_1-m^2$	$-6k_1-m^2+2$	$-4k_1$	$\frac{-m^2\mathrm{sn}(\xi)\mathrm{cn}(\xi)}{k_1+\mathrm{dn}^2(\xi)}$
$\frac{m^2-1}{4(C^2m^2-B^2)}$	$\frac{m^2+1}{2}$	$\frac{(C^2m^2-B^2)(m^2-1)}{4}$	$\frac{\sqrt{\frac{B^2-C^2}{B^2-C^2m^2}}+\mathrm{sn}(\xi)}{B\,\mathrm{cn}(\xi)+C\,\mathrm{dn}(\xi)}$
$\frac{m^4}{4(C^2+B^2)}$	$\frac{m^2}{2}-1$	$\frac{(C^2+B^2)}{4}$	$\frac{\sqrt{\frac{B^2+C^2-C^2m^2}{B^2+C^2}}+\mathrm{dn}(\xi)}{B\,\mathrm{sn}(\xi)+C\,\mathrm{cn}(\xi)}$
$\frac{2m-m^2-1}{B^2}$	$2m^2+2$	$-B^2m^2-B^2-2B^2m$	$\frac{m\,\mathrm{sn}^2(\xi)-1}{B(m\,\mathrm{sn}^2(\xi)+1)}$
$-\frac{2m+m^2+1}{B^2}$	$2m^2+2$	$-B^2m^2-B^2-2B^2m$	$\frac{m\,\mathrm{sn}^2(\xi)+1}{B(m\,\mathrm{sn}^2(\xi)-1)}$
0	1	$\frac{1}{2}$	$\pm\frac{\sqrt{2-2\tanh^2(D-\xi)}}{\tanh(D-\xi)}$
0	0	>0	$-\frac{1}{\sqrt{h_4}\xi}$

$k_1=\sqrt{1-m^2}$, A, B, $C(ABC\neq 0)$, and D are arbitrary constants.

where $B_k = mk(t)x - p^2(1+m)^2\int_0^t k^2(s)\mathrm{d}s - m^2\int_0^t k^2(s)\mathrm{d}s + mc(t) + \xi_k$.

$$\Psi_k = \pm mp\rho_0(t)\sqrt{k(t)}\sqrt{\frac{2D_k}{D}}\mathrm{sn}(\theta)\exp\left[\int_0^t \gamma(s)\mathrm{d}s + i(\varphi_k(x,t)+B_k)\right], \tag{36}$$

where $B_k = mk(t)x - p^2(1+6m+m^2)\int_0^t k^2(s)\mathrm{d}s - m^2\int_0^t k^2(s)\mathrm{d}s + mc(t) + \xi_k$.

$$\Psi_k = \pm mp\rho_0(t)\sqrt{k(t)}\sqrt{\frac{2D_k}{D}}\mathrm{cd}(\theta)\exp\left[\int_0^t \gamma(s)\mathrm{d}s + i(\varphi_k(x,t)+B_k)\right], \tag{37}$$

where $B_k = mk(t)x - p^2(1+m)^2\int_0^t k^2(s)\mathrm{d}s - m^2\int_0^t k^2(s)\mathrm{d}s + mc(t) + \xi_k$.

$$\Psi_k = \pm mp\rho_0(t)\sqrt{k(t)}\sqrt{\frac{2D_k}{D}}\mathrm{cd}(\theta)\exp\left[\int_0^t \gamma(s)\mathrm{d}s + i(\varphi_k(x,t)+B_k)\right], \tag{38}$$

where $B_k = mk(t)x - p^2(1+6m+m^2)\int_0^t k^2(s)\mathrm{d}s - m^2\int_0^t k^2(s)\mathrm{d}s + mc(t) + \xi_k$.

$$\Psi_k = \pm imp\rho_0(t)\sqrt{k(t)}\sqrt{\frac{2D_k}{D}}\mathrm{cn}(\theta)\exp\left[\int_0^t \gamma(s)\mathrm{d}s + i(\varphi_k(x,t)+B_k)\right], \tag{39}$$

where $B_k = mk(t)x + p^2[2m^2 - 2(2\pm1)m\sqrt{m^2-1} - 1]\int_0^t k^2(s)\mathrm{d}s - m^2\int_0^t k^2(s)\mathrm{d}s + mc(t) + \xi_k$.

$$\Psi_k = \pm ip\rho_0(t)\sqrt{k(t)}\sqrt{\frac{2D_k}{D}}\mathrm{dn}(\theta)\exp\left[\int_0^t \gamma(s)\mathrm{d}s + i(\varphi_k(x,t)+B_k)\right], \tag{40}$$

where $B_k = mk(t)x + p^2[2 - m^2 - 2(2\pm1)\sqrt{1-m^2}]\int_0^t k^2(s)\mathrm{d}s - m^2\int_0^t k^2(s)\mathrm{d}s + mc(t) + \xi_k$.

$$\Psi_k = \pm p\rho_0(t)\sqrt{k(t)}\sqrt{\frac{2D_k}{D}}\mathrm{ns}(\theta)\exp\left[\int_0^t \gamma(s)\mathrm{d}s + i(\varphi_k(x,t)+B_k)\right], \tag{41}$$

where $B_k = mk(t)x - p^2[1+m^2+2(2\pm1)m]\int_0^t k^2(s)\mathrm{d}s - m^2\int_0^t k^2(s)\mathrm{d}s + mc(t) + \xi_k$.

$$\Psi_k = \pm p\rho_0(t)\sqrt{k(t)}\sqrt{\frac{2D_k}{D}}\mathrm{dc}(\theta)\exp\left[\int_0^t \gamma(s)\mathrm{d}s + i(\varphi_k(x,t)+B_k)\right], \tag{42}$$

where $B_k = mk(t)x - p^2[1+m^2+2(2\pm1)m]\int_0^t k^2(s)\mathrm{d}s - m^2\int_0^t k^2(s)\mathrm{d}s + mc(t) + \xi_k$.

$$\Psi_k = \pm p\rho_0(t)\sqrt{k(t)}\sqrt{\frac{D_k}{2D}}[\mathrm{ns}(\theta)\pm\mathrm{cs}(\theta)]\exp\left[\int_0^t \gamma(s)\mathrm{d}s + i(\varphi_k(x,t)+B_k)\right], \tag{43}$$

where $B_k = mk(t)x - p^2m^2\int_0^t k^2(s)\mathrm{d}s - m^2\int_0^t k^2(s)\mathrm{d}s + mc(t) + \xi_k$.

$$\Psi_k = \pm p\rho_0(t)\sqrt{k(t)}\sqrt{\frac{D_k}{2D}}[\mathrm{ns}(\theta)\pm\mathrm{cs}(\theta)]\exp\left[\int_0^t \gamma(s)\mathrm{d}s + i(\varphi_k(x,t)+B_k)\right], \tag{44}$$

where $B_k = mk(t)x - p^2(1+m^2)\int_0^t k^2(s)\mathrm{d}s - m^2\int_0^t k^2(s)\mathrm{d}s + mc(t) + \xi_k$.

$$\Psi_k = \pm p\rho_0(t)\sqrt{(1-m^2)k(t)}\sqrt{\frac{D_k}{2D}}[\mathrm{nc}(\theta)\pm\mathrm{sc}(\theta)]\exp\left[\int_0^t \gamma(s)\mathrm{d}s + i(\varphi_k(x,t)+B_k)\right], \tag{45}$$

where $B_k = mk(t)x + p^2m^2\int_0^t k^2(s)\mathrm{d}s - m^2\int_0^t k^2(s)\mathrm{d}s + mc(t) + \xi_k$.

$$\Psi_k = \pm p\rho_0(t)\sqrt{(1-m^2)k(t)}\sqrt{\frac{D_k}{2D}}[\mathrm{nc}(\theta)\pm\mathrm{sc}(\theta)]\exp\left[\int_0^t \gamma(s)\mathrm{d}s + i(\varphi_k(x,t)+B_k)\right], \tag{46}$$

where $B_k = mk(t)x + p^2(2m^2-1)\int_0^t k^2(s)\mathrm{d}s - m^2\int_0^t k^2(s)\mathrm{d}s + mc(t) + \xi_k$.

Since $k(t)$ and $c(t)$ are arbitrary functions, the solution (27) has abundant properties. We show some properties of the bright soliton intensity of Eq. (27) in Figure 1. And the wave propagation of the dark soliton intensity of Eq. (29) is illustrated in Figure 2.

Remark 2.2 We can obtain many other solutions using (25), (26), and Tables 1–3, but we omit them for simplicity.

Remark 2.3 There are some hyperbolic function solutions and trigonometric function solutions can be obtained at the limit case when $m \to 1$ and $m \to 0$, but we also omit them for simplicity.

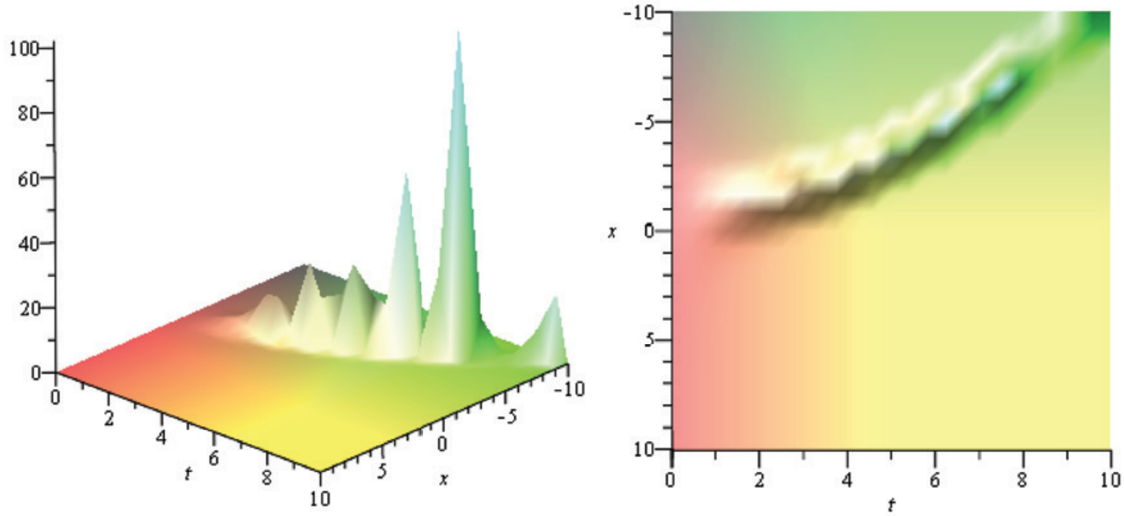

Figure 1: Bright soliton intensity of (27) with $k(t)=c(t)=t$, $h_2=p=1$, $D_k=-D$, $m=-0.15$, $\gamma(t)=-2t$, $\rho_0(t)=e^{t^2}$.

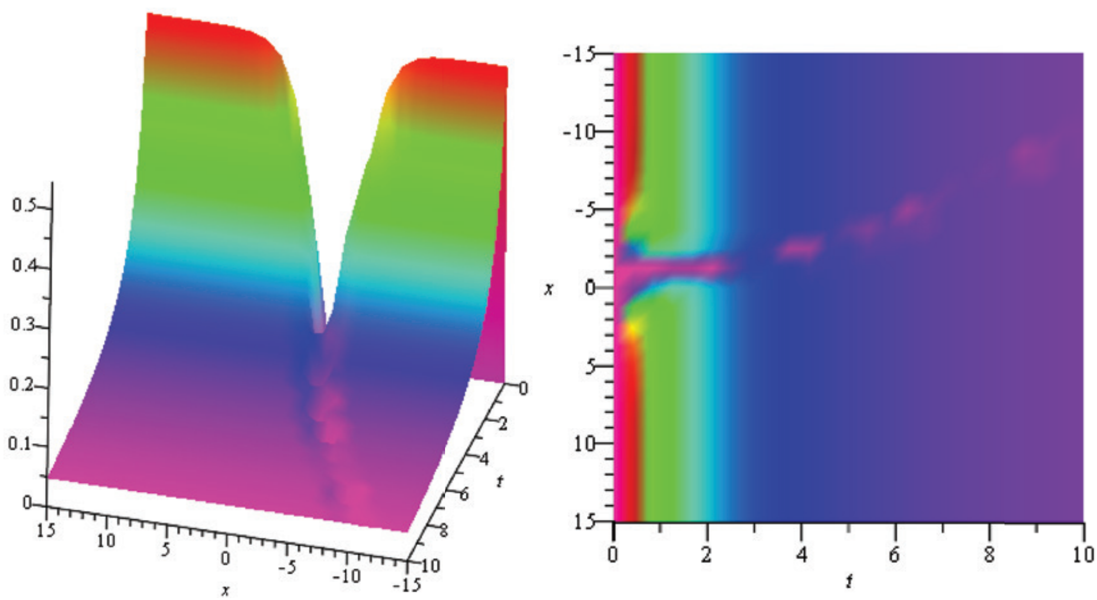

Figure 2: Dark soliton intensity of (29) with $k(t)=c(t)=t$, $h_2=-4$, $p=1$, $D_k=2D$, $m=-0.15$, $\gamma(t)=-2t$, $\rho_0(t)=\frac{1}{\sqrt{t+2t^2}}e^{t^2}$.

3 Conclusion

In this paper, we have obtained many types of exact solutions for the *N*-coupled NLS equations with variable coefficients by means of the combination method of the similarity transformation and auxiliary equation method, which include soliton solutions, combined soliton solutions, triangular periodic solutions, Jacobi elliptic function solutions, and combined Jacobi elliptic function solutions. To the best of our knowledge, the solutions obtained in this letter have not been reported in the previous literature. These solutions may provide more information to further study the mechanisms of the complicated nonlinear physical phenomena.

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