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Decay Mode Solutions for the Supersymmetric Cylindrical KdV Equation

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Abstract: Supersymmetric cylindrical KdV equation is presented. Decay mode solutions for the supersymmetric KdV equation are derived by supersymmetric Hirota operator.

Keywords: Decay Mode Solutions; Superbilinear Form; Supersymmetric Cylindrical KdV Equation; Supersymmetric Hirota Operator.

1 Introduction

Supersymmetric integrable systems have been studied extensively during the past decade. Thus, a large number of well-known integrable equations have been extended into the supersymmetric context, such as KdV equation, KP hierarchy, and Boussinesq equation [1–3]. It has been shown that these supersymmetric integrable systems possess the Bäcklund transformation, the Hamiltonian formalism, Darboux transformation, bilinear form, and multi-soliton solutions [4–8]. The bilinear form of supersymmetric integrable was introduced by Carstea [8], it requires an extension of the Hirota bilinear operator [9, 10] to the supersymmetric systems. In recent years, Carstea, Liu, and Zhang have done a lot of work on the supersymmetric equations [8, 11–13]. However, to our knowledge, the supersymmetric equation for the variable coefficient KdV equation has been considered rarely so far.

The cylindrical KdV

$$u_t+6uu_x+u_{xxx}+\frac{u}{2t}=0 \tag{1}$$

was first studied by Maxon and Viecelli [14]. They solved (1) numerically and obtained that (1) has cylindrical soliton solutions which consists of a pulse moving inward rapidly at an ever increasing speed, leaving behind a flat wake that moves inward at sound speed. In 1980 [15], Nakamura showed analytical soliton solution by Hirota method that agrees with numerical and experiment results.

In this article, we consider the cylindrical KdV equation. We will show the supersymmetric cylindrical KdV equation and obtain the decay mode soliton solutions for it.

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2 Supersymmetrical KdV Equation

The supersymmetric extension of a nonlinear evolution equation refers to a system coupled equations for a bosonic $u(x, t)$ and a fermionic field $\eta(t, x)$, which reduces to the initial equation in the limit where the fermionic field is zero (the bosonic limit). We extend the classical space (x, t) to a large space (superspace) (t, x, θ), where θ is a Grassmann variable and also to extend the pair of fields (u, η) to a large fermionic or bosonic superfield $\phi(t, x, \theta)$. In order to have a nontrivial extension for the cylindrical KdV, we choose ϕ to be fermionic, having the expansion

$$\phi=\eta(t, x)+\theta u(x, t). \tag{2}$$

We consider the space supersymmetric invariance $x\to x-\epsilon\theta$, $\theta\to\theta+\epsilon$ (ϵ is an anticommuting parameter). Multiplying θ in (1), each term in the space supersymmetry is

$$\begin{aligned}&u_t\to\phi_t,\\&uu_x\to(\mathcal{D}\phi)\phi_x \quad\text{or}\quad (\mathcal{D}\phi_x)\phi,\\&u_{xxx}\to\phi_{xxx},\end{aligned} \tag{3}$$

where $\mathcal{D}=\frac{\partial}{\partial\theta}+\theta\frac{\partial}{\partial x}$ is the superderivative and

$$(\mathcal{D}\phi_x)\phi=u_x\eta+\theta uu_x-\theta\eta\eta_{xx}, \tag{4}$$

$$(D\phi)\phi_x=u\eta_x+\theta uu_x. \tag{5}$$

Reduce to the initial equation in the limit where the fermionic field is zero (the bosonic limit). Cylindrical KdV can be extended to the supersymmetric

$$\phi_t+\phi_{xxx}+3(\mathcal{D}\phi)(\phi_x)+3(\mathcal{D}\phi_x)\phi+\frac{\phi}{2t}=0. \tag{6}$$

3 Bilinear Form and Decay Mode Solutions

In order to derive the bilinear form for the cylindrical KdV, we consider the transformation

$$\phi=2\mathcal{D}(\ln f)_x=2\mathcal{D}^3(\ln f), \tag{7}$$

where $f(t, x, \theta)$ is bosonic. Equation (6) can be transformed into the following superbilinear form

$$Ff\cdot f=\left(SD_t+SD_x^3+\frac{\mathcal{D}}{2t}\right)f\cdot f=0, \tag{8}$$

here

$$SD_x^n D_t^m f\cdot g=(\mathcal{D}_{\theta_1}-\mathcal{D}_{\theta_2})\left(\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2}\right)^n\left(\frac{\partial}{\partial t_1}-\frac{\partial}{\partial t_2}\right)^m f(x_1, t_1, \theta_1)g(x_2, t_2, \theta_2)|_{x_1=x_2, t_1=t_2, \theta_1=\theta_2}. \tag{9}$$

We are going to derive the supersoliton solutions through the classical perturbative method. Expanding f into power series of a small parameter ϵ as

$$f=1+\epsilon f^{(1)}+\epsilon^2 f^{(2)}+\epsilon^3 f^{(3)}+\cdots. \tag{10}$$

Substituting (10) into (8) and equating coefficients

$$Ff^{(1)}\cdot 1=0, \tag{11}$$

$$2Ff^{(2)}\cdot 1=-Ff^{(1)}\cdot f^{(1)}, \tag{12}$$

$$Ff^{(3)}\cdot 1=-Ff^{(1)}\cdot f^{(2)}, \tag{13}$$

$$\ldots\ldots. \tag{14}$$

In order to obtain the solutions, we introduce the quantity a_{ij} defined by

$$a_{ij}=\rho_i\rho_j(12t)^{-1/3}[\omega(z_i)\omega'(z_j)-\omega'(z_i)\omega(z_j)]/(z_i-z_j), \tag{15}$$

where ρ_i, ρ_j are arbitrary constant parameters

$$z_k=(x-x_k)(12t)^{-1/3}+(\theta-\theta_k)(12t)^{-1/3},\ k=i, j \tag{16}$$

and $\omega(z_k)$, $k=i, j$ present Airy function Ai or Bi which are two linearly independent solutions of the ordinary differential equation

$$\omega''(z)-z\omega(z)=0.$$

To the definition of a_{ij} for the case $i=j$, we denote as follows

$$a_{ii}=\rho_i^2(12t)^{-1/3}[z_i\omega(z_i)\omega(z_i)-\omega'(z_i)\omega'(z_i)]. \tag{17}$$

By the direct calculation, we can find

$$a'_{il_i}=\rho_i\rho_{l_i}(12t)^{-\frac{2}{3}}\omega_i\omega_{l_i}, \tag{18}$$

$$a''_{il_i}=\rho_i\rho_{l_i}(12t)^{-1}(\omega'_i\omega_{l_i}+\omega_i\omega'_{l_i}). \tag{19}$$

By the direct calculation, we have found

$$Fa_{ii'}\cdot 1=-\frac{3}{(12t)^2}(x_i-x_{i'}+\theta_i-\theta_{i'})^2(1+\theta)a_{ii'}, \tag{20}$$

$$\begin{aligned}Fa_{ii'}\cdot a_{jj'}=(1+\theta)\Big\{&\frac{-3}{(12t)^2}[(x_i-x_{i'}+\theta_i-\theta_{i'})^2\\&+(x_j-x_{j'}+\theta_j-\theta_{j'})^2]a_{ii'}a_{jj'}\\&-\frac{6}{(12t)^2}(x_i-x_j+\theta_i-\theta_j)(x_{i'}-x_{j'}+\theta_{i'}-\theta_{j'})a_{ij}a_{i'j'}\\&+\frac{6}{(12t)^2}(x_i-x_{j'}+\theta_i-\theta_{j'})(x_j-x_{i'}+\theta_j-\theta_{i'})a_{ij'}a_{ji'}\Big\}.\end{aligned} \tag{21}$$

If taking

$$f^{(1)}=a_{11}=\rho_1^2(12t)^{-1/3}[z_1\omega_1(z_1)\omega(z_1)-\omega'(z_1)\omega'(z_1)],$$

we can obtain the one-decay solution. To get further than one solution, we must generalise closed relation (20) and (21). For the arbitrary functions a_1, a_2, b_1, and b_2, we can directly verify the following

$$\begin{aligned}Fa_1a_2\cdot b_1b_2=&(Fa_1\cdot b_1)a_2b_2+(Fa_1\cdot b_2)a_2b_1+(Fa_2\cdot b_1)a_1b_2\\&+(Fa_2\cdot b_2)a_1b_1-(Fa_1\cdot a_2)b_1b_2-(Fb_1\cdot b_2)a_1a_2+c(2, 2),\end{aligned} \tag{22}$$

where

$$\begin{aligned}c(2, 2)=&3\{(D_x^2a_1\cdot b_1)(SD_xa_2\cdot b_2)+(D_x^2a_2\cdot b_2)(SD_xa_1\cdot b_1)\\&+(D_x^2a_1\cdot b_2)(SD_xa_2\cdot b_1)+(D_x^2a_2\cdot b_1)(SD_xa_1\cdot b_2)\\&-(D_x^2a_1\cdot a_2)(SD_xb_1\cdot b_2)-(D_x^2b_1\cdot b_2)(SD_xa_1\cdot a_2)\}\\=&12\theta\{a''_1a''_2b_1b_2+a''_1(a_2b'_1b'_2-a'_2b'_1b_2-a'_2b_1b'_2)\\&+a''_2(a_1b'_1b'_2-a'_1b'_1b_2-a'_1b_1b'_2)+(a\leftrightarrow b, b\leftrightarrow a)\\&+2a'_1a'_2b'_1b'_2\}+\{6a''_1a'_{2,\theta}b_1b_2+6a'_{1,\theta}a''_2b_1b_2\\&+3a''_1a_2(b'_1b_{2,\theta}+b_{1,\theta}b'_2)+3a''_2a_1(b'_1b_{2,\theta}+b_{1,\theta}b'_2)\\&-3a''_1(a'_2b_{1,\theta}b_2+a_{2,\theta}b'_1b_2+a'_2b_{2,\theta}b_1+a_{2,\theta}b'_2b_1)\\&+6a'_{1,\theta}a_2b'_1b'_2+6a'_{2,\theta}a_1b'_1b'_2-3a''_2(a'_1b_{1,\theta}b_2+a_{1,\theta}b'_1b_2\\&+a'_1b_{2,\theta}b_1+a_{1,\theta}b'_2b_1)-6a'_{1,\theta}(a'_2b'_1b_2+a'_2b'_2b_1)\\&-6a'_{2,\theta}(a'_1b'_2b_2+a'_1b'_2b_1)+6a_{1,\theta}a'_2b'_1b'_2+6a_{2,\theta}a'_1b'_1b'_2\\&+(a\leftrightarrow b, b\leftrightarrow a)\}.\end{aligned} \tag{23}$$

Here, prime presents differentiation with respect to x. The relation can be extended to

$$F(a_1\cdots a_n)\cdot(b_1\cdots b_n)=\sum_{i,j=1}^{n}F(a_i\cdot b_j)\prod-\sum_{i,j=1,i\neq j}^{n}F(a_i\cdot a_j)\prod$$
$$-\sum_{i,j=1,i\neq j}^{n}F(b_i\cdot b_j)\prod+c(n,\ n), \tag{24}$$

where

$$F(a_i\cdot b_j)\prod=(Fa_i\cdot b_j)(a_ib_j)^{-1}\prod_{k=1}^{n}a_kb_k,$$
$$F(a_i\cdot a_j)\prod=(Fa_i\cdot a_j)(a_ia_j)^{-1}\prod_{k=1}^{n}a_kb_k,$$
$$F(b_i\cdot b_j)\prod=(Fb_i\cdot b_j)(b_ib_j)^{-1}\prod_{k=1}^{n}a_kb_k, \tag{25}$$

$$\begin{aligned}c(n,\ n)=12\theta\Big\{&\sum_{i<j}a_i''a_j''\prod+\sum_{i\neq j\neq k,j<k}a_i''a_j'a_k'\prod-\sum_{i\neq j}a_i''a_j'b_k'\prod\\&+\sum_{j<k}a_i''b_j'b_k'\prod+2\sum_{i<j<k<l}a_i'a_j'a_k'a_l'\prod-2\sum_{j<k<l}a_i'b_j'b_k'b_l'\prod\\&+(a\leftrightarrow b,\ b\leftrightarrow a)+2\sum_{i<j,k<l}a_i'a_j'b_k'b_l'\prod\Big\}\\&+\Big\{6\sum_{i\neq j}a_i''a'_{j,\theta}\prod+3\sum_{i\neq j\neq k}a_i''a_j'a_{k,\theta}\prod\\&-3\sum_{i\neq j}a_i''(a_j'b_{k,\theta}+a_{j,\theta}b_k')\prod+3\sum_{j\neq k}a_i''b_j'b_{k,\theta}\prod\\&+6\sum_{j<k<l}a_{i,\theta}a_j'a_k'a_l'\prod-6\sum_{j<k<l}a_{i,\theta}b_j'b_k'b_l'\prod\\&-6\sum_{k<l}a_i'b_{j,\theta}b_k'b_l'\prod+6\sum_{j<k}a'_{i,\theta}b_j'b_k'\prod-6\sum_{i\neq j,k}a'_{i,\theta}a_j'b_k'\prod\\&+6\sum_{i\neq j\neq k,j<k}a'_{i,\theta}a_j'a_k'\prod+6\sum_{i\neq j,k<l}a_{i,\theta}a_j'b_k'b_l'+(a\leftrightarrow b,\ b\leftrightarrow a)\Big\}.\end{aligned} \tag{26}$$

We can prove (24–26) by mathematical induction. We consider $(n+1)\times(n+1)$ in (24) product as $F(a_1\cdots a_{n+1})\cdot(b_1\cdots b_{n+1})=F(a_1\cdots a_n)a_{n+1}\cdot(b_1\cdots b_n)b_{n+1}$ and apply 2×2 formula given by (22) to the right-hand side of this. Using relation (24), we see that $F(a_1\cdots a_{n+1})\cdot(b_1\cdots b_{n+1})$ reduces to the form (24, 26) with n replaced by $n+1$.

Let

$$b_{m+1}=\cdots=b_n=1,\ (m\leq n), \tag{27}$$

and rewrite the suffix

$$a_i\to a_{il_i},\ a_j\to a_{jl_j},\ \cdots,\ b_i\to b_{is_i},\ b_j\to b_{js_j},\ \cdots \tag{28}$$

Then

$$\begin{aligned}&F\begin{vmatrix}a_{11}&\cdots&a_{1n}\\\vdots&\cdots&\vdots\\a_{n1}&\cdots&a_{nn}\end{vmatrix}\cdot\begin{vmatrix}b_{11}&\cdots&b_{1m}\\\vdots&\cdots&\vdots\\b_{m1}&\cdots&b_{mm}\end{vmatrix}\\&=F\left[\sum_{l_1l_2\cdots l_n}(-1)^{\tau(l_1,\cdots,l_n)}a_{1l_1}a_{2l_2}\cdots a_{nl_n}\right]\\&\cdot\left[\sum_{s_1s_2\cdots s_m}(-1)^{\tau(s_1,\cdots,s_n)}b_{1s_1}b_{2s_2}\cdots b_{ms_m}\right]\\&=\sum_{\substack{l_1,l_2\cdots l_n,\\s_1,s_2,\cdots s_m}}(-1)^{\tau(l_1,\cdots,l_n)}(-1)^{\tau(s_1,\cdots,s_m)}\\&\Big\{\sum_{i=1}^{n}\sum_{j=1}^{m}F(a_{il_i}\cdot b_{js_j})\prod+(n-m)\sum_{i=1}^{n}F(a_{il_i}\cdot 1)\prod\\&-\sum_{\substack{i,j=1,\\i<j}}^{n}F(a_{il_i}\cdot a_{jl_j})\prod-\sum_{\substack{i,j=1,\\i<j}}^{m}F(b_{is_i}\cdot b_{js_j})\prod\\&-(n-m)\sum_{j=1}^{m}F(1\cdot b_{js_j})\prod+c(n,\ m)\Big\},\end{aligned} \tag{29}$$

where $\sum_{l_1l_2\cdots l_n}$ and $\sum_{s_1s_2\cdots s_m}$ take sum over all possible permutation of $(l_1,\ l_2,\ \ldots,\ l_n)$ and $(s_1,\ s_2,\ \ldots,\ s_m)$. And $c(n,\ m)$ represents the same form as (26) in which replacement of (27) and (28) is made.

In (29), $c(n,\ m)$ only the term

$$\begin{aligned}&\sum_{\substack{(l_1\cdots l_n)\\(s_1\cdots s_m)}}(-1)^{\tau(l_1,\cdots,l_n)+\tau(s_1,\cdots,s_m)}\left[12\theta\left(\sum_{\substack{i,j=1\\i<j}}^{n}a''_{il_i}a''_{jl_j}+\sum_{\substack{i,j=1\\i<j}}^{m}b''_{is_i}b''_{js_j}\right)\right.\\&\left.+6\left(\sum_{\substack{i,j=1\\i\neq j}}^{n}a''_{il_i}a'_{jl_j,\theta}+\sum_{\substack{i,j=1\\i\neq j}}^{m}b''_{is_i}b'_{js_j,\theta}\right)\right],\end{aligned} \tag{30}$$

remain nonvanishing after taking sum over $n!$ and $m!$ permutations. This is seen as follows. Of total $n!$ or $m!$ permutations, there always exist equal number of even and odd permutations, thus we have the relation

$$\begin{aligned}&\sum_{\substack{(l_1\cdots l_n)\\(s_1\cdots s_m)}}\sum_{j<k}(a''_{il_i}b'_{jl_j}b'_{ks_k})(-1)^{\tau(l_1,\cdots,l_n)+\tau(s_1,\cdots,s_m)}\\&=\sum_{\substack{(l_1\cdots l_n)\\\text{all even permutation of}\\(s_1\cdots s_m)}}\sum_{j<k}[a''_{il_i}(b'_{js_j}b'_{ks_k}-b'_{js_k}b'_{ks_j})]\prod(-1)^{\tau(l_1,\cdots,l_n)+\tau(s_1\cdots s_n)}\\&=\sum_{\substack{(l_1\cdots l_n)\\\text{all even permutation of}\\(s_1\cdots s_m)}}[a''_{il_i}\rho_j\rho_{s_j}\rho_k\rho_{s_k}(12t)^{-\frac{4}{3}}\\&(\omega_j\omega_{s_j}\omega_k\omega_{s_k}-\omega_j\omega_{s_k}\omega_k\omega_{s_j})]\prod(-1)^{\tau(l_1,\cdots,l_n)+\tau(s_1\cdots s_n)}=0.\end{aligned} \tag{31}$$

In this way, all the terms in $c(n, m)$ except (30) and $\sum a''_{il_i}a'_{jl_j}b'_{ks_k}$ vanish. As to the remaining two types of terms, the latter also vanishes as

$$\sum_{\substack{(l_1\cdots l_n)\\(s_1\cdots s_m)}}\left\{\sum_{k=1}^{m}\sum_{\substack{i,j=1\\i\neq j}}^{n}(a''_{il_i}a'_{jl_j}b'_{ks_k})\right\}(-1)^{\tau(l_1,\cdots,l_n)+\tau(s_1,\cdots,s_m)}$$
$$=\sum_{\substack{(l_1\cdots l_n)\\\text{all even permutation of}\\(s_1\cdots s_m)}}\sum_{k=1}^{m}\sum_{\substack{i,j=1\\i\neq j}}^{n}[a''_{il_i}a'_{j_j}-a''_{il_j}a'_{jl_j}$$
$$+a''_{jl_j}a'_{il_i}-a''_{jl_i}a'_{il_j}]b'_{ks_k}\prod(-1)^{\tau(l_1,\cdots,l_n)+\tau(s_1\cdots s_n)}$$
$$=0. \tag{32}$$

Thus, in this way, all terms in $c(n, m)$ except (30) vanish. By the (20) and (24), (29) can be written to

$$F\begin{vmatrix}a_{11}&\cdots&a_{1n}\\\vdots&\cdots&\vdots\\a_{n1}&\cdots&a_{nn}\end{vmatrix}\cdot\begin{vmatrix}b_{11}&\cdots&b_{1m}\\\vdots&\cdots&\vdots\\b_{m1}&\cdots&b_{mm}\end{vmatrix}$$
$$=\sum_{\substack{l_1l_2\cdots l_n,\\s_1,s_2,\cdots s_m}}(-1)^{\tau(l_1,\cdots,l_n)+\tau(s_1,\cdots,s_m)}(12t)^{-2}(1+\theta)6$$
$$\left\{\sum_{\substack{i,j=1,\\i<j}}^{n}\left[2(x_i-x_{l_i}+\theta_i-\theta_{l_i})(x_j-x_{l_j}+\theta_j-\theta_{l_j})a_{il_i}a_{jl_j}\prod\right]\right.$$
$$+\sum_{\substack{i,j=1,\\i<j}}^{m}\left[2(x_i-x_{s_i}+\theta_i-\theta_{s_i})(x_j-x_{s_j}+\theta_j-\theta_{is_i})b_{is_i}b_{js_j}\prod\right]$$
$$-\sum_{i=1}^{n}\sum_{j=1}^{m}(x_i-x_j+\theta_i-\theta_j)(x_{l_i}-x_{s_j}+\theta_{l_i}-\theta_{s_j})a_{ij}b_{l_is_j}\prod$$
$$\left.+\sum_{i=1}^{n}\sum_{j=1}^{m}(x_i-x_{s_j}+\theta_i-\theta_{s_j})(x_j-x_{l_i}+\theta_j-\theta_{l_i})a_{is_j}b_{jl_i}\prod\right\}. \tag{33}$$

Especially

$$F\begin{vmatrix}a_{11}&\cdots&a_{1n}\\\vdots&\cdots&\vdots\\a_{n1}&\cdots&a_{nn}\end{vmatrix}\cdot 1=\sum_{l_1\cdots l_n}(-1)^{\tau(l_1\cdots l_n)}\frac{12}{(12t)^2}(1+\theta)$$
$$\left\{\sum_{\substack{i,j=1\\i<j}}^{n}(x_i-x_{l_i}+\theta_i-\theta_{l_i})(x_j-x_{l_j}+\theta_j-\theta_{l_j})a_{il_i}a_{jl_j}\prod\right\}. \tag{34}$$

Several lowest orders of these are written explicitly as

$$Fa_{11}\cdot 1=0, \tag{35}$$

$$F\begin{vmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{vmatrix}\cdot 1=\frac{12}{(12t)^{-2}}(1+\theta)(x_1-x_2+\theta_1-\theta_2)^2a_{12}^2, \tag{36}$$

$$Fa_{11}\cdot a_{22}=-\frac{12}{(12t)^2}(1+\theta)(x_1-x_2+\theta_1-\theta_2)^2a_{12}^2, \tag{37}$$

$$F\begin{vmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{vmatrix}\cdot 1=\frac{12(1+\theta)}{(12t)^2}$$
$$\{(x_1-x_2+\theta_1-\theta_2)^2a_{12}^2a_{33}+(x_1-x_3+\theta_1-\theta_3)^2a_{13}^2a_{22}$$
$$+(x_2-x_3+\theta_2-\theta_3)^2a_{23}^2a_{11}-2[(x_1-x_2+\theta_1-\theta_2)(x_1-x_3+\theta_1-\theta_3)$$
$$+(x_1-x_3+\theta_1-\theta_3)(x_2-x_3+\theta_2-\theta_3)$$
$$+(x_1-x_2+\theta_1-\theta_2)(x_3-x_2+\theta_3-\theta_2)]a_{12}a_{13}a_{23}\}, \tag{38}$$

$$F\begin{vmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{vmatrix}\cdot a_{33}=\frac{12(1+\theta)}{(12t)^2}\{(x_1-x_2+\theta_1-\theta_2)^2a_{12}^2a_{33}$$
$$-(x_1-x_3+\theta_1-\theta_3)^2a_{13}^2a_{22}-(x_2-x_3+\theta_2-\theta_3)^2a_{23}^2a_{11}$$
$$+2(x_1-x_3+\theta_1-\theta_3)(x_2-x_3+\theta_2-\theta_3)a_{12}a_{13}a_{23}\}, \tag{39}$$

$$F\begin{vmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{vmatrix}\cdot a_{11}=\frac{12(1+\theta)}{(12t)^2}(x_2-x_3+\theta_2-\theta_3)^2(a_{12}a_{13}-a_{11}a_{23})^2, \tag{40}$$

$$F\begin{vmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{vmatrix}\cdot\begin{vmatrix}a_{11}&a_{13}\\a_{31}&a_{33}\end{vmatrix}=\frac{-12(1+\theta)}{(12t)^2}(x_2-x_3+\theta_2-\theta_3)^2(a_{12}a_{13}-a_{11}a_{23})^2, \tag{41}$$

$$F\begin{vmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{vmatrix}\cdot\begin{vmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{vmatrix}=0, \tag{42}$$

$$F\begin{vmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{vmatrix}\cdot\begin{vmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{vmatrix}=0. \tag{43}$$

From (35–43), we can obtain the one-, two-, and three-decay mode solutions as following

$$f=1+a_{11},$$
$$f=1+(a_{11}+a_{22})+\begin{vmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{vmatrix}=\begin{vmatrix}1+a_{11}&a_{12}\\a_{21}&1+a_{22}\end{vmatrix},$$
$$f=1+(a_{11}+a_{22}+a_{33})+\left(\begin{vmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{vmatrix}+\begin{vmatrix}a_{22}&a_{23}\\a_{32}&a_{33}\end{vmatrix}+\begin{vmatrix}a_{33}&a_{31}\\a_{13}&a_{11}\end{vmatrix}\right)$$
$$+\begin{vmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{vmatrix}$$
$$=\begin{vmatrix}1+a_{11}&a_{12}&a_{13}\\a_{21}&1+a_{22}&a_{23}\\a_{31}&a_{32}&1+a_{33}\end{vmatrix}. \tag{44}$$

Generally, the N decay mode solution can be shown here

$$f = \begin{vmatrix} 1+a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & 1+a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & 1+a_{nn} \end{vmatrix}. \quad (45)$$

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