Fatemeh Pakdel* and Ali Akbar Rajabi

Studying of the Nucleus-Nucleus Interaction Using Wave Function of the Nucleus and Hyperspherical Formalism

DOI 10.1515/zna-2016-0104
Received March 18, 2016; accepted September 26, 2016; previously published online October 19, 2016

Abstract: The current study presents the results of the use of the wave function of a nucleus in place of the nucleus density distribution to calculate the nucleus-nucleus interaction potential. The wave function is obtained by solving the D-dimensional wave equation using the hyperspherical formalism. The interaction potential between two nuclei is calculated using the double folding model. The numerical results for the interaction potential and the scattering cross section are presented to evaluate the formalism used to calculate the nucleus wave function.

Keywords: D-Dimensional Wave Equation; Double Folding Model; Hyperspherical Formalism; M3Y Interaction; $^{16}O + ^{16}O$ interaction; Yukawa potential.

1 Introduction

The physical world is a many-body-forces world that incorporates solid state, plasma, nuclear and atomic physics, metals, and physical chemistry. To study these many-body systems, their differential equation or their N-particle Schrödinger equation should be solved and their wave function should be obtained. Investigating the wave equation and wave function of these systems is important because it gives important information about them like energy, the phase shift of the scattered wave function, scattering amplitude, cross section, resonance, deformation, polarization, etc.

Various techniques have been used to solve the N-particle Schrödinger equation. The Jacobi coordinates and hyperspherical formalism is one of these methods [1–4]. We calculate the wave function of the nucleus as a many-body system using this method. We use this model as one of the models for solving the wave equation for N-particle systems. In this method, a D-dimensional wave equation must be solved and a D-dimensional wave function must be obtained. The dimension is related to the number of particles in the center of mass coordinate $D = 3N - 3$, where N is the number of particles.

Our goal in this work is to evaluate the hyperspherical formalism and D-dimensional wave equation for describing a nuclear system as a many-body system and calculating the wave function of the nucleus. For this reason, we consider a two-nuclei system. Then we solve a wave equation for each nucleus using the hyperspherical method. The obtained wave functions are used for calculating the interaction potential with the double folding model (DFM) [5, 6]. The DFM has been widely used by many groups in describing light and Heavy-ion scattering [5–8]. The DFM calculates the interaction potential between the two interacting nuclei using their density distributions. In this paper the obtained D-dimensional wave functions of two nuclei are used in place of their densities in the DFM. We are looking for two purposes: first, credit assessment of the hyperspherical formalism and D-dimensional wave equation for calculating the wave function of the nucleus and second, valuation of the DFM with the obtained D-dimensional wave functions of two interacting nuclei in place of their density distributions. We do this by investigating the DFM results for the interaction potential and also comparing the calculated scattering cross section with the experimental data. Finally, comparing our results with the experimental data, we conclude that the hyperspherical formalism and obtained D-dimensional wave function are appropriate for describing the nucleus. So it is obvious that the hyperspherical formalism and obtained wave function can be used to calculate the energy of the nucleus, investigating the scattering process of the nucleus during an interaction and also resonance, deformation, and polarization of the nucleus.

A general definition of the folding model is provided in Section 2. The wave functions of two colliding nuclei are obtained in Section 3. The results and discussions are presented in Section 4. The conclusions are presented in Section 5.

***Corresponding author: Fatemeh Pakdel,** Department of Physics, University of Shahrood, Shahrood, Iran,
E-mail: fatemehpakdel185@yahoo.com
Ali Akbar Rajabi: Department of Physics, University of Shahrood, Shahrood, Iran

2 The Folding Model

The optical model was developed to explain nuclear reactions. The optical potential is a complex potential that represents the intermediate interaction between the projectile and the target nuclei [7, 8]. In this way, a many-body problem can be approximated by an effective nucleon-nucleon (NN) interaction. The folding model is a good method for calculating the real part of the optical potential. If inelastic scattering occurs, the absorption processes should be investigated using the imaginary part of the optical potential. The folding model is microscopic and dependent on the distance between the two nuclei centers of mass and defined by the NN interaction and the nuclear density distributions. The nuclear part of the interaction potential contains two terms: direct term and exchange term. The direct part of the interaction between two nuclei is determined by the DFM as follows [9, 10].

$$V_{\mathrm{ND}}(\vec{R},E_P)=g(E_P)\int \mathrm{d}\vec{\mathrm{r}}_1\int \mathrm{d}\vec{\mathrm{r}}_2\rho_1(\vec{\mathrm{r}}_1)\rho_2(\vec{\mathrm{r}}_2)\mathrm{v}_{\mathrm{NN}(D)}(s) \tag{1}$$

where $\vec{\mathrm{r}}_{1(2)}$ denotes the position of the interacting nucleon of each nucleus, $\rho_{1(2)}(\vec{\mathrm{r}}_{1(2)})$ is the density of each nucleus, v_{NN} is the effective NN potential, $\vec{R}$ is the distance between the colliding nuclei centers of mass, vector $\vec{s}=\vec{R}-\vec{\mathrm{r}}_1+\vec{\mathrm{r}}_2$ corresponds to the distance between two specified interacting points of the projectile and the target, and $g(E_P)=1-k.E_P$ is an energy-dependent coefficient in which $E_p=E_{\mathrm{lab}}/A_P$ is the average energy of each projectile nucleon (E_{lab} is the laboratory energy, and A_p is the nucleon number of the projectile). The exchange part of the nuclear interaction by the DFM is

$$V_{\mathrm{NE}}(\vec{R},E_P)=g(E_P)\int \mathrm{d}\vec{\mathrm{r}}_1\int \mathrm{d}\vec{\mathrm{r}}_2\rho_1(\vec{\mathrm{r}}_1;\,\vec{\mathrm{r}}_1+\vec{s})\rho_2(\vec{\mathrm{r}}_2;\vec{\mathrm{r}}_2-\vec{s})v_{\mathrm{NN}(E)}(s)\\ \exp(ik_{\mathrm{rel}}\vec{s}/A_{\mathrm{red}}) \tag{2}$$

The exchange part densities have been determined using density matrix expansion relations [11, 12] as

$$\rho(\vec{\mathrm{r}};\,\vec{\mathrm{r}}\pm\vec{s})\approx\rho(\vec{\mathrm{r}}\pm\vec{s}/2)\hat{j}_1(|k_{\mathrm{eff}}(\vec{\mathrm{r}}\pm\vec{s}/2)|\cdot s),\ \hat{j}_1(x)\\ =3[\sin(x)-x\cos(x)]/x^3 \tag{3}$$

where $\hat{j}_1(x)$ is the first degree spherical Bessel function and k_{eff} is the effective Fermi momentum, which has been extracted from the extended Tomas-Fermi approximation [13] as

$$k_{\mathrm{eff}}^2(\vec{r})=\left(\frac{3\pi^2\rho(\vec{r})}{2}\right)^{2/3}+\frac{5C_s}{3}\left(\frac{\vec{\nabla}\rho(\vec{r})}{\rho(\vec{r})}\right)^2+\frac{5\nabla^2\rho(\vec{r})}{36\rho(\vec{r})} \tag{4}$$

where C_s determines the strength of the Weizsäcker correction term for the kinetic energy density [14, 15]. We use $C_s=1/36$, which provides good results [14]. Wave number k_{rel} is added in consideration of the relative motion of colliding nuclei as follows

$$k_{\mathrm{rel}}^2=2m_nA_{\mathrm{red}}[E_{c.m.}-V_{\mathrm{tot}}(R)]/\hbar^2 \tag{5}$$

where $A_{\mathrm{red}}=A_PA_T/(A_P+A_T)$ is the reduced mass number and m_n is the bare nucleon mass.

The NN interaction is widely assumed for the finite-range approximation as the M3Y potential for both the direct and the exchange parts and is defined with the specified coefficients [16] as

$$v_{D(E)}(s)=\sum_{i=1}^{3}G_{D(E)i}v_i(s),\ v_i(s)=\frac{\exp[-s/r_{vi}]}{[s/r_{vi}]} \tag{6}$$

It is known that the M3Y NN interaction must be density dependent. We use the generalized density dependence of the M3Y interaction introduced elsewhere [17]. It is employed as a multiplier in the nuclear part of the interaction as follows

$$F(\rho)=C\{1+\alpha\exp(-\beta\rho)-\gamma\rho\} \tag{7}$$

In place of the finite-range NN interaction, a zero range was used in the earlier works [6] as

$$v_{E\delta}(s)=J(E)\delta(s),\ J(E)=G_{E\delta}g(E_P) \tag{8}$$

In this work we use the finite-range CDM3Y5-Reid density-dependent interaction. All required constant values are derived from Ref. [16].

Because the total interaction potential is the sum of the nuclear, Coulomb, and rotational potentials denoted by $V_{\mathrm{total}}(\vec{R})=V_N(\vec{R})+V_C(\vec{R})+V_{\mathrm{rot}}(\vec{R})$, the Coulomb potential is also required (we have ignored the rotational part in the present study). The Coulomb part assumes that v_{NN} is in the form of the point-point Coulomb potential as

$$v_C(\mathrm{r})=\frac{Z_1Z_2e^2}{4\pi\varepsilon_0}\frac{1}{r} \tag{9}$$

where Z_1 and Z_2 are the nuclei charges. The Coulomb part has a form similar to the direct interaction in (1) [18, 19]

$$V_C(\vec{R})=\int \mathrm{d}\vec{\mathrm{r}}_1\int \mathrm{d}\vec{\mathrm{r}}_2\rho_1(\vec{\mathrm{r}}_1)\rho_2(\vec{\mathrm{r}}_2)\mathrm{v}_C(s) \tag{10}$$

where $\rho_{1(2)}(\vec{\mathrm{r}}_{1(2)})$ are the charge densities of the projectile and the target nucleus.

This paper uses wave functions instead of densities; therefore, the density distributions for the direct and the Coulomb parts are

$$|\psi_1(\vec{\mathrm{r}}_1)\psi^*{}_1(\vec{\mathrm{r}}_1)||\psi_2(\vec{\mathrm{r}}_2)\psi^*{}_2(\vec{\mathrm{r}}_2)| \tag{11}$$

and the density distributions for the exchange part are

$$\left|\psi(\vec{\mathrm{r}};\ \vec{\mathrm{r}}\pm\vec{s})\psi^{*}(\vec{\mathrm{r}};\vec{\mathrm{r}}\pm\vec{s})\right|\approx\left|\psi(\vec{\mathrm{r}}\pm\vec{s}/2)\psi^{*}(\vec{\mathrm{r}}\pm\vec{s}/2)\right| \hat{j}_1(\left|k_{eff}(\vec{r}\pm\vec{s}/2)\right|\cdot s) \tag{12}$$

where $\psi_1(\vec{\mathrm{r}}_1), \psi(\vec{\mathrm{r}}+\vec{s}/2)$ and $\psi_2(\vec{\mathrm{r}}_2), \psi(\vec{\mathrm{r}}-\vec{s}/2)$ represent the wave functions of the projectile and the target nucleus, respectively, and $\nu_{\mathrm{NN}}(s)$ shows the NN interaction between the two nuclei; thus, $V_F(\vec{R})$ is the interaction potential between the two nuclei.

To calculate the wave function of each interacting nucleus, we assume that the interaction between the nucleons of each nucleus has a Yukawa form and solve the many-body equation using this potential. The wave function obtained for each nucleus is used to calculate the interaction potential using the DFM. In real physical processes, the interaction between two compound particles such as nuclei, atoms, and molecules is much more complicated than the interaction described by the central potential; however, the use of even such a simple model in many cases allows us to find the most important features of the processes under consideration. Moreover, the results can be directly generalized for more difficult cases.

3 Wave Equation

As stated, we must calculate the wave functions of two nuclei to find the interaction potential using the DFM. Because the nucleus is a many-body system, we examine the many-body wave equation. The Schrödinger equation for a system of the N fixed identical particles is

$$\left\{-\frac{1}{2m}\sum_{i=1}^{N-1}\nabla_{r_i}^2+\sum_{i,j>i}V(r_{ij})-E\right\}\psi(\mathrm{r}_{ij})=0 \tag{13}$$

where $r_{ij}=r_j-r_i$ is the relative coordinate of the particles and $V(r_{ij})$ is the interaction potential between the particles in terms of the relative distance of the pair in the two-body subsystem. It is known that the relative distance between the particles by the Jacobi coordinate transformation is

$$\xi_i=\sqrt{\frac{i}{i+1}}\left(\mathrm{r}_{i+1}-\frac{1}{i}\sum_{j=1}^{i}r_j\right)\ i=1,\ 2,\ \ldots,\ N-1 \tag{14}$$

There are N–1 Jacobi coordinates for the N-particle system. In this way, one can consider the center of mass of two arbitrary particles; the third particle location is considered to be relative to the center of mass of the two first particles and similarly for all particles. We now define the hyperradius [20] as

$$x=(\xi_1+\xi_2+\ldots+\xi_{N-1})^{1/2} \tag{15}$$

The hyperradius is composed of terms of the Jacobi coordinates where each one contacts the center of mass of the subsystem for the remaining particles. The hyperradius is, thus, the relative distance of the particles from each other and from the center of mass. For the hyperspherical coordinates and D-dimensional space, the Laplacian is [21, 22]

$$\sum_{i=1}^{N-1}\nabla_{\xi_i}^2=\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{D-1}{x}\frac{\mathrm{d}}{\mathrm{d}x}+\frac{\Gamma^2(\Omega)}{x^2}\right) \tag{16}$$

where $\frac{\Gamma^2(\Omega)}{x^2}$ is a generalization of the centrifugal barrier and $\Gamma^2(\Omega)$ is the grand orbital operator with the following eigenvalues [23, 24]

$$\Gamma^2(\Omega)=-\gamma(\gamma+D-2) \tag{17}$$

where $\gamma=2n+l_{\xi_1}+l_{\xi_2}+\ldots$ is the grand angular quantum number, n is any non-negative integer, and $l_{\xi_1}, l_{\xi_2}, \ldots$ are the angular momenta associated with relative Jacobi coordinates ξ_1 and ξ_2,.. It then follows that the hyperradial Schrödinger equation in the D dimensions is [20–24]

$$\left\{\frac{-1}{2m}\frac{1}{x^{D-1}}\left[\frac{\mathrm{d}}{\mathrm{d}x}x^{D-1}\frac{\mathrm{d}}{\mathrm{d}x}\right]+\frac{\Gamma^2(\Omega)}{x^2}+V(x)\right\}\psi(\mathrm{x})=\mathrm{E}\psi(\mathrm{x}) \tag{18}$$

In this formalism, the potential between the particles depends only on the relative distance between them (hyperradius); thus, it is called the hypercentral potential. Here, the dimension is related to the number of particles in the center of mass coordinate D = 3N – 3, where N is the number of particles.

We intend to solve this D-dimensional wave equation for investigating each nucleus of two interacting nuclei system. Our focus in this work is on the hyperradial part of the wave equation.

3.1 Finding the Wave Function

This section investigates a nucleus to obtain the radial wave function. We must solve the wave equation shown in (18). To do this, we must first identify the potential. When solving (18) with the specified potential for a nucleus, we assume that the potential is the internal interaction of the nucleus particles; thus, a short-range potential is required. Herein, we consider it to be the Yukawa potential as

$$V(\mathrm{x})=-v_o\frac{e^{-\alpha x}}{x} \tag{19}$$

The potential placed in the D-dimensional equation is dependent on the hyperradius. The dependence on hyperradius x means, in general, that the potential has an N-body character. It is dependent only on the relative distance of the particles and it is invariant for anyrotation in D-dimensional space. The nature of the potential is two-body, but it is dependent on the relative distance between the available particles. Herein, this potential is dependent on the relative distance between the N particles.

We are faced with two potentialsin this paper. The potential of this section is different from the potential used in Section 2. The M3Y interaction used in Section 2 is the NN interaction potential between two interacting nuclei, while the potential used in the wave equation is the imaginary potential for considering the internal interaction between the nucleons of each nucleus. We consider that this interaction is the Yukawa potential. In other words, we consider that the internal interaction of nucleons of each interacting nucleus is Yukawa type and the interaction of them with the nucleons of another interacting nucleus is M3Y type.

In this way, and according to the eigenvalues of the centrifugal barrier, (18) becomes

$$\left\{\frac{-1}{2m}\frac{1}{x^{D-1}}\left[\frac{\mathrm{d}}{\mathrm{d}x}x^{D-1}\frac{\mathrm{d}}{\mathrm{d}x}\right]-\frac{\gamma(\gamma+D-2)}{x^2}-v_o\frac{e^{-\alpha x}}{x}\right\}\psi(\mathrm{x})=\mathrm{E}\,\psi(\mathrm{x}) \tag{20}$$

To simplify the equation, we require the following approximation and change the variable [25–27]

$$\frac{1}{x}\approx 2\alpha\frac{e^{-\alpha x}}{(1-e^{-2\alpha x})}\quad y=1-e^{-2\alpha x} \tag{21}$$

Substituting these into (20) produces

$$y(1-y)\frac{\mathrm{d}^2\psi(\mathrm{y})}{\mathrm{d}y^2}-y\frac{\mathrm{d}\,\psi(\mathrm{y})}{\mathrm{d}y}-\left[\left(\frac{(D-1)(D-3)}{4}+\gamma(\gamma+D-2)\right)\frac{1}{y}-\frac{m(E_{n,l}-2)}{4\alpha^2}\frac{y}{1-y}-\frac{mv_0}{\alpha}\right]\psi(\mathrm{y})=0 \tag{22}$$

A solution for this wave equation [28, 29] is

$$\psi(\mathrm{y})=y^{\mu}(1-y)^{\nu}f(y) \tag{23}$$

Rewriting Eq. (22) with this function produces

$$y(1-y)f''(y)+[2\mu(1-y)-2\nu y-y]f'(y)+\left[\mu(\mu-1)\frac{(1-y)}{y}-2\mu\nu+\nu(\nu-1)\frac{y}{(1-y)}-\mu+\nu\frac{y}{(1-y)}-\left(\frac{(\mathrm{D}-1)(\mathrm{D}-3)}{4}+\gamma(\gamma+D-2)\right)\frac{1}{y}+\frac{m(E-2)}{4\alpha^2}\frac{y}{(1-y)}+\frac{mV_0}{\alpha}\right]f(y)=0 \tag{24}$$

The general form of the hypergeometric equation [29] is

$$y(1-y)f''(y)+[c-(a+b+1)y]f'(y)-abf(y)=0,\quad f(y)={}_2\mathrm{F}_1(a,b,c;y) \tag{25}$$

According to this equation, (24) is a hypergeometric equation with solution $f(y)={}_2\mathrm{F}_1(a,\,b,\,c;\,y)$. We have solved this hypergeometric equation; the coefficients are calculated as follows

$$a=\mu+\nu+\sqrt{\frac{-m(E-2)}{4\alpha^2}},\quad b=\mu+\nu-\sqrt{\frac{-m(E-2)}{4\alpha^2}},\quad c=2\mu \tag{26}$$

$$\mu=\frac{1}{2}\pm\frac{1}{2}\sqrt{1+(D-1)(D-3)+\frac{\gamma(\gamma+D-2)}{4}},\quad \nu=i\sqrt{\frac{m(E-2)}{4\alpha^2}}$$

Using these coefficients, the radial wave function of a nucleus with a Yukawa internal interaction (N is the normalization constant) is

$$\psi(\mathrm{x})=N(1-e^{-2\alpha x})^{\mu}e^{-2\alpha\nu x}\,{}_2\mathrm{F}_1(a,\,b,\,c;\,1-e^{-2\alpha x}) \tag{27}$$

The wave function of each nucleus with a Yukawa internal interaction can be achieved by changing the number of nucleus particles and, consequently, the dimensions of Hilbert space ($D=3N-3$) using the above relation.

In the present work, we considered the $^{16}\mathrm{O}+^{16}\mathrm{O}$ interaction. Using the wave function obtained for each nucleus, we numerically calculated the DFM integral. The nuclear, Coulomb, and total interaction potentials were

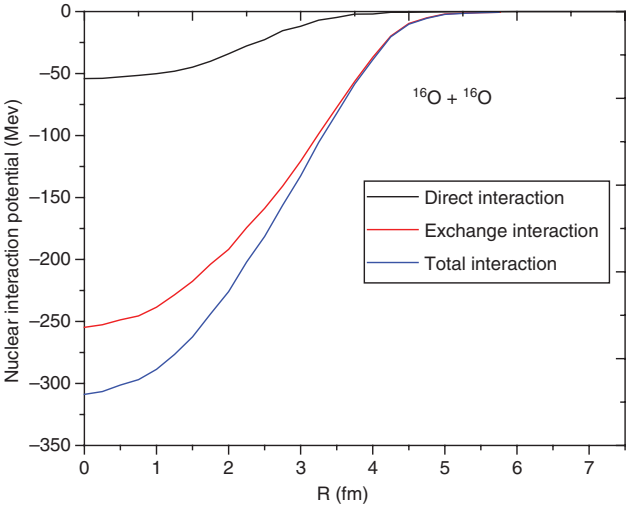

Figure 1: The direct, exchange, and total nuclear interaction potential for $^{16}\mathrm{O}+^{16}\mathrm{O}$ in terms of CDM3Y5-Reid density-dependent interaction using the DFM and D-dimensional calculations.

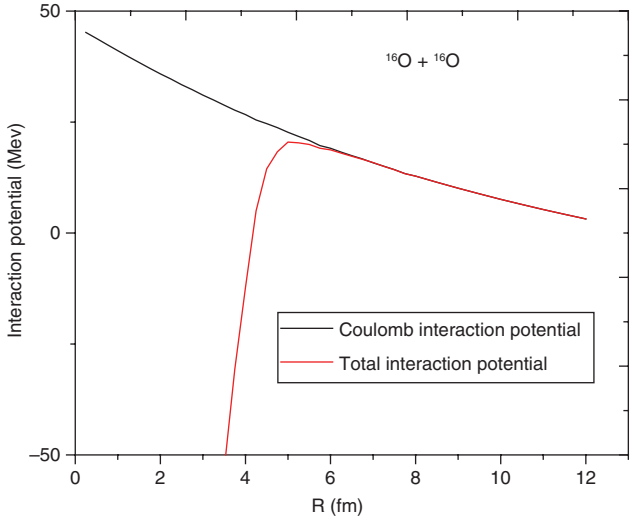

Figure 2: The Coulomb and total interaction potential for $^{16}O + ^{16}O$ obtained from DFM using the D-dimensional space calculations.

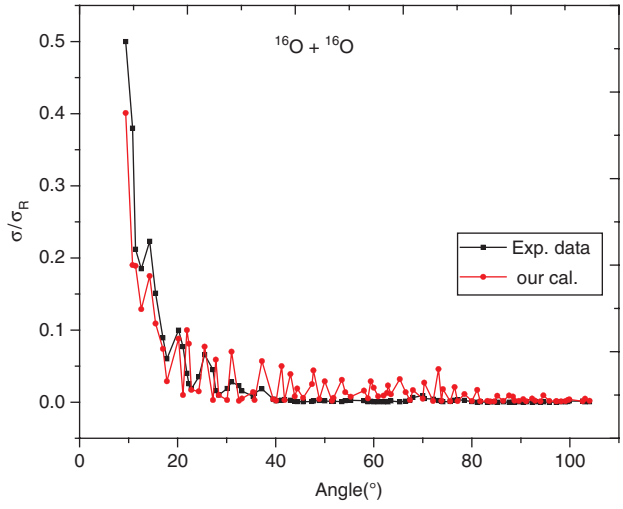

Figure 3: The obtained elastic $^{16}O + ^{16}O$ scattering data at 124 MeV compared with the experimental data [32].

obtained for $^{16}O + ^{16}O$, and the numerical results are shown in Figures 1 and 2.

On the other hand, we have investigated the scattering problem using the partial wave method and have calculated the elastic scattering cross section [30, 31]. The numerical results are shown in Figure 3.

4 Results and Discussion

The interaction potential between two colliding nuclei has been calculated using their wave functions. The goal was to evaluate the validity of substitution of the wave functions of two colliding nuclei for their density distributions in the DF integral. The wave functions were obtained by solving the D-dimensional wave equation in hyperspherical space. The $^{16}O + ^{16}O$ interaction was investigated, and the exchange part of the nuclear interaction was taken to be in the finite range. The density dependence of the NN interaction was accounted for. The numerical results for the nuclear, Coulomb, and total interaction potentials are presented in Figures 1 and 2. Figure 1 shows the difference between the direct and exchange parts of the nuclear interaction. It can be seen that the direct part has the larger values and tends to have a smaller slope toward zero. The slope is greater for the exchange part and the total nuclear interaction amounts tend toward zero slower than the exchange part amounts. Figure 2 shows the Coulomb interaction potential and the total interaction potential. It can be seen that the height of the Coulomb barrier in $R = 5$ fm is 20.4 Mev.

Also the elastic scattering cross section was studied for $E = 124$ Mev, and the numerical results are compared with the experimental data in Figure 3 [32].

5 Conclusions

The goal of this paper was to evaluate the usage of the wave functions of two interacting nuclei obtained from solving the D-dimensional wave equation using the hyperspherical formalism instead of their densities to calculate the interaction potential. According to Figures 1 and 2 of our previous work [33], we can see that usage of the different forms of the NN interaction (M3Y-Reid or Paris, zero-range or finite-range, density-dependent or independent) affects the results of our work. For example, it can be seen in that paper that the direct part of the interaction potential for the considered reaction has larger values and tends to have a smaller slope toward zero, while the slope is greater for the zero-range exchange part, and the finite-range amounts tend toward zero slower than the zero-range amounts [33]. All of these affect the height of the Coulomb barrier and so cross section values. Reid and Paris interactions produce very close results for the interaction potential and so have very similar results for the scattering cross section in our work that is almost indiscernible. Thus, in this work, we have used the finite-range CDM3Y5-Reid density-dependent interaction, and its results are closer to the experimental data according to the results of our previous work.

It is also well known that the elastic scattering cannot be described correctly by using a purely real potential [34]. This is probably why our work got a rather erratic

differential cross section in Figure 3. To consider this point, we must resolve the D-dimensional wave equation with the imaginary part of the potential, and the imaginary part of the wave function must be found. Then the DFM integral must be solved for this part with the obtained wave function. Because of the relation $|\psi\psi^*|$ in the DFM integral, the imaginary parameter (i) will be removed, and the interaction potential will be calculated with the real amount. Thus, the interaction potential will be the summation of the two parts.

The nuclear interaction potential does not vanish for large separation distances, say above 11 fm. It may be because of not removing some effects of DFM, like the contribution of the center of mass motion. Considering these points could possibly lead to more favorable results in our calculations. Investigation of these items will be the subject of our next work. According to the results of the current work, it seems that more favorable results may be obtained by applying some needed corrections.

References

[1] F. J. Dyson, Phys. Rev. **75,** 486 (1949).
[2] R. P. Feynman, Phys. Rev. **76,** 749 (1949).
[3] M. M. Giannini, E. Santopinto, and A. Vassallo, Nucl. Phys. A**699,** 308 (2002).
[4] M. M. Giannini, E. Santopinto, and A. Vassallo, Prog. Part. Nucl. Phys. **50,** 263 (2003).
[5] B. Sinha, Phys. Rep. C **20,** 1 (1975).
[6] G. R. Satchler and W. G. Love, Phys. Rep. **55,** 184 (1979).
[7] C. H. Dasso and G. Pollarolo, Phys. Rev. C **68,** 054604 (2003).
[8] I. I. Gontchar, D. J. Hinde, M. Dasgupta, and J. O. Newton, Phys. Rev. C **69,** 024610 (2004).
[9] I. I. Gontchar and M. V. Chushnyakova, Comput. Phys. Commun. **181,** 168 (2010).
[10] M. Aygun, Chinese J. Phys. **53,** 080301 (2015).
[11] D. T. Khoa, W. von Oertzen, and H. G. Bohlen, Phys. Rev. C **49,** 1652 (1994).
[12] J. W. Negele and D. Vautherin, Phys. Rev. C **5,** 1472 (1972).
[13] X. Campi and A. Bouyssy, Phys. Lett. B **73,** 263 (1978).
[14] D. T. Khoa, Phys. Rev. C **63,** 034007 (2001).
[15] M. Ismail and K. A. Ramadan, J. Phys. G **26,** 1621 (2000).
[16] G. Bertsch, J. Borysowicz, H. McManus, and W. G. Love, Nucl. Phys. A **284,** 399 (1977).
[17] N. Anantaraman, H. Toki, and G. F. Bertsch, Nucl. Phys. A **398,** 269 (1983).
[18] D. T. Khoa and W. von Oertzen, Phys. Lett. B **304,** 8 (1993).
[19] D. T. Khoa, G. R. Satchler, and W. von Oertzen, Phys. Rev. C **56,** 954 (1997).
[20] M. Fabre de la Ripelle, Ann. Phys. N.Y. **147,** 281 (1983).
[21] O. I. Tolstikhin, V. N. Ostrovsky, and H. Nakamura, Phys. Rev. Lett. **80,** 41 (1998).
[22] S. H. Dong, Wave Equations in Higher Dimensions, Springer, Netherlands 2011.
[23] J. L. Ballot and M. Fabre de la Ripelle, Ann. Phys. N.Y. **127,** 62 (1980).
[24] A. A. Rajabi, Few-Body Syst. **37,** 197 (2005).
[25] W. C. Qiang and S. H. Dong, Phys. Lett. A **368,** 13 (2007).
[26] S. H. Dong, W. C. Qiang, G. H. Sun, and V. B. Bezerra, J. Phys. A **40,** 10535 (2007).
[27] S. Dong, G. H. Sun, and S. H. Dong, Int. J. Mod. Phys. E **22,** 1350036 (2013).
[28] S. H. Dong, Factorization Method In Quantum Mechanics, Vol. 150, Springer, Netherlands 2007, p. 297.
[29] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Vol. 55, Courier Corporation, New York 1965, p. 1064.
[30] R. Navarro Pérez, J. E. Amaro, and E. Ruiz Arriola, Phys. Rev. C **88,** 024002 (2013).
[31] L. D. Landau and E. M. Lifshitz, Course of Theoretical Physics 1 Mechanics, 3rd ed, (Eds. J. B. Sykes and J. S. Bell), Pergamon press, Oxford, New York, Toronto, Sydney, Paris, Frankfurt 1976, XVII-169 p.
[32] M. P. Nicoli, F. Haas, R. M. Freeman, N. Aissaoui, C. Beck, A. Elanique, R. Nouicer, A. Morsad, S. Szilner, Z. Basrak, M. E. Brandan, and G. R. Satchler, Phys. Rev. C **60,** 064608 (1999).
[33] F. Pakdel and A. A. Rajabi, Chinese J. Phys. **54,** 385 (2016).
[34] P. E. Hodgson, Nuclear Heave-Ion Reactions, Oxford University Press, Clarendon and New York 1978.