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Bäcklund Transformation and Soliton Solutions for a (3+1)-Dimensional Variable-Coefficient Breaking Soliton Equation

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Abstract: In this article, a (3+1)-dimensional variable-coefficient breaking soliton equation is investigated. Based on the Bell polynomials and symbolic computation, the bilinear forms and Bäcklund transformation for the equation are derived. One-, two-, and three-soliton solutions are obtained via the Hirota method. *N*-soliton solutions are also constructed. Propagation characteristics and interaction behaviors of the solitons are discussed graphically: (i) solitonic direction and position depend on the sign of the wave numbers; (ii) shapes of the multisoliton interactions in the scaled space and time coordinates are affected by the variable coefficients; (iii) multisoliton interactions are elastic for that the velocity and amplitude of each soliton remain unchanged after each interaction except for a phase shift.

Keywords: (3+1)-Dimensional Variable-Coefficient Breaking Soliton Equation; Bäcklund Transformation; Bell Polynomials; Hirota Method; *N*-Soliton Solutions.

1 Introduction

Nonlinear evolution equations (NLEEs) have been used as the models to describe some nonlinear phenomena in fluids, plasmas, fibre optics, etc. [1–6]. To understand the mechanism of those phenomena, people have tried to seek their analytic solutions, especially the soliton solutions [7–12]. Methods to obtain their solutions have been proposed, such as the inverse scattering transformation method [13, 14], Hirota method [15, 16], Bäcklund transformation [17], Darboux transformation [18], and Bell polynomials [19–21]. Among them, Hirota method has been used to obtain the bilinear forms and analytic solutions via the dependent-variable transformation and formal parameter expansion [15, 16, 22, 23]. Bäcklund transformations have also been derived through the Hirota method [15]. Connection between the Bell polynomials and Hirota method has been proved [19–21]. Bell-polynomial-typed Bäcklund transformations have been transformed into the bilinear forms [16, 24].

Among the NLEEs, a (2+1)-dimensional breaking soliton equation [25],

$$u_{xt}-4u_x u_{xy}-2u_{xx}u_y-u_{xxxy}=0 \tag{1}$$

has been used to describe the interaction between the Riemann wave propagating along the y-axis and a long wave propagating along the x-axis, where u is a function of the scaled space coordinates x and y with time coordinate t [26, 27]. Equation (1) has been extended to a (3+1)-dimensional breaking soliton equation as follows [28, 29]:

$$u_{xt}-4u_x(u_{xy}+u_{xz})-2u_{xx}(u_y+u_z)-(u_{xxxy}+u_{xxxz})=0, \tag{2}$$

where z is a scaled space coordinate. Similarly to (1), (2) can describe the interactions among two Riemann waves propagating along the y- or z-axis and a long wave propagating along the x-axis. Multisoliton solutions for (2) have been derived via the modified form of Hirota method [28]. New exact solutions for (2) have also been obtained via the Hirota method [29].

However, to describe the phenomena in the realistic world, the variable-coefficient NLEEs have been constructed and studied [30–32]. In the context of ocean waves, the temporal variability of the variable coefficients may be due to the pressure dependence of thermal expansion coefficient of seawater coupled with the large-scale meridional variation of the oceanic temperature–salinity relation, topography of the continental shelf, changing hydrography from deep to shallow water and other dynamical conditions [33–35]. Thus, in this article, we investigate a variable-coefficient extension of (2), i.e.

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$$u_{xt}+\alpha(t)u_x(u_{xy}+u_{xz})+\beta(t)u_{xx}(u_y+u_z)+\gamma(t)(u_{xxxy}+u_{xxxz})=0, \tag{3}$$

where $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ are the analytic functions of t. Special cases of (3) in fluids and plasmas have been investigated:

1. When $\alpha(t)=\frac{1}{2}$, $\beta(t)=\frac{1}{4}$, and $\gamma(t)=\frac{1}{8}$, by virtue of a dimensional reduction $\partial_y=\partial_z$ and a transformation $h(x, z, t)=u_x(x, z, t)$, (3) has been degenerated into the Calogero–Bogoyavlenskii–Schiff equation [25, 36]:
$$h_t+hh_z+\frac{1}{2}h_x\int h_z\mathrm{d}x+\frac{1}{4}h_{xxz}=0, \tag{4}$$
which has been used to describe the interaction between a Riemann wave propagating along the z-axis and a long wave propagating along the x-axis [37].
2. With a dimensional reduction $\partial_z=\partial_y$, (2) has been seen to reduce to a (2+1)-dimensional variable-coefficient breaking soliton equation [38],
$$u_{xt}+\alpha(t)u_xu_{xy}+\beta(t)u_{xx}u_y+\gamma(t)u_{xxxy}=0, \tag{5}$$
with the relevant soliton solutions discussed [38].
3. When $\alpha(t)=-2$, $\beta(t)=-1$ and $\gamma(t)=\frac{1}{2}$, by virtue of a dimensional reduction $\partial_z=\partial_y$, (3) has become (1) [25].
4. When $\alpha(t)=-4$, $\beta(t)=-2$, and $\gamma(t)=1$, (3) has become (2) [28, 29].

However, to our knowledge, neither the multisoliton solutions nor the Bäcklund transformation for (3) have been obtained. In Section 2, concepts and identities about the Bell polynomials will be introduced. In Section 3, bilinear form and multisolition solutions for (3) will be presented. In Section 4, Bäcklund transformation for (3) will be derived. In Section 5, multisoliton propagation and interaction will be disscussed graphically. Section 6 will be our conclusions.

2 Bell Polynomials

The Bell polynomials [19–21] are defined as

$$Y_{\kappa x}(\phi)=Y_\kappa(\phi_1, \ldots, \phi_\kappa)=e^{-\phi}\partial_x^\kappa e^\phi,\ (\kappa=1, 2, \ldots), \tag{6}$$

where ϕ is a C^∞ function of x and $\phi_\kappa=\partial_x^\kappa\phi$. For example,

$$Y_x(\phi)=\phi_1,\quad Y_{2x}(\phi)=\phi_2+\phi_1^2,\quad Y_{3x}(\phi)=\phi_3+3\phi_1\phi_2+\phi_1^3,\ \ldots \tag{7}$$

Similarly, two-dimensional extension of the Bell polynomials [19–21] is introduced as

$$Y_{nx,pt}(\varphi)=Y_{n,p}(\varphi_{1,1}, \ldots, \varphi_{1,p}, \ldots, \varphi_{n,1}, \ldots,\varphi_{n,p})=e^{-\varphi}\partial_x^n\partial_t^pe^\varphi, \tag{8}$$

where φ is a C^∞ function of x and t, $\varphi_{i,j}=\partial_x^i\partial_t^j\varphi$, with $i=1, \ldots, n$ and $j=1, \ldots, p$, while n and p are the nonnegative integers. Two-dimensional binary Bell polynomials [20, 21] are

$$\mathscr{Y}_{nx,pt}(V, W)=Y_{nx,pt}(v_{i,j})\Big|_{v_{i,j}=\begin{cases}V_{i,j},\ i+j=\text{odd}\\ W_{i,j},\ i+j=\text{even}\end{cases}}, \tag{9}$$

where v, V and W are all the C^∞ functions of x and t, $v_{i,j}=\partial_x^i\partial_t^jv$, $V_{i,j}=\partial_x^i\partial_t^jV$, and $W_{i,j}=\partial_x^i\partial_t^jW$. The link between the Bell polynomials and Hirota D-operators can be given as [20, 21]

$$\mathscr{Y}_{nx,pt}[V=\ln(G/F),\ W=\ln(GF)]=(GF)^{-1}D_x^nD_t^pG\cdot F, \tag{10}$$

where G and F are the functions of x and y, D_x and D_y are the Hirota bilinear operators defined by

$$D_x^nD_t^pG\cdot F=\left(\frac{\partial}{\partial_x}-\frac{\partial}{\partial_{x'}}\right)^n\left(\frac{\partial}{\partial_t}-\frac{\partial}{\partial_{t'}}\right)^p\times G(x, t)F(x', t')\big|_{x'=x,t'=t}, \tag{11}$$

x' and t' are the formal variables. Particulary, with the assumption that $q=W-V=2\ln F$, we can get the $\mathscr{P}$-polynomials as

$$\mathscr{P}_{nx,pt}(q)=\mathscr{Y}_{nx,pt}(0, q). \tag{12}$$

For example,

$$\mathscr{P}_{x,t}=q_{xt},\quad \mathscr{P}_{2x}(q)=q_{xx},\quad \mathscr{P}_{3x,t}=q_{3x,t}+3q_{2x}q_{x,t},\quad \mathscr{P}_{2x,2t}=q_{2x,2t}+q_{2x}q_{2t}+2q_{x,t}^2,\ \ldots \tag{13}$$

When $q=2\ln G$, the $\mathscr{P}$-polynomials can be linked to the Hirota D-operator according to the following identity [19–21]:

$$\mathscr{P}_{nx,pt}(q=2\ln G)=G^{-2}D_x^nD_t^pG\cdot G. \tag{14}$$

3 Bilinear Forms and Soliton Solutions for (3)

3.1 Bilinear Forms for (3) under Constraints (18)

In this section, via the Bell polynomials, bilinear form for (3) will be deduced. Under the scale transformations,

$$x\to\lambda^{\rho_1}x,\ \ y\to\lambda^{\rho_2}y,\ \ z\to\lambda^{\rho_3}z,\ \ t\to\lambda^{\rho_4},\ \ u\to\lambda^{\zeta}u, \tag{15}$$

(3) keeps invariant when $\rho_3=\rho_2$, $\rho_4=3\rho_1+\rho_2$, and $\zeta=-\rho_1$, where λ, ρ_1, ρ_2, ρ_3, ρ_4 and ζ are all the non-zero real constants to be determined. Therefore, we set that

$$u=cQ_x, \tag{16}$$

where Q is a function of x, y, z, and t with c as a nonzero real constant. Substituting (16) into (3), we obtain

$$Q_{xxt}+c\alpha(t)Q_{xx}(Q_{xxy}+Q_{xxz})+c\beta(t)Q_{xxx}(Q_{xy}+Q_{xz})+\gamma(t)(Q_{xxxxy}+Q_{xxxxz})=0. \tag{17}$$

In order to transform (17) into the $\mathscr{P}$-polynomials, we set the constraints

$$\alpha(t)=2\beta(t),\ \ \gamma(t)=\frac{1}{2}c\beta(t). \tag{18}$$

Considering (13), we integrate (17) with respect to x and let the integration constant equal to zero, so as to get

$$\mathscr{P}_{xt}(Q)+\frac{1}{3}c\beta(t)\mathscr{P}_{3x,y}(Q)+\frac{1}{3}c\beta(t)\mathscr{P}_{3x,z}(Q)+\frac{1}{6}c\beta(t)\partial_y\int\mathscr{P}_{4x}(Q)\mathrm{d}x+\frac{1}{6}c\beta(t)\partial_z\int\mathscr{P}_{4x}(Q)\mathrm{d}x=0. \tag{19}$$

Introducing an auxiliary independent variable s, we can transform (19) into

$$E_1(Q)=\mathscr{P}_{4x}(Q)-m\mathscr{P}_{xs}(Q)=0, \tag{20a}$$

$$E_2(Q)=\mathscr{P}_{xt}(Q)+\frac{1}{3}c\beta(t)\mathscr{P}_{3x,y}(Q)+\frac{1}{3}c\beta(t)\mathscr{P}_{3x,z}(Q)+\frac{m}{6}c\beta(t)\mathscr{P}_{ys}(Q)+\frac{m}{6}c\beta(t)\mathscr{P}_{zs}(Q)=0, \tag{20b}$$

where m is a nonzero constant and E_1 and E_2 are the polynomials of Q. Setting $Q=2\ln f(x,y,z,t)$, with f as a real function of x, y, z, and t, we can get the bilinear forms of (3) under Constraints (18) as follows:

$$(D_x^4-mD_xD_s)f\cdot f=0, \tag{21a}$$

$$\left[D_xD_t+\frac{1}{3}c\beta(t)D_x^3D_y+\frac{1}{3}c\beta(t)D_x^3D_z+\frac{m}{6}c\beta(t)D_yD_s+\frac{m}{6}c\beta(t)D_zD_s\right]f\cdot f=0. \tag{21b}$$

The relation between u and f is

$$u=2c(\ln f)_x. \tag{22}$$

Via the symbolic computation [39–41] and Hirota method, we expand f as the power series of a small parameter ε:

$$f=1+\varepsilon f_1(x,y,z,t)+\varepsilon^2 f_2(x,y,z,t)+\varepsilon^3 f_3(x,y,z,t)+\ldots, \tag{23}$$

where $f_\chi\,(x,y,z,t)$ ($\chi=1,2,\ldots$) are the functions of x, y, z and t. Substituting Expression (23) into (21) and collecting the coefficients of the same power of ε, we have

$$\varepsilon^0:(D_x^4-mD_xD_s)1\cdot1=0, \tag{24a}$$

$$\varepsilon^0:\left[D_xD_t+\frac{1}{3}c\beta(t)D_x^3D_y+\frac{1}{3}c\beta(t)D_x^3D_z+\frac{m}{6}c\beta(t)D_yD_s+\frac{m}{6}c\beta(t)D_zD_s\right]1\cdot1=0, \tag{24b}$$

$$\varepsilon^1:(D_x^4-mD_xD_s)(1\cdot f_1+f_1\cdot1)=0, \tag{24c}$$

$$\varepsilon^1:\left[D_xD_t+\frac{1}{3}c\beta(t)D_x^3D_y+\frac{1}{3}c\beta(t)D_x^3D_z+\frac{m}{6}c\beta(t)D_yD_s+\frac{m}{6}c\beta(t)D_zD_s\right]\times(1\cdot f_1+f_1\cdot1)=0, \tag{24d}$$

$$\varepsilon^2:(D_x^4-mD_xD_s)(1\cdot f_2+f_1\cdot f_1+f_2\cdot1)=0, \tag{24e}$$

$$\varepsilon^2:\left[D_xD_t+\frac{1}{3}c\beta(t)D_x^3D_y+\frac{1}{3}c\beta(t)D_x^3D_z+\frac{m}{6}c\beta(t)D_yD_s+\frac{m}{6}c\beta(t)D_zD_s\right]\times(1\cdot f_2+f_1\cdot f_1+f_2\cdot1)=0, \tag{24f}$$

$$\varepsilon^3:(D_x^4-mD_xD_s)(1\cdot f_3+f_1\cdot f_2+f_2\cdot f_1+f_3\cdot1)=0, \tag{24g}$$

$$\varepsilon^3:\left[D_xD_t+\frac{1}{3}c\beta(t)D_x^3D_y+\frac{1}{3}c\beta(t)D_x^3D_z+\frac{m}{6}c\beta(t)D_yD_s+\frac{m}{6}c\beta(t)D_zD_s\right]\times(1\cdot f_3+f_1\cdot f_2+f_2\cdot f_1+f_3\cdot1)=0,\ \vdots \tag{24h}$$

3.2 Soliton Solutions for (3) under Constraints (18)

In order to obtain the one-soliton solutions for (3), we assume that

$$f_1=e^{\xi_1}, \tag{25}$$

where $\xi_1=a_1x+b_1y+l_1y+k_1s+\omega_1(t)+\delta_1$, while a_1, b_1, l_1, k_1, and δ_1 are all the nonzero real constants, and $\omega_1(t)$ is a function of t. Truncating Expression (23) with $f_\vartheta=0$ ($\vartheta=2$, 3, 4, …) and solving (24), we have

$$k_1=\frac{a_1^3}{m}, \tag{26a}$$

$$\omega_1(t)=-\frac{1}{2}a_1^2c(b_1+l_1)\int\beta(t)\mathrm{d}t. \tag{26b}$$

Thus, the one-soliton solutions for (3) under Constraints (18) can be written as

$$u=2c[\ln(1+e^{\xi_1})]_x=\frac{2ca_1e^{\xi_1}}{1+e^{\xi_1}}. \tag{27}$$

Similarly, to construct the two-soliton solutions, we set that

$$f=e^{\xi_1}+e^{\xi_2}+e^{\xi_1+\xi_2+A_{12}}, \tag{28}$$

where $\xi_j=a_jx+b_jy+l_jz+k_js+\omega_j(t)+\delta_j$ ($j=1$, 2), while a_2, b_2, l_2, k_2, δ_2 and A_{12} are all the nonzero real constants, and $\omega_2(t)$ is a function of t. Truncating Expression (23) with $f_\theta=0$ ($\theta=3$, 4, 5, …) and substituting (28) into (24), we obtain the two-soliton solutions under Constraints (18) as

$$u=2c[\ln(1+e^{\xi_1}+e^{\xi_2}+e^{\xi_1+\xi_2+A_{12}})]_x, \tag{29}$$

with

$$\begin{aligned}&k_j=\frac{a_j^3}{m},\\&\omega_j(t)=-\frac{1}{2}ca_j^2(b_j+l_j)\int\beta(t)\mathrm{d}t,\\&e^{A_{12}}=\frac{(a_1-a_2)^2}{(a_1+a_2)^2}.\end{aligned} \tag{30}$$

In order to derive the three-soliton solutions, we set

$$\begin{aligned}f=&1+e^{\xi_1}+e^{\xi_2}+e^{\xi_3}+e^{\xi_1+\xi_2+A_{12}}+e^{\xi_1+\xi_3+A_{13}}\\&+e^{\xi_2+\xi_3+A_{23}}+e^{\xi_1+\xi_2+\xi_3+A_{123}},\end{aligned} \tag{31}$$

where $\xi_j=a_jx+b_jy+l_jz+k_js+\omega_j(t)+\delta_j$ ($j=1$, 2, 3), while a_3, b_3, l_3, k_3, δ_3, A_{13}, A_{23}, and A_{123} are all the nonzero real constants, and $\omega_3(t)$ is a function of t. Truncating Expression (23) with $f_\tau=0$ ($\tau=4$, 5, 6, …) and substituting (31) into (24), we obtain the three-soliton solutions under Constraints (18) as

$$\begin{aligned}u=&2c[\ln(1+e^{\xi_1}+e^{\xi_2}+e^{\xi_3}+e^{\xi_1+\xi_2+A_{12}}+e^{\xi_1+\xi_3+A_{13}}\\&+e^{\xi_2+\xi_3+A_{23}}+e^{\xi_1+\xi_2+\xi_3+A_{123}})]_x,\end{aligned} \tag{32}$$

with

$$\begin{aligned}&k_j=\frac{a_j^3}{m},\\&\omega_j(t)=-\frac{1}{2}ca_j^2(b_j+l_j)\int\beta(t)\mathrm{d}t,\\&e^{A_{jr}}=\frac{(k_j-k_r)^2}{(k_j+k_r)^2}\ (j,\ r=1,\ 2,\ 3,\ \text{and}\ j<r),\\&e^{A_{123}}=\frac{(a_1-a_2)^2(a_1-a_3)^2(a_2-a_3)^2}{(a_1+a_2)^2(a_1+a_3)^2(a_2+a_3)^2}.\end{aligned} \tag{33}$$

Continuing this progress, N-soliton solutions for (3) under Constraints (18) can be obtained as

$$u=2c\left\{\ln\left[\sum_{\mu=0,1}\exp\left(\sum_{j=1}^{N}\mu_j\xi_j+\sum_{1\le j<\varrho}^{N}\mu_j\mu_\varrho A_{j\varrho}\right)\right]\right\}_x, \tag{34}$$

with

$$\begin{aligned}&k_j=\frac{a_j^3}{m},\\&\xi_j=a_jx+b_jy+l_jz+k_js+\omega_j(t)+\delta_j,\\&\omega_j(t)=-\frac{1}{2}ca_j^2(b_j+l_j)\int\beta(t)\mathrm{d}t,\\&A_{j\varrho}=\frac{(k_j-k_\varrho)^2}{(k_j+k_\varrho)^2}(j,\ \varrho=1,\ 2,\ \dots,\ N),\end{aligned} \tag{35}$$

and the sum is taken over all the possible combinations of $\mu_j=0$, 1 ($j=1$, 2, …, N).

4 Bäcklund Transformation for (3)

In order to obtain the Bäcklund transformation for (3), let $P=2\ln g(x, y, z, t)$ be another solution of (20) with $g(x, y, z, t)$ as a function of x, y, z and t, and introduce the following relations:

$$\begin{aligned}&w(x,\ y,\ z,\ t)=\ln[g(x,\ y,\ z,\ t)f(x,\ y,\ z,\ t)],\\&v(x,\ y,\ z,\ t)=\ln[g(x,\ y,\ z,\ t)/f(x,\ y,\ z,\ t)].\end{aligned} \tag{36}$$

We have

$$\begin{aligned}E_1(P)-E_1(Q)=&2\partial x[\mathscr{Y}_{3x}(v,\ w)-m\mathscr{Y}_s(v,\ w)]\\&+2\text{Wronskian}[\mathscr{Y}_{2x}(v,\ w),\ \mathscr{Y}_x(v,\ w)]=0,\end{aligned} \tag{37a}$$

$$E_2(P)-E_2(Q)=2v_{xt}+\frac{2}{3}c\beta(t)v_{3x,y}+2c\beta(t)w_{xx}v_{xy}+2c\beta(t)v_{xx}w_{xy}$$
$$+\frac{2}{3}c\beta(t)v_{3x,z}+2c\beta(t)w_{xx}v_{xz}+2c\beta(t)v_{xx}w_{xz}$$
$$+\frac{m}{3}c\beta(t)v_{ys}+\frac{m}{3}c\beta(t)v_{zs}=0, \quad (37b)$$

which can be decomposed into the Bell-polynomial-type Bäcklund transformation with

$$\mathscr{Y}_{2x}(v,w)=\lambda\,\mathscr{Y}_x(V)+k, \quad (38a)$$

$$\mathscr{Y}_{3x}(v,w)+3k\mathscr{Y}_x(v,w)-m\mathscr{Y}_s(v,w)=0, \quad (38b)$$

$$\mathscr{Y}_t(v,w)+\frac{1}{2}c\beta(t)[(\lambda^2+3k)\mathscr{Y}_y(v,w)+(\lambda^2+3k)\mathscr{Y}_z(v,w)$$
$$+\mathscr{Y}_{2x,y}(v,w)+\mathscr{Y}_{2x,z}(v,w)-\lambda\,\mathscr{Y}_{xy}(v,w)$$
$$-\lambda\,\mathscr{Y}_{xz}(v,w)]=0. \quad (38c)$$

With the application of (10), (38) can be transformed into the bilinear form of Bäcklund transformation as

$$(D_x^2-\lambda D_x-k)g\cdot f=0, \quad (39a)$$

$$(D_x^3+3kD_x-mD_s)g\cdot f=0, \quad (39b)$$

$$\left[D_t+\frac{1}{2}c\beta(t)(\lambda^2+3k+D_{2x}-\lambda D_x)(D_y+D_z)\right]g\cdot f=0. \quad (39c)$$

5 Analysis and Discussions

In this section, propagation and interaction of the solitons will be analyzed. Via One-Soliton Solutions (27), intensity of the soliton can be obtained as

$$|u|=\left|\frac{2ca_1e^{\xi_1}}{1+e^{\xi_1}}\right|=\left|ca_1\left(1+\tanh\frac{\xi_1}{2}\right)\right|, \quad (40)$$

from which we can obtain that the solitonic amplitude has a range of $[0, |ca_1|]$. For c and a_1 are both the real constants, the solitonic amplitude has a finite value. We also find that the variable coefficient $\beta(t)$ has no effects on the maximum amplitude of the soliton. To study the solitonic velocity, we employ the concept of characteristic line [42, 43] for the solitonic propagation based on (40). Characteristic-line equation can be written as

$$a_1x+b_1y+l_1z+k_1s+\omega_1(t)+\delta_1=C, \quad (41)$$

where C is a real constant. Differentiating (41) with respect to t, we can obtain the solitonic velocity vector, $\mathbf{V}$, as

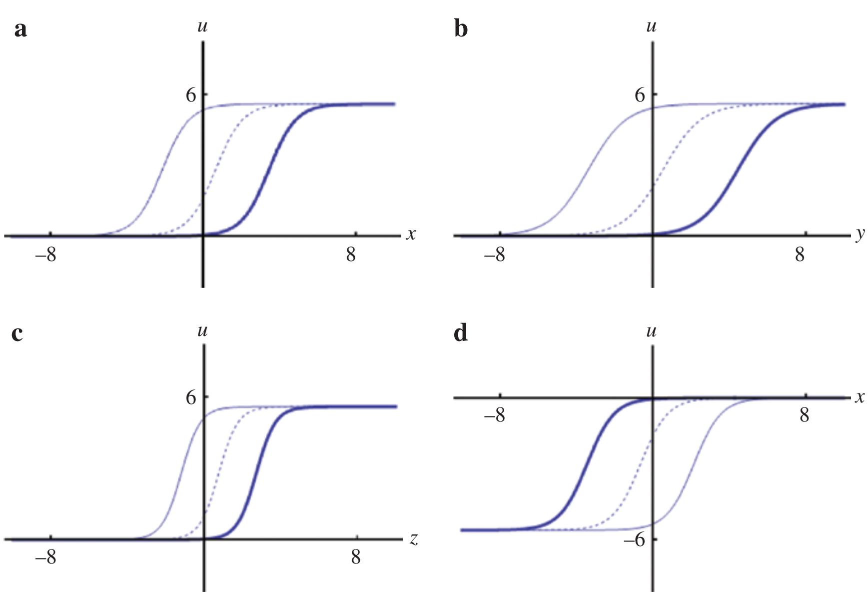

Figure 1: One solitons via Solutions (27), with $s=0$, $c=2$, $b_1=1$, $l_1=2$, $\beta(t)=1$, $\delta_1=0$: (a) $a_1=\frac{5}{7}$, $y=z=1$, $t=0$ (solid line), $t=1$ (dashed line), $t=2$ (bold solid line); (b) $a_1=\frac{5}{7}$, $x=z=1$, $t=0$ (solid line), $t=1$ (dashed line), $t=2$ (bold solid line); (c) $a_1=\frac{5}{7}$, $x=y=1$, $t=0$ (solid line), $t=1$ (dashed line), $t=2$ (bold solid line); (d) $a_1=-\frac{5}{7}$, $y=z=1$, $t=0$ (solid line), $t=1$ (dashed line), $t=2$ (bold solid line).

$$\mathbf{V}=\left(\frac{a_1c(b_1+l_1)\beta(t)}{2},\ \frac{a_1^2c(b_1+l_1)\beta(t)}{2b_1},\ \frac{a_1^2c(b_1+l_1)\beta(t)}{2l_1}\right)^T, \quad (42)$$

which indicates that the solitonic velocity is related to $\beta(t)$.

Figures 1 and 2 describe the one-soliton solutions when the variable coefficient $\beta(t)$ takes different values. With $\beta(t)=1$, Figure 1a and d show that the soliton is right going and locates at $u>0$ when a is positive; The soliton is left going and locates at $u<0$ when a is negative. Similarly, Figure 1b and c show that the solitons are right going when b and l are positive, respectively.

With $\beta(t)=\sin(t)$, Figure 2a–c show the periodic-shaped solitons on the $x-t$, $y-t$, and $z-t$ planes, while Figure 2d–f show the kink-shaped solitons on the $x-y$, $x-z$, and $y-z$ planes.

Figure 3 depicts the interaction between the two solitons via Solutions (29) when the variable coefficient $\beta(t)=\sin(t)$. Figure 3a–c show the periodic-shaped properties of the two-soliton solutions on the $x-t$, $y-t$, and $z-t$ planes, while Figure 3d–f show the kink-shaped properties on the $x-t$, $x-z$, and $y-z$ planes. From Figure 3, we know that the velocities and amplitudes of the solions remain unchanged after the interaction except for a phase shift.

For the effects of the variable coefficient $\beta(t)$ on the solitons, Figure 4 is depicted to show the two-soliton intereactions when the variable coefficient $\beta(t)$ takes different values. Comparing Figure 3a with Figure 4, we can see that the two-solition solutions have different shapes when $\beta(t)$ varies. In Figure 3a, when $\beta(t)$ is a periodic function of t, the two-soliton interaction has the periodic properties on the $x-t$ plane. In Figure 4a, when $\beta(t)=t^2$, the two-soliton interaction has the cubic properties on the $x-t$ plane, while in Figure 4b, with $\beta(t)=t$, the two-soliton interaction has the quadric properties on the $x-t$ plane. In Figure 4c, with $\beta(t)=1$, the two-soliton interaction has the kink-shaped properties on the $x-t$ plane.

As seen in Figure 5, the three-soliton solutions have the similar properties as those of the two solitons. Figure 5 shows that the three-soliton solutions have the periodic properties on the $x-t$, $y-t$, and $z-t$ planes and have the kink-shaped properties on the $x-y$, $x-z$, and $y-z$ planes. Velocities and amplitudes of the three solitons also remain unchanged after the interaction except for a phase shift.

6 Conclusions

In this article, via the Bell polynomials, Hirota method and symbolic computation, we have investigated (3), which covers (1), (2), (4), and (5) with different values of

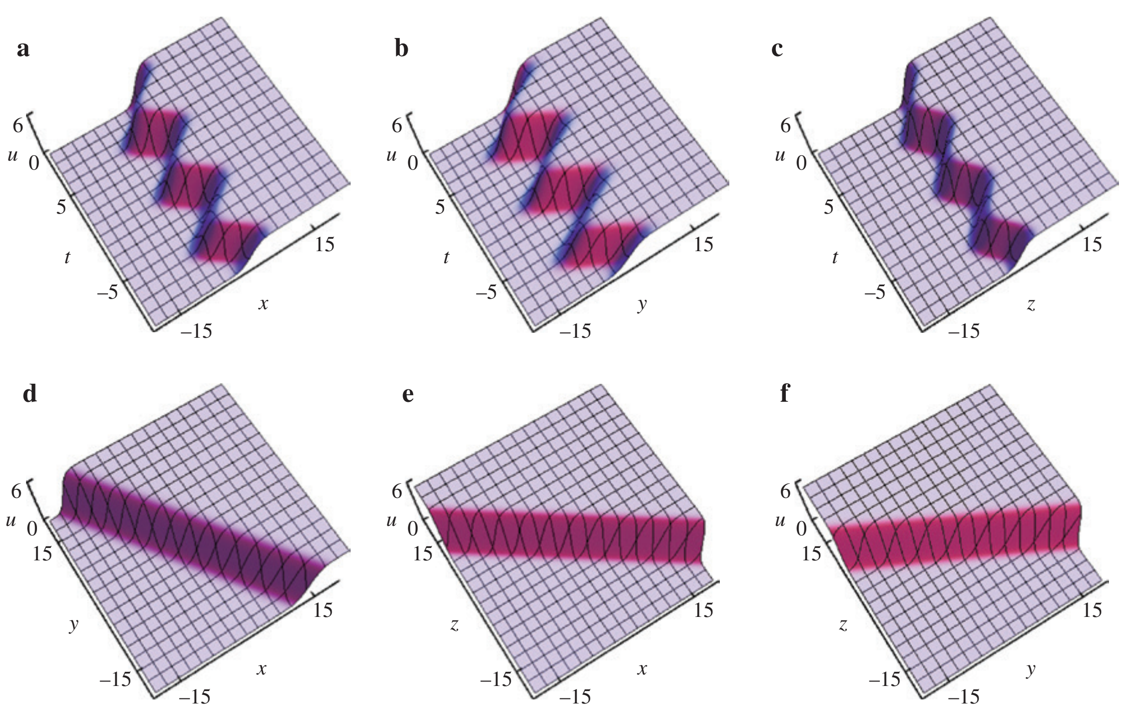

Figure 2: One soliton via Solutions (27), with $s=0$, $c=2$, $a_1=\frac{7}{5}$, $b_1=1$, $l_1=2$, $\beta(t)=\sin(t)$, $\delta_1=0$: (a) $y=z=1$; (b) $x=z=1$; (c) $x=y=1$; (d) $t=0$, $z=1$; (e) $t=0$, $y=1$; (f) $t=0$, $x=1$.

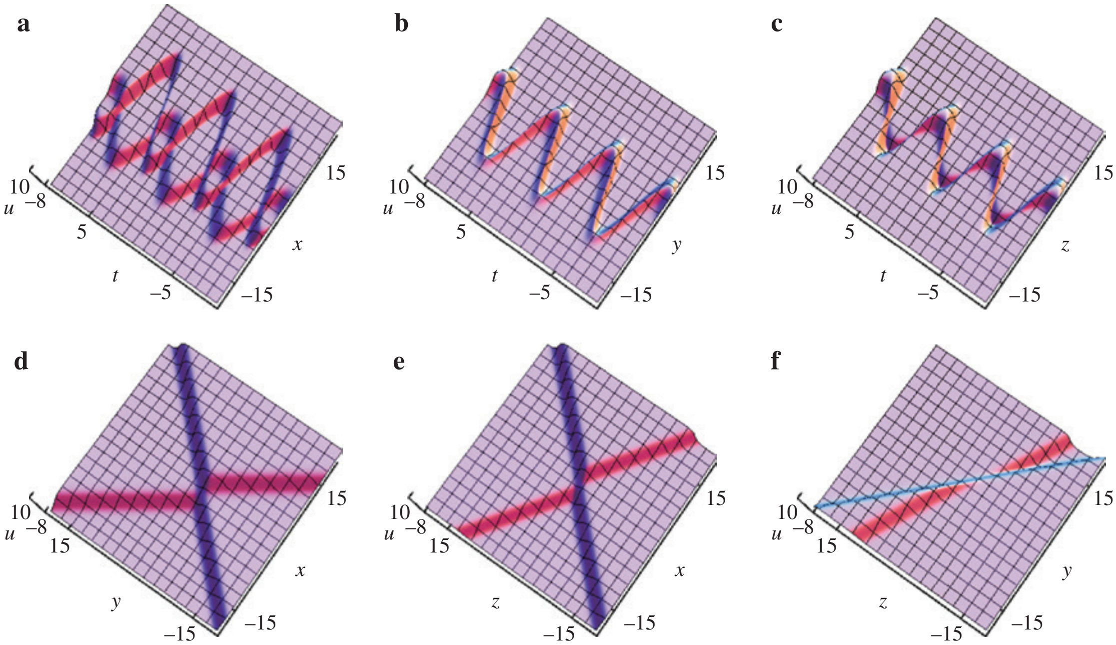

Figure 3: Interaction between the two solitons via Solutions (29), with $s=0$, $c=2$, $a_1=\frac{7}{5}$, $a_2=-2$, $b_1=1$, $b_2=2$, $l_1=l_2=2$, $\beta(t)=\sin(t)$, $\delta_1=\delta_2=0$: (a) $y=z=1$; (b) $x=z=1$; (c) $x=y=1$; (d) $t=0$, $z=1$; (e) $t=0$, $x=1$; (f) $t=0$, $y=1$.

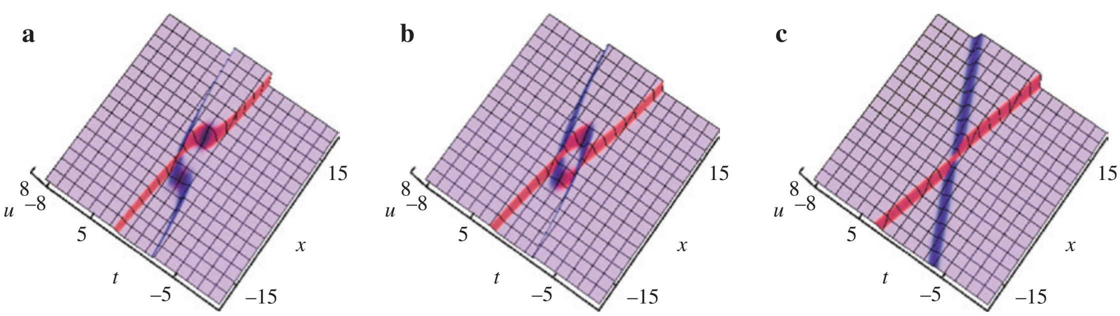

Figure 4: The same as Figure 3a except that (a) $\beta(t)=t^2$; (b) $\beta(t)=t$; (c) $\beta(t)=1$.

the variable coefficients. The results can be concluded as follows:

1. In virtue of the Bell polynomials, we have obtained Bilinear Forms (20) and Bäcklund Transformation (38) for (3) under Constraints (18).
2. With the help of the Hirota method and symbolic computation, we have obtained the one-, two-, and three-soliton solutions for (3) under Constraints (18), i.e., Solutions (27), (29), and (32). Besides, *N*-Soliton Solutions (34) under Constraints (18) have also been constructed.

Based on the above-mentioned results, we have graphically illustrated the propagation and interaction of the solitons as follows:

i. Figures 1 and 2 have been depicted to show the one-soliton evolution. Figure 1 has shown that the solitonic direction is determined by the sign of the wave numbers a_1, b_1, and l_1. From Figure 2, we have found that the one-soliton solutions have the periodic properties on the $x-t$, $y-t$, and $z-t$ planes, while they have the kink-shaped properties on the $x-t$, $x-z$, and $y-z$ planes.

ii. Interactions based on the multisoliton solutions have been presented on the $x-t$, $y-t$, $z-t$, $x-t$, $x-z$, and $y-z$ planes in Figures 3–5. Figures 3 and 4 have illustrated the interactions between the two solitons. In Figure 3, periodic properties on the $x-t$, $y-t$, and $z-t$ planes have been depicted, while kink-shaped properties on the $x-t$, $x-z$, and $y-z$ planes have been presented in Figure 3d–f. Besides, interactions between the two solitons have been seen to be elastic because the velocities and amplitudes of the two solitons have kept unchanged after each interaction

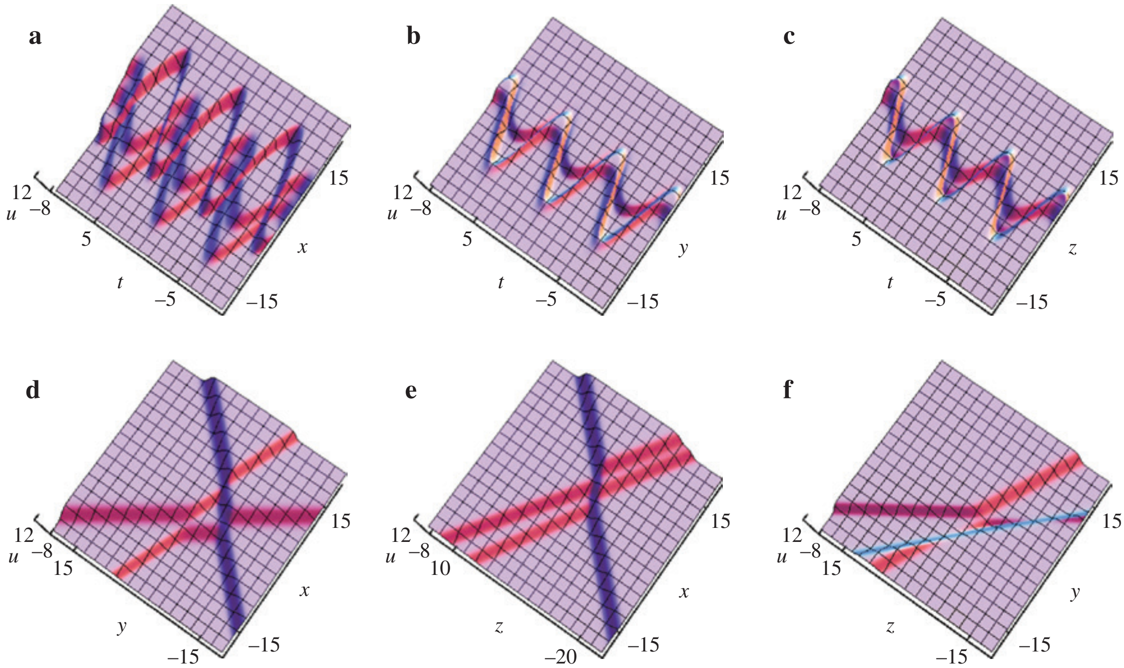

Figure 5: Three soliton solutions via Solutions (32), with $s=0$, $c=2$, $a_1=\frac{7}{5}$, $a_2=-2$, $a_3=\frac{6}{5}$, $b_1=1$, $b_2=2$, $b_3=3$, $l_1=l_2=l_3=2$, $\beta(t)=\sin(t)$, $\delta_1=\delta_2=\delta_3=0$: (a) $y=z=1$; (b) $x=z=1$; (c) $x=y=1$; (d) $t=0$, $z=1$; (e) $t=0$, $y=1$; (f) $t=0$, $x=1$.

except for a phase shift. Figure 4 has shown that the two solitons' shapes are determined by the variable coefficient $\beta(t)$.

iii. Figure 5 has shown that when $\beta(t)$ is a periodic function of t, the three-soliton solutions on the $x-t$, $y-t$, and $z-t$ planes have the periodic properties while have the kink-shape properties on the $x-y$, $x-z$, and $y-z$ planes. Meanwhile, we have found that the three solitons' velocities and amplitudes have kept unchanged after each interaction except for a phase shift, which means that the interactions among the three solitons are elastic.

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