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Method of Multiple Scales and Travelling Wave Solutions for (2+1)-Dimensional KdV Type Nonlinear Evolution Equations

DOI 10.1515/zna-2016-0123
Received March 30, 2016; accepted June 2, 2016; previously published online June 24, 2016

Abstract: In this article, we applied the method of multiple scales for Korteweg–de Vries (KdV) type equations and we derived nonlinear Schrödinger (NLS) type equations. So we get a relation between KdV type equations and NLS type equations. In addition, exact solutions were found for KdV type equations. The $\left(\frac{G'}{G}\right)$-expansion methods and the $\left(\frac{G'}{G}, \frac{1}{G}\right)$-expansion methods were proposed to establish new exact solutions for KdV type differential equations. We obtained periodic and hyperbolic function solutions for these equations. These methods are very effective for getting travelling wave solutions of nonlinear evolution equations (NEEs).

Keywords: Exact Travelling Wave Solution; $\left(\frac{G'}{G}\right)$-Expansion Method; $\left(\frac{G'}{G}, \frac{1}{G}\right)$-Expansion Method; Method of Multiple Scales.

PACS (2006): 02.30.Ik; 02.70.Wz; 05.45.Yv.

1 Introduction

Nonlinear evolution equations (NEEs) are the mathematical models of problems that arise in many field of science. In recent years, NEEs have become an important field of study in applied mathematics [1–3]. Nonlinear phenomena come up in a variety of scientific applications such as optical fibres, fluid dynamics, plasma physics, chemical physics, and in engineering applications as well. Nonlinear equations that model these scientific phenomena have long been a major concern for research work [4]. The nonlinear Schrödinger (NLS) equation is an example of a universal nonlinear model that describes many physical nonlinear systems. The equation can be applied to hydrodynamics, nonlinear optics, nonlinear acoustics, quantum condensates, plasma physics, heat pulses in solids, and various other nonlinear instability phenomena [5]. It is well known that a multiple scales analysis of the KdV equation (and, indeed a wide variety of equations) leads to the NLS equation for the modulated amplitude [6–10]. In [6], Zakharov and Kuznetsov showed a much deeper correspondence between these integrable equations not only at the level of the equation but also at the level of the linear spectral problem by showing that a multiple scales analysis of the Schrödinger spectral problem leads to the Zakharov–Shabat problem for the NLS equation. Dag and Özer also showed the same relationship between the NLS equation and integrable fifth-order NEEs [11]. On the other hand, similar methods could also be used in [12–15].

Travelling waves appear in many physical structures in solitary wave theory such as solitons, kinks, compactons, peakons, cuspons, and others. In recent years, lots of methods have been improved to find exact travelling wave solutions of NEEs [16, 17]. Through these methods, the $\left(\frac{G'}{G}\right)$-expansion method and the $\left(\frac{G'}{G}, \frac{1}{G}\right)$-expansion method are very simple, convenient, and straightforward to construct the travelling wave solutions of NEEs. Zhang et al. [18] and Aslan [19] used the $\left(\frac{G'}{G}\right)$-expansion method to address some physically important NLDDEs. Zhang et al. [20] proposed a generalised $\left(\frac{G'}{G}\right)$-expansion method to improve and extend the works of Wang et al. [21] and Tang et al. [22] for solving variable-coefficient equations and high-dimensional equations. More recently, the $\left(\frac{G'}{G}\right)$-expansion method has been applied to fractional differential difference equation to get new exact travelling wave solutions in the sense of modified Riemann Liouville derivative [23]. On the other hand, $\left(\frac{G'}{G}, \frac{1}{G}\right)$-expansion

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method was first improved by Li et al. [24] to find travelling wave solutions of NEEs and then the method has been applied to different type equations in [25, 26].

2 Description of the Method of Multiple Scales for NEEs

Multiple scales method was proposed by Zakharov and Kuznetsov [6] to reduce the KdV equation into the NLS equation and apply to a class of NEEs. They showed that using this method, conventionally employed in the theory of nonlinear waves, integrable systems are reduced to other integrable systems. If the initial system is nonintegrable, the result can be either integrable or nonintegrable. But if we treat an integrable system properly, we must always get an integrable system as the result of our analysis. This is the main purpose in the application of this method to integrable systems.

In this article, we consider the application of the multiple scales method to NEEs. Using the technique of Zakharov and Kuznetsov [6], getting the NLS type equations from KdV type equations was shown step by step.

Let consider the general evolution equation

$$u_t = K(u, u_x, u_y \dots) \tag{1}$$

General NEEs $K[u]$ are a function of u and its derivatives with respect to the x-spatial variables. The well known of this type equations is KdV equation.

$L[\partial_x, \partial_y]u$ is the linear part of $K[u]$. So, using $K[u]$, we can reach the dispersion relation for the (1). Substituting the plane wave solution

$$\begin{aligned} u_k &= Ae^{i(kx+ry-\omega(k,r)t)} \\ &\equiv Ae^{i\theta} \end{aligned} \tag{2}$$

into the linear part of (1)

$$u_t = L[\partial_x, \partial_y]u \tag{3}$$

we get dispersion relation

$$\omega(k, r) = iL[ik, ir] \tag{4}$$

Then, dispersion relation (4) is substituted in (1). We assume the following series expansions for the solution of the (1):

$$u(x, y, t) = \sum_{n=1}^{\infty} \epsilon^n U_n(x, y, t, \xi, \eta, \tau)$$

Based on this solution, we also define slow spaces ξ, η and multiple time variable τ with respect to the scaling parameter $\epsilon > 0$, respectively, as follows. A nonlinear equation modulates the amplitude of this plane wave solution in such a way that may consider it depend on the slow variables:

$$\begin{aligned} \xi &= \epsilon\left(x - \frac{d\omega(k, r)}{dk}t\right) \\ \eta &= \epsilon\left(y - \frac{d\omega(k, r)}{dr}t\right) \\ \tau &= -\frac{1}{2}\epsilon^2\left(\frac{d^2\omega(k, r)}{dk^2} + \frac{d^2\omega(k, r)}{dr^2}\right)t \end{aligned} \tag{5}$$

If we choose the slow variables different forms, we can derive higher-order NLS equations. The multiple scales analysis starts with the assumption:

$$u(x, y, t) = U(x, y, t, \xi, \eta, \tau) \tag{6}$$

and solution of U is in the form

$$U(x, y, t, \xi, \eta, \tau) = \epsilon U_1 + \epsilon^2 U_2 + \epsilon^3 U_3 + \dots \tag{7}$$

Then, considering transformation (6) and solution (7), using dispersion relation (4) and slow variables (5), we get u and its derivatives with respect to ϵ in (1). And we substitute these terms with (6) and (7) in (1). Collecting all terms with the same order of ϵ together, the left-hand side of (1) is converted into a polynomial in ϵ. Then setting each coefficient of this polynomial to zero, we get a set of algebraic eqautions. Using wave solution space (2) and dispersion relation (4), these equations can be solved by iteration and by use of Maple. So, we can get NLS type equations from (1). In addition, from this procedure, we can reach numerical solutions of KdV type equations.

3 Description of the $\left(\frac{G'}{G}\right)$-Expansion Method for NEEs

$\left(\frac{G'}{G}\right)$-expansion method has been described first by Wang et al. [27] and in ([23, 28–32]), this method has been applied to a lot of different type nonlinear partial differential equations. $\left(\frac{G'}{G}\right)$-expansion method is very practicable and effective to find travelling wave solutions of NEEs. Using this method, hyperbolic and trigonometric solutions of NEEs are get.

We can describe the $\left(\frac{G'}{G}\right)$-expansion method step by step as follows:

Let consider nonlinear partial differential equation in the form

$$P(u, u_t, u_x, u_y, u_{yy}, u_{tt}, u_{xx}, \ldots)=0, \tag{8}$$

where P is a polynomial of $u(x, y, t)$ and its derivatives.

Step 1: Let consider

$$u(x, y, t)=U(\xi),\ \xi=kx+ry-ct \tag{9}$$

travelling wave transformation for the travelling wave solutions of (8). With this wave transformation, (8) can be reduced to

$$Q(U, -cU', kU', rU', c^2U'', r^2U'', k^2U'', \ldots)=0. \tag{10}$$

ordinary differential equation where $U=U(\xi)$ and prime denotes derivatives with respect to ξ.

Step 2: We predict the solution of (10) in the finite series form

$$U(\xi)=\sum_{l=0}^{m}\alpha_l\left(\frac{G'(\xi)}{G(\xi)}\right)^l,\ \alpha_m\neq 0 \tag{11}$$

where m and α_l's are constants to be determined later, $G(\xi)$ satisfies a second-order linear ordinary differential equation:

$$\frac{d^2G(\xi)}{d\xi^2}+\lambda\frac{dG(\xi)}{d\xi}+\mu G(\xi)=0, \tag{12}$$

where λ and μ are arbitrary constants. Using the general solutions of (12), we get following cases:

$\left(\frac{G'(\xi)}{G(\xi)}\right)$, the left-hand side of (3) is converted into a polynomial in $\left(\frac{G'(\xi)}{G(\xi)}\right)$. For example, for $m=1$ in (12) is in the form

$$\frac{d^2G(\xi)}{d\xi^2}+\lambda\frac{dG(\xi)}{d\xi}+\mu G(\xi)=0\Rightarrow G''(\xi)=-\lambda G'(\xi)-\mu G(\xi)$$

where

$$U(\xi)=a_0+a_1\left(\frac{G'(\xi)}{G(\xi)}\right) \tag{14}$$

so we get

$$U'=a_1\left(-\lambda\frac{G'}{G}-\mu-\left(\frac{G'}{G}\right)^2\right) \tag{15}$$

and

$$U''=a_1\left(\lambda\mu+(\lambda^2+2\mu)\left(\frac{G'}{G}\right)+3\lambda\left(\frac{G'}{G}\right)^2+2\left(\frac{G'}{G}\right)^3\right) \tag{16}$$

$\vdots$

as polynomials of $\left(\frac{G'(\xi)}{G(\xi)}\right)$. In addition, we can obtain these for higher order values of m. As a final step, substituting (11) and (14–16) together with (12) into (10) and collecting all terms with the same order of $\left(\frac{G'(\xi)}{G(\xi)}\right)$ together,

$$\frac{G'(\xi)}{G(\xi)}=\begin{cases}\dfrac{\sqrt{\lambda^2-4\mu}}{2}\left(\dfrac{C_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+C_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{C_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}\right)-\dfrac{\lambda}{2}, & \lambda^2-4\mu>0\\[2ex] \dfrac{\sqrt{4\mu-\lambda^2}}{2}\left(\dfrac{-C_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+C_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}{C_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+C_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}\right)-\dfrac{\lambda}{2}, & \lambda^2-4\mu<0\end{cases} \tag{13}$$

where C_1 and C_2 are arbitrary constants.

Step 3: We can easily determine the degree m of (11) using the homogeneous balance principle for the highest order nonlinear term(s) and the highest order partial derivative of $U(\xi)$ in (10).

Step 4: As a final step, substituting (11) together with (12) into (10) and collecting all terms with the same order of the left-hand side of (10) is converted into a polynomial in $\left(\frac{G'(\xi)}{G(\xi)}\right)$. Equating each coefficient of $\left(\frac{G'(\xi)}{G(\xi)}\right)^l$ $(l=0, 1, 2, \ldots)$ to zero yields a set of algebraic equations for "α_l $(l=0, 1, 2, \ldots, N)$, k, c". Solving these algebraic equations system, we can define "α_l $(l=0, 1, 2, \ldots, N)$, k, c". Finally, we substitute these values into expression (11) and obtain various kinds of exact solutions to (8) by use of Maple.

4 Description of the $\left(\frac{G'}{G}, \frac{1}{G}\right)$-Expansion Method for NEEs

$\left(\frac{G'}{G}, \frac{1}{G}\right)$-Expansion method was first improved by Li et al. [24] to find travelling wave solutions of NEEs and it can be considered as a generalisation of the original $\left(\frac{G'}{G}\right)$-expansion method. Then this method was applied to another nonlinear partial differential equations by a lot of scientists [25, 26].

The key idea of the $\left(\frac{G'}{G}\right)$-expansion method is that the exact solutions of NEEs can be expressed by a polynomial in one variable $\left(\frac{G'(\xi)}{G(\xi)}\right)$ in which $G=G(\xi)$ satisfies a second-order linear ODE

$$G''(\xi)+\lambda G'(\xi)+\mu G(\xi)=0 \tag{17}$$

where λ and μ are constants. The key idea of the $\left(\frac{G'}{G}, \frac{1}{G}\right)$-expansion method is that the exact travelling wave solutions of NEEs can be expressed by a polynomial in the two variables $\left(\frac{G'(\xi)}{G(\xi)}\right)$ and $\left(\frac{1}{G(\xi)}\right)$, in which $G=G(\xi)$ satisfies a second-order linear ODE

$$G''(\xi)+\lambda G(\xi)=\mu \tag{18}$$

where λ and μ are constants. Using homogeneous balance procedure between the highest order derivatives and nonlinear terms in the NEE, we can determine the degree of the polynomial. Finally, using the method, we can find the coefficients of the polynomial and we reach the exact travelling wave solutions of the NEEs.

Before describe the main steps of the method, we consider the following remarks:

Let

$$\phi=\left(\frac{G'(\xi)}{G(\xi)}\right) \text{ and } \psi=\left(\frac{1}{G(\xi)}\right) \tag{19}$$

Using (18) and (19), we get

$$\begin{aligned}\phi'&=\frac{G''G-G'G'}{G^2}\\&=\frac{G''}{G}-\left(\frac{G'}{G}\right)^2\\&=\frac{\mu-\lambda G}{G}-\phi^2\\&=\mu\psi-\lambda-\phi^2\end{aligned} \tag{20}$$

and

$$\psi'=-\frac{G'}{G^2}=-\phi\psi \tag{21}$$

Then, considering the general solution of linear ordinary differential equation (18), we get

$$\begin{cases}G(\xi)=C_1\sinh(\sqrt{-\lambda}\xi)+C_2\cosh(\sqrt{-\lambda}\xi)+\dfrac{\mu}{\lambda}, & \lambda<0\\ G(\xi)=C_1\sin(\sqrt{\lambda}\xi)+C_2\cos(\sqrt{\lambda}\xi)+\dfrac{\mu}{\lambda}, & \lambda>0\end{cases} \tag{22}$$

where C_1 and C_2 are arbitrary constants. Thus, we get

$$\begin{cases}\psi^2=\dfrac{-\lambda(\phi^2-2\mu\psi+\lambda)}{\lambda^2\sigma+\mu^2}, & \sigma=C_1^2-C_2^2 \text{ and } \lambda<0\\ \psi^2=\dfrac{-\lambda(\phi^2-2\mu\psi+\lambda)}{\lambda^2\sigma-\mu^2}, & \sigma=C_1^2+C_2^2 \text{ and } \lambda>0\end{cases} \tag{23}$$

Let consider nonlinear partial differential equation in the form

$$F(u, u_x, u_y, u_t, u_{xx}, u_{tt}, u_{xt}, \ldots)=0 \tag{24}$$

where F is a polynomial of u and its partial derivatives.

Step 1: Using travelling wave transformation

$$u(x, y, t)=U(\xi), \; \xi=kx+ry-ct$$

nonlinear partial differential equation (24) can be reduced to ordinary differential equation

$$P(U, U', U'', \ldots)=0 \tag{25}$$

where P is a polynomial of U and its total derivatives, while $\left(' = \frac{d}{d\xi}\right)$.

Step 2: Solution of the (25) can be expressed by a polynomial in the variables in ϕ ve ψ as follows:

$$U(\xi)=\sum_{i=0}^{N} a_i\phi^i+\sum_{i=1}^{N} b_i\phi^{i-1}\psi \tag{26}$$

where a_i $(i=0, 1, 2, \ldots, N)$ and b_i $(i=0, 1, 2, \ldots, N)$ are constants to be determined later.

Step 3: According to the homogeneous balance procedure, balancing the highest order derivatives, and the nonlinear terms in (25), we can find the positive integer N in (26).

Step 4: Substituting (26) into (25) along with (20), (21), and (23), the left-hand side of (25) convert to a polynomial in ϕ and ψ, in which the degree of ψ is not longer than 1.

Then, equating the coefficients of this polynomial to zero, we get a system of algebraic equations.

Step 5: Solving the algebraic equation system with the aid of Maple, we find a_i, b_i, k, r, c, μ, C_1 C_2 values for $\lambda<0$, $\lambda>0$. Finally substituting this values in (25), hyperbolic and trigonometric solutions can be found, respectively.

5 Applications

In this section, we apply $\left(\frac{G'}{G}\right)$-expansion method and $\left(\frac{G'}{G}, \frac{1}{G}\right)$-expansion method summarised in Section 2 to (2+1)-dimensional KdV4 equation.

5.1 (2+1)-Dimensional KdV4 Equation

Using symmetries of KdV equation, a new class of partial differential equations was obtained in (2+1) dimension by Gürses and Pekcan [33]. Thus, the authors presented a useful work and developed (2+1)-dimensional generalisation of the KdV equations. Then using this work, (2+1) dimensional KdV4 equation has been obtained by Wazwaz [34] as follows:

$$u_{xy}+u_{xxxt}+u_{xxxx}+3(u_x^2)_x+4u_xu_{xt}+2u_{xx}u_t=0 \tag{27}$$

5.1.1 Method of Multiple Scales

(2+1)-Dimensional KdV4 equation is in the form (27). To find dispersion relation for (27), we consider the linear part of (27) in the form

$$u_{xy}+u_{xxxt}+u_{xxxx}=0 \tag{28}$$

and linear differential equation (28) satisfies the solution

$$u(x, y, t)=e^{i\theta},\ \theta=kx+ry-\omega(k, r)t \tag{29}$$

Substituting the solution (29) into the linear differential equation (28), we get

$$\begin{aligned}&\Rightarrow i^2kr+i^4k^3(-\omega)+i^4k^4=0\\&\Rightarrow -kr-\omega k^3+k^4=0\end{aligned}$$

and from this we reach the

$$\omega(k, r)=k-\frac{r}{k^2} \tag{30}$$

dispersion relation. Thus, the solution of linear differential equation (28) is as follows:

$$u(x, y, t)=e^{i\left(kx+ry-\left(k-\frac{r}{k^2}\right)t\right)} \tag{31}$$

Let the solution of (27) is in the form

$$\begin{aligned}u(x, y, t)&=U(x, y, t, \xi, \eta, \tau),\ U(x, y, t, \xi, \eta, \tau)\\&=\epsilon U_1+\epsilon^2U_2+\epsilon^3U_3+\dots\end{aligned} \tag{32}$$

and slow variables are in the form

$$\begin{aligned}\xi&=\epsilon\left(x-\frac{d\omega(k, r)}{dk}t\right)\\&=\epsilon\left(x-\left(1+\frac{2r}{k^3}\right)t\right)\\\eta&=\epsilon\left(y-\frac{d\omega(k, r)}{dr}t\right)\\&=\epsilon\left(y+\frac{1}{k^2}t\right)\\\tau&=-\frac{1}{2}\epsilon^2\left(\frac{d^2\omega(k, r)}{dk^2}t+\frac{d^2\omega(k, r)}{dr^2}t\right)\\&=-\frac{3r\epsilon^2}{k^4}t\end{aligned} \tag{33}$$

Then using (30)–(33), the total derivatives in (27) are obtained as follows:

$$\begin{aligned}D_x&=(\partial_x+\epsilon\partial_\xi)\\D_{xx}&=\partial_{xx}+2\epsilon\partial_{\xi x}+\epsilon^2\partial_{\xi\xi}\\D_{xxx}&=\partial_{xxx}+3\epsilon\partial_{\xi xx}+3\epsilon^2\partial_{\xi\xi x}+\epsilon^3\partial_{\xi\xi\xi}\\D_{xxxx}&=\partial_{xxxx}+4\epsilon\partial_{\xi xxx}+6\epsilon^2\partial_{\xi\xi xx}+4\epsilon^3\partial_{\xi\xi\xi x}+\epsilon^4\partial_{\xi\xi\xi\xi}\end{aligned} \tag{34}$$

$$\begin{aligned}D_y&=(\partial_y+\epsilon\partial_\eta)\\D_t&=\left(\partial_t-\left(1+\frac{2r}{k^3}\right)\epsilon\partial_\xi+\frac{\epsilon}{k^2}\partial_\eta+\frac{3r}{k^4}\epsilon^2\partial_\tau\right)\\D_{xy}&=(\partial_{xy}+\epsilon\partial_{x\eta}+\epsilon\partial_{y\xi}+\epsilon^2\partial_{\xi\eta})\\D_{xt}&=\partial_{xt}-\left(1+\frac{2r}{k^3}\right)\epsilon\partial_{x\xi}+\frac{3r}{k^4}\epsilon^2\partial_{x\tau}+\frac{\epsilon}{k^2}\partial_{x\eta}+\epsilon\partial_{t\xi}-\\&\quad\epsilon^2\left(1+\frac{2r}{k^3}\right)\partial_{\xi\xi}+\frac{3r}{k^4}\epsilon^3\partial_{\xi\tau}+\frac{\epsilon^2}{k^2}\partial_{\xi\eta}\\D_{xxxt}&=\partial_{xxxt}-\left(1+\frac{2r}{k^3}\right)\epsilon\partial_{xxx\xi}+\frac{3r}{k^4}\epsilon^2\partial_{xxx\tau}+\frac{\epsilon}{k^2}\partial_{xxx\eta}+\\&\quad 3\epsilon\partial_{xxt\xi}-3\epsilon^2\left(1+\frac{2r}{k^3}\right)\partial_{xx\xi\xi}+\frac{9r}{k^4}\epsilon^3\partial_{xx\xi\tau}+\frac{3\epsilon^2}{k^2}\partial_{xx\xi\eta}\\&\quad+3\epsilon^2\partial_{xt\xi\xi}-3\epsilon^3\left(1+\frac{2r}{k^3}\right)\partial_{x\xi\xi\xi}+\frac{9r}{k^4}\epsilon^4\partial_{x\xi\xi\tau}+\frac{3\epsilon^3}{k^2}\partial_{x\xi\xi\eta}\\&\quad+\epsilon^3\partial_{t\xi\xi\xi}-\epsilon^4\left(1+\frac{2r}{k^3}\right)\partial_{\xi\xi\xi\xi}+\frac{3r}{k^4}\epsilon^5\partial_{\xi\xi\xi\tau}+\frac{\epsilon^4}{k^2}\partial_{\xi\xi\xi\eta}\end{aligned} \tag{35}$$

Substituting (32), (34), and (35) into the (27), we get a polynomial in ϵ. Equating each coefficients of this polynomial to zero, we find

$$\epsilon : u_{1xxxx} + u_{1xxxt} + u_{1xy} = 0 \quad (36)$$

$$\epsilon^2 : u_{2xxxx} + u_{2xxxt} + u_{2xy} + u_{1x\eta} + u_{1y\xi} + 4u_{1xxx\xi} + 3u_{1\xi xxt} + 6u_{1x}u_{1xx} + 4u_{1x}u_{1xt} + 2u_{1xx}u_{1t} - \left(1 + \frac{2r}{k^3}\right)u_{1xxx\xi} + \frac{1}{k^2}u_{1xxx\eta} = 0 \quad (37)$$

$$\epsilon^3 : u_{3xxxx} + u_{3xxxt} + u_{3xy} + \frac{3r}{k^4}u_{1xxx\tau} - 3\left(1 + \frac{2r}{k^3}\right)u_{1xx\xi\xi} + \frac{3}{k^2}u_{1\xi xx\eta} + u_{1\xi\eta} + u_{2y\xi} + 4u_{2\xi xxx} + 6u_{1\xi\xi xx} + 3u_{2\xi xxt} + 6u_{1x}(u_{2xx} + 2u_{1\xi x})6u_{1xx}(u_{2x} + u_{1\xi}) + 4u_{1x}\left(u_{2xt} + \frac{1}{k^2}u_{1x\eta} - \left(1 + \frac{2r}{k^3}\right)u_{1\xi x} + u_{1\xi t}\right) + 4u_{1xt}(u_{2x} + u_{1\xi}) + u_{2x\eta} + 2u_{1xx}\left(u_{2t} + \frac{1}{k^2}u_{1\eta} - \left(1 + \frac{2r}{k^3}\right)u_{1\xi}\right) + 3u_{1\xi\xi xt} + 2u_{1t}(u_{2xx} + 2u_{1\xi x}) - \left(1 + \frac{2r}{k^3}\right)u_{2\xi xxx} + \frac{1}{k^2}u_{2xxx\eta} = 0 \quad (38)$$

$$\vdots \quad (39)$$

Then, we can find the solution of (36) as follows:

$$u_1(x, y, t, \xi, \eta, \tau) = v_1(\xi, \eta, \tau)e^{i\left(kx + ry - \left(k - \frac{r}{k^2}\right)t\right)} + c.c. \quad (40)$$

where c.c. is complex conjugate of v_1. Substituting the solution (40) into (37), the solution of (37) is in the form

$$u_2(x, y, t, \xi, \eta, \tau) = v_2(\xi, \eta, \tau)e^{2i\left(kx + ry - \left(k - \frac{r}{k^2}\right)t\right)} + c.c. + f(\xi, \eta, \tau) \quad (41)$$

where $f(\xi, \eta, \tau)$ is integration constant. Thus, we get

$$v_2(\xi, \eta, \tau) = \frac{iv_1^2(\xi, \eta, \tau)}{2k}, \; v_{-2}(\xi, \eta, \tau) = \frac{-iv_{-1}^2(\xi, \eta, \tau)}{2k} \quad (42)$$

where v_{-1} is the complex conjugate of v_1 and v_{-2} is the complex conjugate of v_2. Substituting solutions (40)–(42) into the (38), we find the solution of (38) in the form

$$u_3(x, y, t, \xi, \eta, \tau) = v_3(\xi, \eta, \tau)e^{3i\left(kx + ry - \left(k - \frac{r}{k^2}\right)t\right)} + f_2(\xi, \eta, \tau)e^{2i\left(kx + ry - \left(k - \frac{r}{k^2}\right)t\right)} + c.c. \quad (43)$$

where $f_2(\xi, \eta, \tau)$ is integration constant. Then, we get

$$v_3(\xi, \eta, \tau) = \frac{-v_1^3(\xi, \eta, \tau)}{4k^2}, \; v_{-3}(\xi, \eta, \tau) = \frac{-v_{-1}^3(\xi, \eta, \tau)}{4k^2} \quad (44)$$

and

$$f_2(\xi, \eta, \tau) = \frac{-v_1 v_{1\xi}}{2k^2}$$
$$f_3(\xi, \eta, \tau) = \frac{-v_{-1} v_{-1\xi}}{2k^2}$$
$$f(\xi, \eta, \tau) = 0 \quad (45)$$

where v_{-3} and f_3 are the complex conjugates of v_3 and f_2, respectively. Thus, the solutions of (36)–(38) are obtained as

$$u_1 = v_1(\xi, \eta, \tau)e^{i\theta} + c.c.$$
$$u_2 = i(2k)^{-1}v_1^2(\xi, \eta, \tau)e^{2i\theta} + c.c.$$
$$u_3 = -\frac{1}{2}(k)^{-2}v_1(\xi, \eta, \tau)v_{1\xi}(\xi, \eta, \tau)e^{2i\theta} - (2k)^{-2}v_1^3(\xi, \eta, \tau)e^{3i\theta} + c.c \quad (46)$$

where

$$\theta = \left(kx + ry - \left(k - \frac{r}{k^2}\right)t\right) \quad (47)$$

Finally, substituting the solutions (46) into (38), we get

$$\Rightarrow iv_{1\tau} = v_{1\xi\xi} - 2v_1^2 v_{-1} - \frac{2k}{3r}v_{1\xi\eta}$$
$$\Rightarrow iv_{1\tau} = v_{1\xi\xi} - 2v_1|v_1|^2 - \frac{2k}{3r}v_{1\xi\eta} \quad (48)$$

and

$$\Rightarrow -iv_{-1\tau} = v_{-1\xi\xi} - 2v_{-1}^2 v_1 - \frac{2k}{3r}v_{-1\xi\eta}$$
$$\Rightarrow -iv_{-1\tau} = v_{-1\xi\xi} - 2v_{-1}|v_{-1}|^2 - \frac{2k}{3r}v_{-1\xi\eta} \quad (49)$$

Describing as $q = v_1$, $q^* = v_{-1}$ and $\left(\lambda = \frac{2k}{3r}\right)$, from (48) and (49), we get the (2+1) dimensional NLS type equations in the form

$$iq_\tau = q_{\xi\xi} - 2q|q|^2 - \lambda q_{\xi\eta} \quad (50)$$

and

$$-iq_\tau^* = q_{\xi\xi}^* - 2q^*|q^*|^2 - \lambda q_{\xi\eta}^* \quad (51)$$

In addition, numerical solution of the (2+1)-dimensional KdV4 equation (27) is found as

$$u(x,y,t)=\epsilon kq(\xi,\eta,\tau)e^{i\theta}+i\epsilon^2(2k)^{-1}q^2(\xi,\eta,\tau)e^{2i\theta} -\frac{1}{2}k^{-2}\epsilon^3 q(\xi,\eta,\tau)q_\xi(\xi,\eta,\tau)e^{2i\theta}- (2k)^{-2}\epsilon^3 q_1^3(\xi,\eta,\tau)e^{3i\theta}+c.c.+\ldots \tag{52}$$

where q is solution of NLS equation [35].

5.1.2 $\left(\frac{G'}{G}\right)$-Expansion Method

Using the travelling wave transformation,

$$u(x,y,t)=U(\xi),\ \xi=kx+ry-ct$$

(27) turns into

$$krU''-k^3cU^{(4)}+k^4U^{(4)}+6k^2(k-c)U'U''=0 \tag{53}$$

ordinary differential equation. Suppose that the travelling wave solution of (53) is in the form

$$U(\xi)=\sum_{l=0}^{m}\alpha_l\left(\frac{G'(\xi)}{G(\xi)}\right)^l,\ \alpha_m\neq 0$$

According the homogeneous balance procedure, in (53) balancing the highest order derivative term $U^{(4)}$ and the highest order nonlinear term $U'U''$, we get

$$U^{(4)}\sim U'U'' \tag{54}$$

$$\begin{aligned}&\Rightarrow m+4=m+1+m+2\\&\Rightarrow m=1\end{aligned} \tag{55}$$

Thus substituting (55) into (11), the travelling wave solution of (53) is found as

$$U(\xi)=\alpha_0+\alpha_1\left(\frac{G'(\xi)}{G(\xi)}\right),\ \alpha_1\neq 0 \tag{56}$$

where $\alpha_l\,(l=0,1)$ are constants to be determined later and $G(\xi)$ satisfies second-order linear ordinary differential equation

$$G''(\xi)+\lambda G'(\xi)+\mu G(\xi)=0, \tag{57}$$

where λ and μ are arbitrary constants. Using (57) and (56), the derivatives in (53) are obtained as follows:

$$\begin{aligned}U'&=-\alpha_1\left(\lambda\frac{G'}{G}+\mu+\frac{G'^2}{G}\right)\\ U''&=\lambda^2\frac{G'}{G}+\alpha_1\lambda\mu-\mu\frac{G'}{G}+3\alpha_1\lambda\frac{G'^2}{G}+3\alpha_1\frac{G'}{G}\mu+2\alpha_1^3\frac{G'}{G}\end{aligned} \tag{58}$$

$$U^{(4)}=\alpha_1\left(15\lambda^3\frac{G'^2}{G}+\lambda^4\frac{G'}{G}+\lambda^3\mu+22\lambda^2\mu\frac{G'}{G}+60\lambda\mu\frac{G'^2}{G} +8\lambda\mu^2+16\mu^2\frac{G'}{G}+50\frac{G'^3}{G}\lambda^2+40\frac{G'^3}{G}\mu+60\frac{G'^4}{G}\lambda +24\frac{G'^5}{G}\right) \tag{59}$$

Substituting these terms together with (56) and (57), into (53), clearing the denominator and setting the coefficients of $\left(\frac{G'(\xi)}{G(\xi)}\right)^i$ to zero we have algebraic system for "α_0, α_1, k, r, c" in the form:

$$\begin{aligned}
&\left(\frac{G'(\xi)}{G(\xi)}\right)^0: 6k^2(c-k)\mu^2\lambda\alpha_1^2-k\lambda\mu(-8k^3\mu-k^3\lambda^2+8k^2c\mu+k^2c\lambda^2-r)\alpha_1\\
&\left(\frac{G'(\xi)}{G(\xi)}\right)^1: 12k^2\mu(\lambda^2+\mu)(c-k)\alpha_1^2-k(-k^3\lambda^4-22k^3\mu\lambda^2-16k^3\mu^2+22k^2\mu c\lambda^2+16k^2\mu^2c+k^2c\lambda^4-r\lambda^2-2r\mu)\alpha_1\\
&\left(\frac{G'(\xi)}{G(\xi)}\right)^2: 6k^2\lambda(\lambda^2+6\mu)(c-k)\alpha_1^2-3k\lambda(-5k^3\lambda^2-20k^3\mu+5k^2c\lambda^2+20k^2c\mu-r)\alpha_1\\
&\left(\frac{G'(\xi)}{G(\xi)}\right)^3: 24k^2(\lambda^2+\mu)(c-k)\alpha_1^2-2k(-25k^3\lambda^2-20k^3\mu+25k^2c\lambda^2+20k^2c\mu-r)\alpha_1\\
&\left(\frac{G'(\xi)}{G(\xi)}\right)^4: 30k^2(c-k)\lambda\alpha_1^2-60k^3(c-k)\lambda\alpha_1\\
&\left(\frac{G'(\xi)}{G(\xi)}\right)^5: 12k^2(c-k)\alpha_1^2-24k^3(c-k)\alpha_1
\end{aligned} \tag{60}$$

Solving the algebraic equation system (60) for α_0, α_1, k, r, c by use of Maple, we get

$$\begin{aligned}&\alpha_0=\alpha_0,\ \alpha_1=2k,\\&c=c,\ k=k,\ r=k^2(c-k)(\lambda^2-4\mu)\end{aligned} \tag{61}$$

Thus, hyperbolic and trigonometric solutions of (53) are found as follows [35]:

(i) For $\lambda^2-4\mu>0$,

Substituting (61) and (13) into (56), we get hyperbolic function solutions of (53)

$$U(\xi)=\alpha_0+k\left(\sqrt{\lambda^2-4\mu}\,\frac{C_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+C_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{C_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}-\lambda\right) \quad (62)$$

where C_1 and C_2 are arbitrary constants. From (62) using $\xi=(kx+k^2(c-k)(\lambda^2-4\mu)-ct)$, we obtain

$$u(x,\,y,\,t)=\alpha_0+k(\sqrt{\lambda^2-4\mu}\;v(x,\,y,\,t)-\lambda)$$

where

$$v(x,\,y,\,t)=\frac{C_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(kx+k^2(c-k)(\lambda^2-4\mu)-ct)\right)+C_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(kx+k^2(c-k)(\lambda^2-4\mu)-ct)\right)}{C_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(kx+k^2(c-k)(\lambda^2-4\mu)-ct)\right)+C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(kx+k^2(c-k)(\lambda^2-4\mu)-ct)\right)}$$

(ii) For $\lambda^2-4\mu<0$,

Substituting (61) and (13) into (56), we find trigonometric function solutions of (53)

$$U(\xi)=\alpha_0+k\left(\sqrt{4\mu-\lambda^2}\,\frac{-C_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+C_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}{C_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+C_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}-\lambda\right) \quad (63)$$

where C_1 and C_2 are arbitrary constants. From (63) using $\xi=(kx+k^2(k-c)(4\mu-\lambda^2)-ct)$, we obtain

$$u(x,\,y,\,t)=\alpha_0+k(\sqrt{4\mu-\lambda^2}\;v(x,\,y,\,t)-\lambda)$$

where

$$v(x,\,y,\,t)=\frac{-C_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(kx+k^2(k-c)(4\mu-\lambda^2)-ct)\right)+C_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(kx+k^2(k-c)(4\mu-\lambda^2)-ct)\right)}{C_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(kx+k^2(k-c)(4\mu-\lambda^2)-ct)\right)+C_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(kx+k^2(k-c)(4\mu-\lambda^2)-ct)\right)}$$

5.1.3 $\left(\frac{G'}{G},\frac{1}{G}\right)$-Expansion Method

Using the travelling wave transformation

$$u(x,\,y,\,t)=U(\xi),\;\xi=kx+hy-ct \quad (64)$$

(27) turns into

$$khU''+k^3(k-c)U^{(4)}+6k^2(k-c)U'U''=0 \quad (65)$$

ordinary differential equation. Assume that the solution of (65) is in the form (26). Then using the homogeneous balance procedure like (54) and (55), we find

$$N=1$$

Then the solution of (65) is found as

$$U(\xi)=a_0+a_1\phi(\xi)+b_1\psi(\xi) \quad (66)$$

where "a_0, a_1, b_1" are constants to be determined later. Using (66) and

$$\begin{aligned}\psi'&=-\phi\psi\\ \phi'&=\mu\psi-\lambda-\phi^2\end{aligned} \quad (67)$$

we get the derivatives in (65) as follows:

$$\begin{aligned}U'&=-a_1\phi^2+a_1\mu\psi-a_1\lambda-b_1\phi\psi\\ U''&=2a_1\phi^3-3a_1\mu\phi\psi+2a_1\phi\lambda+2b_1\phi^2\psi-b_1\psi^2\mu+b_1\psi\lambda\\ U^{(4)}&=-60\mu\phi^3\psi+30a_1\mu^2\phi\psi^2-36b_1\mu\phi^2\psi^2+28b_1\lambda\phi^2\psi\\ &\quad-11b_1\mu\lambda\psi^2-45a_1\mu\lambda\phi\psi+24b_1\phi^4\psi+40a_1\phi^3\lambda\\ &\quad+16a_1\phi\lambda^2+6b_1\mu^2\psi^3+5b_1\psi\lambda^2+24a_1\phi^5\end{aligned} \quad (68)$$

Thus, new and different types hyperbolic and trigonometric function solutions for (2+1) dimensional KdV4 equation are obtained for $\lambda<0$ and $\lambda>0$ [35].

(i) For $\lambda<0$;

Substituting (66), (68), and

$$\psi^2=\frac{-\lambda(\phi^2-2\mu\psi+\lambda)}{\lambda^2\nu+\mu^2},\;\nu=C_1^2-C_2^2\text{ and }\lambda<0 \quad (69)$$

into (65) and using (69), it can be seen that the powers of ψ are not higher than 1 in (65). Thus, the left-hand side of (65) becomes a polynomial in ϕ and ψ. Setting the each coefficient of equation to zero yields a system of algebraic equations:

$$
\begin{aligned}
\phi^0:\ & b_1\lambda^2\mu(k^2c\lambda\mu^2-k^3\lambda\mu^2+h\mu^2-11k^2c\lambda^3\nu+11k^3\lambda^3\nu\\
&+12\lambda^3ka_1\nu(c-k)+h\lambda^2\nu)\\
\psi:\ & -b_1\lambda(6\lambda^5ka_1\nu^2(k-c)+5k^2\lambda^5\nu^2(c-k)-h\lambda^4\nu^2\\
&+18\lambda^3k^2\mu^2\nu(k-c)+18\lambda^3ka_1\mu^2\nu(c-k)+k^2\lambda\mu^4(c-k)\\
&+k^2\lambda\mu^4(c-k)+h\mu^4)\\
\phi:\ & 2\lambda(6ka_1^2\lambda^5\nu^2(c-k)+8k^2a_1\lambda^5\nu^2(k-c)+3kb_1^2\lambda^4\nu(k\\
&-c)+ha_1\lambda^4\nu^2+3ka_1^2\mu^2\lambda^3\nu(c-k)+k^2a_1\lambda^3\mu^2\nu(k-c)\\
&+2ha_1\mu^2\lambda^2\nu+3kb_1^2\lambda^2\mu^2(c-k)+7k^2a_1\lambda\mu^4(c-k)\\
&-3kca_1^2\lambda\mu^4+ha_1\mu^4)\\
\phi\psi:\ & -3\mu(5k^2a_1\lambda\mu^4(c-k)+2ka_1^2\lambda\mu^4(k-c)+8ka_1^2\lambda^3\nu\mu^2(c\\
&-k)+2kb_1^2\lambda^2\mu^2(c-k)+10ka_1^2\lambda^5\nu^2(c-k)+(k\\
&-c)10k^2a_1\lambda^3\mu^2\nu+6kb_1^2\lambda^4\nu(k-c)+15k^2a_1\lambda^5\nu^2(k-c)\\
&+ha_1(\lambda^4\nu^2+\mu^4)\\
\phi^2:\ & -b_1\mu\lambda(35k^2\lambda\mu^2(c-k)+36\mu^2\lambda a_1k(k-c)+47k^2\lambda^3\nu(c\\
&-k)+48\lambda^3k\nu a_1(k-c)-h\mu^2-h\lambda^2\nu)\\
\phi^2\psi:\ & 2b_1(\lambda^2\nu+\mu^2)(24ka_1\lambda\mu^2(k-c)+25k^2\lambda\mu^2(c-k)\\
&+14k^2\lambda^3\nu(k-c)+15ka_1\lambda^3\nu(c-k)+h\mu^2+h\lambda^2\nu\\
\phi^3:\ & 18kb_1^2\lambda^4\nu(k-c)+6kb_1^2\lambda^2\mu^2(k-c)+6ka_1^2\lambda\mu^4(c-k)\\
&+2ha_1\mu^4+10k^2a_1\lambda\mu^4(k-c)+40k^2a_1\lambda^5\nu^2(k-c)\\
&-24ka_1^2\lambda^5\nu^2(k-c)+50k^2a_1\lambda^3\mu^2\nu(k-c)+2ha_1\lambda^2\nu(\lambda^2\nu\\
&+2\mu^2)-30ka_1^2\mu^2\lambda^3\nu(k-c)\\
\phi^3\psi:\ & 30k(\lambda^2\nu+\mu^2)\mu(2\mu^2ka_1-a_1^2(\lambda^2\nu+\mu^2)+2ka_1\lambda^2\nu\\
&+b_1^2\lambda)(c-k)\\
\phi^4:\ & -36kb_1(\lambda^2\nu+\mu^2)(-a_1+k)(c-k)\mu\lambda\\
\phi^4\psi:\ & -24kb_1(\lambda^2\nu+\mu^2)^2(-a_1+k)(c-k)\\
\phi^5:\ & -12k(\lambda^2\nu+\mu^2)((\lambda^2\nu+\mu^2)(2ka_1-a_1^2)+b_1^2\lambda)(c-k)
\end{aligned}
\tag{70}
$$

Solving the algebraic equations by Maple, we find

$$
\begin{aligned}
&a_0=a_0,\ a_1=a_1,\ b_1=\pm\sqrt{-\frac{\lambda^2\nu+\mu^2}{\lambda}}a_1\\
&k=a_1,\ h=(a_1-c)\lambda a_1^2,\ c=c
\end{aligned}
\tag{71}
$$

From (71), we get

$$
\begin{aligned}
\phi(\xi)&=\frac{\sqrt{-\lambda}(A_1\cosh(\sqrt{-\lambda}\xi)+A_2\sinh(\sqrt{-\lambda}\xi))}{(A_1\sinh(\sqrt{-\lambda}\xi)+A_2\cosh(\sqrt{-\lambda}\xi))+\frac{\mu}{\lambda}}\\
\psi(\xi)&=\frac{1}{A_1\sinh(\sqrt{-\lambda}\xi)+A_2\cosh(\sqrt{-\lambda}\xi)+\frac{\mu}{\lambda}}
\end{aligned}
\tag{72}
$$

where $(\phi,\psi)=\left(\frac{G'}{G},\frac{1}{G}\right)$ and

$$G(\xi)=A_1\sinh(\sqrt{-\lambda}\xi)+A_2\cosh(\sqrt{-\lambda}\xi)+\frac{\mu}{\lambda},\ \lambda<0 \tag{73}$$

Finally substituting (71) and (72) into (66), the exact travelling wave solution of (65) is found as

$$
\begin{aligned}
U(\xi)=a_0+a_1&\frac{\sqrt{-\lambda}(A_1\cosh(\sqrt{-\lambda}\xi)+A_2\sinh(\sqrt{-\lambda}\xi))}{(A_1\sinh(\sqrt{-\lambda}\xi)+A_2\cosh(\sqrt{-\lambda}\xi))+\frac{\mu}{\lambda}}\\
&\pm\sqrt{-\frac{\lambda^2\nu+\mu^2}{\lambda}}a_1\frac{1}{A_1\sinh(\sqrt{-\lambda}\xi)+A_2\cosh(\sqrt{-\lambda}\xi)+\frac{\mu}{\lambda}}
\end{aligned}
$$

where

$$\xi=a_1x+(a_1-c)\lambda a_1^2y-ct \tag{74}$$

And using (64) hyperbolic solution of (27) is obtained as follows:

$$u(x,y,t)=a_0+a_1.v(x,y,t)$$

where

$$
\begin{aligned}
v(x,y,t)=(&\sqrt{-\lambda}(A_1\cosh(\sqrt{-\lambda}(a_1x+(a_1-c)\lambda a_1^2y-ct))+\\
&A_2\sinh\left(\sqrt{-\lambda}(a_1x+(a_1-c)\lambda a_1^2y-ct))\right)\pm\sqrt{-\frac{\lambda^2\nu+\mu^2}{\lambda}}\Bigg)/.\\
&((A_1\sinh(\sqrt{-\lambda}(a_1x+(a_1-c)\lambda a_1^2y-ct))+\\
&A_2\cosh\left(\sqrt{-\lambda}(a_1x+(a_1-c)\lambda a_1^2y-ct))\right)+\frac{\mu}{\lambda}\Bigg)
\end{aligned}
$$

(ii) For $\lambda>0$;
Substituting (66), (68) and

$$\psi^2=\frac{-\lambda(\phi^2-2\mu\psi+\lambda)}{\lambda^2\sigma-\mu^2},\ \sigma=C_1^2+C_2^2\ \text{and}\ \lambda>0 \tag{75}$$

into (65) and using (75), it can be seen that the powers of ψ are not higher than 1 in (65). Thus, the left-hand side of (65) becomes a polynomial in ϕ and ψ. Setting the each coefficient of equation to zero yields a system of algebraic equations:

$$\phi^0\psi^0: b_1\lambda^2\mu((23k^2\lambda\mu^2-24k\lambda\mu^2a_1-11k^2\lambda^3\nu+12\lambda^3ka_1\nu)(c-k)+h(\lambda^2\nu-\mu^2))(\lambda^2\nu+\mu^2)^2$$

$$\psi: -b_1\lambda((57k^2\lambda\mu^4-60\lambda ka_1\mu^4+5k^2\lambda^5\nu^2-6ka_1\lambda^5\nu^2-38\lambda^3k^2\mu^2\nu+42\lambda^3ka_1\mu^2\nu)(c-k)-h(3\mu^4+\lambda^4\nu^2))(\lambda^2\nu+\mu^2)^2$$

$$\phi: -2\lambda((8k^2a_1\lambda^5\nu^2-6ka_1^2\lambda^5\nu^2+3kb_1^2\lambda^4\nu-31k^2a_1\mu^2\lambda^3\nu+21ka_1^2\lambda^3\mu^2\nu-9kb_1^2\lambda^2\mu^2+23k^2a_1\lambda\mu^4-15ka_1^2\mu^4\lambda)(c-k)+ha_1\lambda^2\nu(2\mu^2-\lambda^2\nu)-ha_1\mu^4)(\lambda^2\nu+\mu^2)^2$$

$$\phi\psi: 3\mu((35k^2a_1\lambda\mu^4-22ka_1^2\lambda\mu^4-50k^2a_1\lambda^3\mu^2\nu-14kb_1^2\lambda^2\mu^2+32ka_1^2\lambda^3\nu\mu^2-10ka_1^2\lambda^5\nu^2+6kb_1^2\lambda^4\nu+15k^2a_1\lambda^5\nu^2)(c-k)+ha_1\lambda^2\nu(2\mu^2-\lambda^2\nu)-ha_1\mu^4)(\lambda^2\nu+\mu^2)^2$$

$$\phi^2: b_1\mu\lambda((59k^2\lambda\mu^2-60\mu^2\lambda a_1k+48\lambda^3k\nu a_1-47k^2\lambda^3\nu)(c-k)-h\mu^2+h\lambda^2\nu)(\lambda^2\nu+\mu^2)^2$$

$$\phi^2\psi: -2b_1(-\lambda^2\nu+\mu^2)((-54ka_1\lambda\mu^2+53k^2\lambda\mu^2-14k^2\lambda^3\nu+15ka_1\lambda^3\nu)(c-k)-h\mu^2+h\lambda^2\nu)(\lambda^2\nu+\mu^2)^2$$

$$\phi^3: -2((35k^2a_1\lambda\mu^4+20k^2a_1\lambda^5\nu^2-12ka_1^2\lambda^5\nu^2-21ka_1^2\lambda\mu^4-55k^2a_1\lambda^3\mu^2\nu+33ka_1^2\mu^2\lambda^3\nu+9kb_1^2\lambda^4\nu-15kb_1^2\lambda^2\mu^2)(c-k)+ha_1\lambda^2\nu(2\mu^2-\lambda^2\nu)-ha_1\mu^4)(\lambda^2\nu+\mu^2)^2$$

$$\phi^3\psi: 30k(-\lambda^2\nu+\mu^2)\mu(2\mu^2ka_1+a_1^2(\lambda^2\nu-\mu^2)-2ka_1\lambda^2\nu-b_1^2\lambda)(c-k)(\lambda^2\nu+\mu^2)^2$$

$$\phi^4: 36kb_1(-\lambda^2\nu+\mu^2)(-a_1+k)(c-k)\mu\lambda(\lambda^2\nu+\mu^2)^2$$

$$\phi^4\psi: -24kb_1(-\lambda^2\nu+\mu^2)^2(-a_1+k)(c-k)(\lambda^2\nu+\mu^2)^2$$

$$\phi^5: -12k(-\lambda^2\nu+\mu^2)(-a_1+k)(c-k)(\lambda^2\nu+\mu^2)^2((2k-a_1)a_1(-\lambda^2\nu+\mu^2)-b_1^2\lambda)$$

Solving the algebraic equation systems by use of Maple, we get

$$a_0=a_0,\ a_1=a_1,\ b_1=\pm\sqrt{-\frac{\lambda^2\nu-\mu^2}{\lambda}}a_1$$
$$k=a_1,\ h=(a_1-c)\lambda a_1^2,\ c=c \tag{76}$$

Then, we find

$$\phi(\xi)=\frac{\sqrt{\lambda}(A_1\cos(\sqrt{\lambda}\xi)-A_2\sin(\sqrt{\lambda}\xi))}{(A_1\sin(\sqrt{\lambda}\xi)+A_2\cos(\sqrt{\lambda}\xi))+\frac{\mu}{\lambda}}$$
$$\psi(\xi)=\frac{1}{(A_1\sin(\sqrt{\lambda}\xi)+A_2\cos(\sqrt{\lambda}\xi))+\frac{\mu}{\lambda}} \tag{77}$$

where $(\phi,\psi)=\left(\frac{G'}{G},\frac{1}{G}\right)$ and

$$G(\xi)=A_1\sin(\sqrt{\lambda}\xi)+A_2\cos(\sqrt{\lambda}\xi)+\frac{\mu}{\lambda},\ \lambda>0 \tag{78}$$

Substituting (76) and (77) into (66), the exact travelling wave solution of (65) is found as

$$U(\xi)=a_0+a_1\frac{\sqrt{\lambda}(A_1\cos(\sqrt{\lambda}\xi)-A_2\sin(\sqrt{\lambda}\xi))}{(A_1\sin(\sqrt{\lambda}\xi)+A_2\cos(\sqrt{\lambda}\xi))+\frac{\mu}{\lambda}}\pm\sqrt{-\frac{\lambda^2\nu-\mu^2}{\lambda}}a_1\frac{1}{(A_1\sin(\sqrt{\lambda}\xi)+A_2\cos(\sqrt{\lambda}\xi))+\frac{\mu}{\lambda}} \tag{79}$$

where

$$\xi=a_1x+(a_1-c)\lambda a_1^2y-ct \tag{80}$$

Finally using (64), trigonometric solution of (27) is obtained in the form

$$u(x,y,t)=a_0+a_1.v(x,y,t)$$

where

$$v(x,y,t)=(\sqrt{\lambda}(A_1\cos(\sqrt{\lambda}(a_1x+(a_1-c)\lambda a_1^2y-ct))-A_2\sin\left(\sqrt{\lambda}(a_1x+(a_1-c)\lambda a_1^2y-ct))\right)\pm\sqrt{-\frac{\lambda^2\nu-\mu^2}{\lambda}}\Bigg)$$
$$/.((A_1\sin(\sqrt{\lambda}(a_1x+(a_1-c)\lambda a_1^2y-ct))+A_2\cos\left(\sqrt{\lambda}(a_1x+(a_1-c)\lambda a_1^2y-ct))\right)+\frac{\mu}{\lambda}\Bigg)$$

6 Conclusion

The solutions of NEEs have many potential applications in physics and engineering. Through this type NEEs, we investigate the solutions of (2+1)-dimensional KdV4 equation and the relation between KdV4 equation and NLS equation. Applying the method of multiple scales to (2+1)-dimensional KdV4 equation, we get (2+1)-dimensional NLS type equations. Moreover, numerical solutions have been obtained for KdV4 equation by this method. In this article, we only studied on deriving NLS type equations from KdV type equations and their solutions by use of multiple scales method. In addition, we can seek the application of the method to the spectral problems and recursion operators of KdV4 equation in further works. By this way, we can get the spectral problems and recursion operators of NLS equation. At the same time if we choose the slow variables different forms, we can derive higher-order NLS type equations. However, this method can be applied not only this type NEEs but also nonlinear differential difference equations and higher-order NEEs as a future work. On the other hand, we have used the $\left(\frac{G'}{G}\right)$-expansion method and $\left(\frac{G'}{G}, \frac{1}{G}\right)$-expansion method to find the exact travelling wave solutions of KdV4 equation in this work. We concluded that the performances of these methods are reliable, simple, and these methods give many new trigonometric and hyperbolic type exact solutions. Finally, it is worth mentioning that the implementation of these proposed methods is very simple and straightforward, and it can also be applied to many other NEEs, differential difference equations, and fractional differential equations. The details about these methods and their applications to other NEEs are given in [35].

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