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The Wheeler–DeWitt Equation in Filćhenkov Model: The Lie Algebraic Approach

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Abstract: The Wheeler–DeWitt equation in Filćhenkov model with terms related to strings, dust, relativistic matter, bosons and fermions, and ultra stiff matter is solved in a quasi-exact analytical manner via the Lie algebraic approach. In the calculations, using the representation theory of *sl*(2), the general ($N+1$)-dimensional matrix equation is constructed whose determinant yields the solutions of the problem.

Keywords: Filćhenkov Model; Lie Algebraic Approach; Quasi-Exactly Solvable Model; Wheeler–DeWitt Equation.

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1 Introduction

For years, the standard model of cosmology has been an acceptable framework to account for many observed phenomena. Nevertheless, it ought be modified due to the challenging problems such as singularity [1], flatness [2], horizon [3], small-scale inhomogeneity [4], entropy [5], monopole [6], cosmological constant [7, 8], and age [9]. Most important of all, however, is the unified description of quantum and gravity theories. A step forward to this aim was the so-called Wheeler–DeWitt (WDW) equation [10–12]. Here, we focus on the WDW equation proposed by Filćhenkov [13].

The model was originally formulated in an attempt to discuss the pre-de-Sitter in terms of quantum mechanics. The de-Sitter vacuum corresponding to the $p=-\varepsilon$ state is considered as an initial state in the universe evolution. In other words, the universe is assumed to be born from this vacuum by tunneling through a potential barrier which separates the de-Sitter domain from its so-called pre-de-Sitter stage [13]. Filćhenkov used a novel idea and considered the quantum birth of flat, closed, and open universes at Planckian densities which could introduce other kinds of matter than vacuum, namely, domain walls, strings, dust, relativistic matter, bosons and fermions, and ultra stiff matter [13]. Through his model, various issues have been addressed including the possibility of quantum birth of a hot universe, the monopole rest mass, vacuum dark energy, etc. [13–15]. It should be noted that the study of other kinds of matter than the vacuum can be alternatively done via the spinor fields [16–19].

Till now, the corresponding differential equation has been analyzed in its special cases [10, 20, 21]. In this work, we solve the corresponding differential equation by using the Lie algebraic approach of quasi-exact solvability [22–24] when the two terms related to vacuum and domain walls are neglected. To perform a comprehensive study, we construct the ($N+1$)-dimensional matrix representation of the problem with the help of the representation theory of *sl*(2). Therefore, the ($N+1$) set of solutions can be determined exactly by diagonalizing the corresponding matrix without the cumbersome numerical and analytical procedures.

Our paper is organized as follows. In Section 2, we briefly review the WDW equation within the model proposed in [10]. Section 3 is devoted to algebraization of the corresponding differential equation. We show that the problem possesses *sl*(2) algebraic structure which allows us to use the representation theory of *sl*(2) to construct the general matrix whose determinant yields the solutions of the problem. We end with conclusions in Section 4.

2 Wheeler–Dewitt Equation

Assuming the scalar factor a and the scalar field ϕ as independent variables, the WDW in minisuperspace takes the form [13]

$$\left[-\frac{1}{a^r}\frac{\partial}{\partial a}a^r\frac{\partial}{\partial a}+\frac{1}{a^2}\frac{\partial^2}{\partial\phi^2}+V(a,\phi)\right]\Psi=0, \tag{1}$$

where Ψ is the cosmic wave function. Here, we assume the factor ordering $r=0$ and $\frac{\partial\Psi}{\partial\phi}=0,$ which implies the

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homogeneity in the Lorentzian domain and stationarity in the Euclidean one. Therefore, (1) reduces to the ordinary form

$$\left[-\frac{d^2}{da^2}+V(a)\right]\Psi=0. \tag{2}$$

In Friedmann universe, the choice

$$V(a)=\frac{2m_{pl}}{\hbar^2}[U(a)-E] \tag{3}$$

transforms (2) into the Schrödinger form [13]

$$\left[\frac{d^2}{da^2}-U(a)\right]\Psi(a)=0, \tag{4}$$

where

$$U(a)=\frac{9\pi^2}{4G^2}\left(ka^2-\frac{8\pi G}{3}a^4\tilde{\rho}\right), \tag{5}$$

and

$$\tilde{\rho}=\rho_{pl}\sum_{n=0}^{6}B_n\left(\frac{l_{pl}}{a}\right)^n. \tag{6}$$

In the above relations, a, k, l_{pl}, and ρ_{pl} are the cosmic scale factor, the reciprocal of Planck mass, Planck length, and the Planck density, respectively. $n=3(1+\omega)$ can take the values 0, 1, 2, 3, 4, 5, and 6, which, respectively correspond to vacuum, domain walls, strings, dust, relativistic matter, bosons and fermions, and ultra stiff matter [10]. Here, we neglect the first two terms related to vacuum and domain walls. Substituting (5) and (6) into (4) leads to

$$\left(\frac{d^2}{da^2}+\xi_0+\xi_1 a+\xi_2 a^2+\frac{\xi_3}{a}+\frac{\xi_4}{a^2}\right)\Psi(a)=0, \tag{7}$$

where

$$\begin{aligned}\xi_0&=\frac{6\pi^3\rho_{pl}B_4 l_{pl}{}^4}{G},\quad \xi_1=\frac{6\pi^3\rho_{pl}B_3 l_{pl}{}^3}{G},\\ \xi_2&=\frac{6\pi^3\rho_{pl}B_2 l_{pl}{}^2}{G}-\frac{9\pi^2 k}{4G^2},\quad \xi_3=\frac{6\pi^3\rho_{pl}B_5 l_{pl}{}^5}{G},\\ \xi_4&=\frac{6\pi^3\rho_{pl}B_6 l_{pl}{}^6}{G}.\end{aligned} \tag{8}$$

In the next section, we solve the problem using the Lie algebraic approach of quasi-exact solvability.

3 The Lie Algebraic Method and Quasi-Exact Solution

First, we recall the basic idea of the Lie algebraic approach of quasi-exact solvability [22, 23] (for elementary discussions, see Appendix). A quantum system is called exactly solvable (ES) if all the eigenvalues and the corresponding eigenfunctions are known exactly. In contrast, a quantum system is called quasi-exactly solvable (QES) if only a finite number of eigenvalues and eigenfunctions can be determined exactly. As the number of ES states can be selected as large as desired, a QES model could be applied as a good alternative to an ES model. Exact and quasi-exact solvability are closely related [25]. A differential operator H is said to be Lie algebraic if it is an element of the universal enveloping algebra U_g of a finite-dimensional Lie algebra g. In one dimension, the only Lie algebra of first-order differential operators which possesses finite-dimensional representations is $sl(2)$, whose generators

$$\begin{aligned}J_j^+&=x^2\frac{d}{dx}-2jx,\\ J_j^0&=x\frac{d}{dx}-j,\\ J_j^-&=\frac{d}{dx},\quad j=0,\ \frac{1}{2},\ 1,\ \ldots,\end{aligned} \tag{9}$$

obey the commutation relations

$$[J_j^+,J_j^-]=-2J_j^0,\quad [J_j^\pm,J_j^0]=\mp J_j^\pm \tag{10}$$

and preserve the space of polynomials of finite order

$$\mathrm{P}_{2j+1}=\langle 1,\ x,\ x^2,\ \ldots,\ x^{2j}\rangle. \tag{11}$$

On the other hand, the most general quadratic differential equation which preserves the space P_{2j+1} is given by

$$H=\sum_{p,q=0,\pm}C_{pq}J^pJ^q+\sum_{p=0,\pm}C_pJ^p+C, \tag{12}$$

where C_{pq}, C_p, $C\in R$. This operator can be always reduced to a Schrödinger-type operator (and vice versa) by making proper variable and/or gauge transformation (see Appendix). In applying the method to the present problem, i.e. (7), we use the following transformation

$$\begin{aligned}\Psi(a)&=G(a)\cdot\tilde{\Psi}(a),\\ G(a)&=\exp(\alpha a^2+\beta a+\delta\ln(a)),\end{aligned} \tag{13}$$

where

$$\alpha=-\frac{\sqrt{-\xi_2}}{2},\quad \beta=\frac{\xi_1}{2\sqrt{-\xi_2}},\quad \delta=\frac{1+\sqrt{1-4\xi_4}}{2}. \tag{14}$$

Inserting the gauge transformation (13) in (7), we get

$$\begin{aligned}&H\tilde{\Psi}(a)=0,\\ &H=a\frac{\mathrm{d}^2}{\mathrm{d}a^2}+(4\alpha a^2+2\beta a+2\delta)\frac{\mathrm{d}}{\mathrm{d}a}\\ &\quad+(2\alpha+\beta^2+4\alpha\delta+\xi_0)a+(2\delta\beta+\xi_3),\end{aligned} \tag{15}$$

where we have multiplied the equation by a on the left for later convenience. In order to show how $sl(2)$ Lie algebra underlies the quasi-exact solvability of the problem, we consider the action of the operator H on monomials of a. It is easy to verify that

$$\begin{aligned}Ha^{2j}&=(2\alpha(4j+2\delta+1)+\beta^2+\xi_0)a^{2j+1}\\ &\quad+\text{lower-order terms},\end{aligned} \tag{16}$$

which implies that the operator H cannot be solved exactly due to the existence of a^{2j+1} term. However, if this term vanishes from the right-hand side of (16), the operator H preserves the space of polynomials P_{2j+1}, which is related to quasi-exact solvability. In order to relate (15) to QES differential equations, it is necessary to introduce the nonnegative finite integer $N=2j$ such that

$$2\alpha(2N+2\delta+1)+\beta^2+\xi_0=0, \tag{17}$$

which gives the exact solutions of the problem restricted to the invariant subspace P_{N+1}. On the other hand, from (12), any QES differential operator can be represented as

$$\begin{aligned}H&=C_{++}J_N^+J_N^+ +C_{+0}J_N^+J_N^0+C_{+-}J_N^+J_N^-\\ &\quad+C_{0-}J_N^0J_N^-+C_{--}J_N^-J_N^-+C_+J_N^++C_0J_N^0+C_-J_N^-+C.\end{aligned} \tag{18}$$

Substituting (9) into (18) yields the following differential form

$$\left\{P_4(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2}+P_3(x)\frac{\mathrm{d}}{\mathrm{d}x}+P_2(x)\right\}\tilde{\psi}(x)=0, \tag{19}$$

where $P_i(x)$ are the polynomials of degree i

$$\begin{aligned}P_4(x)&=C_{++}x^4+C_{+0}x^3+C_{+-}x^2+C_{0-}x+C_{--},\\ P_3(x)&=C_{++}(2-2N)x^3+\left(C_++C_{+0}\left(1-\frac{3N}{2}\right)\right)x^2\\ &\quad+(C_0-NC_{+-})x+\left(C_--\frac{N}{2}C_{0-}\right),\\ P_2(x)&=C_{++}N(N-1)x^2+\left(\frac{N^2}{2}C_{+0}-NC_+\right)x+\left(C-\frac{N}{2}C_0\right).\end{aligned} \tag{20}$$

Making the change of variable $x=a$ and comparing (20) with (15), we find

$$\begin{aligned}&C_{++}=C_{+0}=C_{+-}=C_{--}=0,\\ &C_{0-}=1,\quad C_+=4\alpha,\quad C_0=2\beta,\\ &-NC_+=2\alpha+\beta^2+4\alpha\delta+\xi_0,\\ &C-\frac{N}{2}C_0=2\delta\beta+\xi_3\\ &C_--\frac{N}{2}C_{0-}=2\delta.\end{aligned} \tag{21}$$

Therefore, the differential operator H is rewritten as an element of the universal enveloping algebra of $sl(2)$ as

$$\begin{aligned}H&=J_N^0J_N^-+4\alpha J_N^++2\beta J_N^0+\left(2\delta+\frac{N}{2}\right)J_N^-\\ &\quad+(2\beta\delta+N\beta+\xi_3).\end{aligned} \tag{22}$$

Since this operator possesses the generator of positive grading, it is therefore a QES operator according to the Turbiner's theorem [22]. This algebraization allows us to use the representation theory of $sl(2)$ to study the model with the most general case. As a result of quasi-exact solvability, the operator H preserves the finite $(N+1)$-dimensional invariant subspace P_{N+1} spanned by the basis $\{1, a, a^2, \ldots, a^N\}$. The corresponding eigenfunction has therefore the form

$$\Psi_N(a)=\exp(\alpha a^2+\beta a+\delta\ln(a))\sum_{m=0}^{N}c_ma^m, \tag{23}$$

where the c_m's are the expansion constants. After some algebra, we find that the action of the operator H on the eigenfunctions $\Psi_N(a)$ can be represented as the following matrix equation

$$\begin{pmatrix}2\beta\delta+\xi_3 & 2\delta & 0 & 0 & 0\\ -4N\alpha & 2\beta(\delta+1)+\xi_3 & 4\delta+2 & 0 & 0\\ 0 & -4(N-1)\alpha & 2\beta(\delta+2)+\xi_3 & \ddots & \vdots\\ 0 & 0 & \ddots & \ddots & N(2\delta+N-1)\\ 0 & 0 & \ldots & -4\alpha & 2\beta(\delta+N)+\xi_3\end{pmatrix}\begin{pmatrix}c_0\\ c_1\\ \vdots\\ c_{N-1}\\ c_N\end{pmatrix}=0, \tag{24}$$

where the expansion coefficients c_m ($m=0, 1, 2, \ldots, N$) satisfy the following recursion relation

$$c_{m+1}=\frac{(8\alpha)c_{m-1}-(2\beta(\delta+m)+\xi_3)c_m}{(m^2+(2\delta+1)m+2\delta)}, \tag{25}$$

with the boundary conditions $c_{-1}=0$ and $c_{m+1}=0$. In more explicit form, the eigenvalue equation from (17) is

$$\sqrt{-\frac{6\pi^3\rho_{pl}B_2l_{pl}^{\ 2}}{G}+\frac{9\pi^2k}{4G^2}}\left(2N+2+\sqrt{1-\frac{24\pi^3\rho_{pl}B_6l_{pl}^{\ 6}}{G}}\right)+\frac{\left(\frac{6\pi^3\rho_{pl}B_3l_{pl}^{\ 3}}{G}\right)^2}{\left(\frac{24\pi^3\rho_{pl}B_2l_{pl}^{\ 2}}{G}-\frac{9\pi^2k}{G^2}\right)}-\frac{6\pi^3\rho_{pl}B_4l_{pl}^{\ 4}}{G}=0. \tag{26}$$

We note that the constraints obtaining from the nontrivial solutions of the matrix (24), together with (26), provide the quasi-exact solutions of the problem. Therefore, we have succeeded in finding a general solution for an arbitrary finite value of N which enables us to obtain the $N+1$ number of eigenvalues and eigenfunctions exactly. Here, N is a nonnegative integer which determines the dimension of the invariant module.

4 Conclusion

Using the Lie algebraic approach of quasi-exact solvability, we solved the WDW equation in the Filćhenkov model when terms related to vacuum and domain walls are not taken into account. With the help of the representation theory of $sl(2)$, we constructed the general $(N+1)$-dimensional matrix equation whose nontrivial solutions yield the analytical solutions of the problem. For the special case of the ground state, our results yield the results of [20]. To our best knowledge, the Filćhenkov model has not been analytically solved, while all terms are present for higher states. Our work, despite neglecting two terms, which is of course its failure, provides the higher-state solutions, while five terms of the model are present. This is particularly important as the model describes the universe evolution as a succession of transitions to progressively higher energy levels so that the presently observable universe is considered as the quantum system in a highly excited state. Having calculated the spectrum and the wave function of the system, the results can be used to study the cosmic microwave anisotropy, limits on the temperature of a universe born in a tunneling effect, the monopole abundance, quantum birth of a hot universe, miniholes, vacuum dark energy, etc.

Appendix

Lie Algebraic Approach of Quasi-Exact Solvability

From the Lie algebraic point of view, the underlying idea behind the solvability (exact solvability and quasi-exact solvability) is the existence of a hidden Lie algebraic structure [22, 23]. In the case of ES models, the Hamiltonian of the system can be diagonalized completely due to the fact that algebraic symmetry is complete, but in the case of QES models, the Hamiltonian is only block diagonalized. Such a finite block can always be diagonalized, which yields a finite part of the spectrum algebraically. In one dimension, every QES differential operator H can be represented as the quadratic combination $\sum_{p,q=0,\pm} C_{pq}J^pJ^q+\sum_{p=0,\pm} C_pJ^p+C$ where the operators J^p ($p=0, \pm$) are $sl(2)$ generators given in (9). Also, according to Turbiner's theorem [22], a QES operator has no terms of positive grading, if and only if it is an ES operator. Substitution of (9) into H gives the second-order differential equation

$$H=-P(x)\frac{d^2}{dx^2}-Q(x)\frac{d}{dx}-R(x), \tag{A-1}$$

where P, Q, and R are polynomials of degree at most 4, 3, and 2, respectively. Now, consider the eigenvalue problem $H\psi(x)=E\psi(x)$. The operator $\tilde{H}$ is equivalent to H, if they are related by the following transformation

$$\tilde{H}(x)=e^{\mu(x)}H(x)e^{-\mu(x)}, \tag{A-2}$$

$$z=\zeta(x), \tag{A-3}$$

where $\mu(x)$ and $\zeta(x)$ are smooth functions. Now, introducing the new function

$$\tilde{\psi}(z)=\tilde{\psi}(\zeta(x))=e^{\mu(x)}\psi(x) \tag{A-4}$$

leads to the eigenvalue problem with the same eigenvalue E and the following Hamiltonian

$$\tilde{H}=-P\zeta'^2\frac{d^2}{dz^2}+(2P\mu'\zeta'-Q\zeta'-P\zeta'')\frac{d}{dz}+(P\mu''-P\mu'^2+Q\mu'-R). \tag{A-5}$$

This operator is equivalent to the Sturm–Liouville-type problem with the Schrödinger operator

$$\tilde{H}=-\frac{d^2}{dz^2}+V(z), \quad \text{(A-6)}$$
$$V(z)=\frac{3P'^2-8QP'+4Q^2}{16P}-R+\frac{1}{2}Q'-\frac{1}{4}P'',$$

under the following transformations

$$z=\zeta(x)=\int^x \frac{dx}{\sqrt{P(x)}}, \quad \text{(A-7)}$$

$$\mu(x)=-\frac{1}{4}\log|P(x)|+\int^x \frac{Q(x)dx}{2p(x)}. \quad \text{(A-8)}$$

Hence, taking H as an element of the universal enveloping algebra $sl(2)$, the corresponding potential $V(z)$ and, afterwards, the explicit expression for the change of variable and the functional form of the eigenfunctions are obtained algebraically. A complete classification of the one-dimensional QES systems associated with the Lie algebra $sl(2)$ can be found in [22]. We now apply the above results to the two most well-known QES and ES potentials, the QES sextic oscillator and the ES harmonic oscillator, respectively.

1. The QES sextic oscillator

The simplest example of a QES operator which is not ES is the following non-linear combination in the generators (9)

$$H_{QES}=4J^0J^-+4aJ^++4bJ^0-2(2j+1+2k)J^-+4bj, \quad \text{(A-9)}$$

which has the differential form

$$H\left(x,\frac{d}{dx}\right)=-4x\frac{d^2}{dx^2}+2(2ax^2+2bx-1-2k)\frac{d}{dx} -8ajx, \quad \text{(A-10)}$$

where $x\in\mathbb{R}$, $a\geq 0$, and $b>0$. From (A-7) and (A-8), by using a change of variable and a gauge function

$$x=z^2, \quad \text{(A-11)}$$

$$\mu(x)=\frac{ax^2}{4}+\frac{bx}{2}-\frac{k}{2}\ln(x), \quad \text{(A-12)}$$

we arrive at the spectral problem (A-6) with the QES sextic anharmonic potential

$$V_{QES}(z)=a^2z^6+2abz^4 +(b^2-(8j+3+2k)a)z^2-b(1+2k). \quad \text{(A-13)}$$

2. The ES harmonic oscillator

Now, consider the following non-linear combination of the $sl(2)$ generators

$$H_{ES}=-J^-J^--\omega J^0, \quad \text{(A-14)}$$

which has the differential form

$$H\left(x,\frac{d}{dx}\right)=-\frac{d^2}{dx^2}-\omega x\frac{d}{dx}+j\omega. \quad \text{(A-15)}$$

From (A-7), by introducing the following gauge function

$$\mu(x)=\frac{\omega x^2}{4}, \quad \text{(A-16)}$$

we arrive at the spectral problem (A-6) with the ES harmonic oscillator potential

$$V_{ES}(z)=\frac{1}{2}\omega^2z^2+\text{const}, \quad \text{(A-17)}$$

where ω is the oscillation frequency.

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