Lihua Wu* and Xianguo Geng

Quasi-periodic Solutions to the $K(-2, -2)$ Hierarchy

DOI 10.1515/zna-2016-0101
Received March 17, 2016; accepted April 27, 2016; previously published online May 26, 2016

Abstract: With the help of the characteristic polynomial of Lax matrix for the $K(-2, -2)$ hierarchy, we define a hyperelliptic curve $\mathcal{K}_{n+1}$ of arithmetic genus $n+1$. By introducing the Baker–Akhiezer function and meromorphic function, the $K(-2, -2)$ hierarchy is decomposed into Dubrovin-type differential equations. Based on the theory of hyperelliptic curve, the explicit Riemann theta function representation of meromorphic function is given, and from which the quasi-periodic solutions to the $K(-2, -2)$ hierarchy are obtained.

Keywords: Hyperelliptic Curve; Quasi-periodic Solutions; The $K(-2, -2)$ Hierarchy.

1 Introduction

It is well known that soliton equations have very wide applications in fields of fluid dynamics, plasma physics, optical fibers, biology, and many more. Quasi-periodic solutions of soliton equations are of great importance for it reveals inherent structure of solutions and describes quasi-periodic actions of nonlinear phenomenon, especially can be used to find multi-soliton solutions and elliptic function solutions, and similar ones to these. Since the first research on finite-gap solutions of Korteweg–de Vries equation around 1975, there has been numerous works devoted to constructing quasi-periodic solutions for soliton equations [1–25] and references therein.

In study of the role of nonlinear dispersion in the formation of patterns in liquid drops, Rosenau and Hyman [26] proposed in 1993 the nonlinear dispersive $K(m, n)$ equation

$$u_t + a(u^m)_x + (u^n)_{xxx} = 0,\ m,\ n > 1 \tag{1}$$

where a is a constant. It was found in [27–33] that nonlinear dispersion can compactify solitary waves and generate compactons, which are solitons with compact support. In addition, the compacton structure of $K(m, n)$ equation was discussed thoroughly. In addition to compatons, the $K(m, n)$ equation has also other kinds of solutions, such as kinks, peakons, and cuspons [29, 34–36]. Furthermore, several generalizations of the $K(m, n)$ equation have also been considered in the literature [37, 38].

In 1991, by introducing a nonconfocal generator of finite-dimensional integrable systems, Cao and Geng [39] obtained a new soliton hierarchy and corresponding Lax pairs. The first nontrivial member in the hierarchy is as follows

$$u_t + 4\epsilon(u^{-2})_x - (u^{-2})_{xxx} = 0, \quad \epsilon = \pm 1, \tag{2}$$

which was also found by Olver and Rosenau [27], and Qiao [40] later and can be denoted as $K(-2, -2)$. Subsequently, Geng [41] established the generalised Hamiltonian structure for the $K(-2, -2)$ hierarchy and decomposed it into finite-dimensional Liouville integrable system using the nonlinearization approach, from which its solutions were reduced to solving the compatible Hamiltonian systems of ordinary differential equations. Moreover, Sakovich [42] proposed a transformation, which relates the $K(-2, -2)$ equation with the modified Korteweg–de Vries equation. Based on this transformation, the N-soliton solutions of $K(-2, -2)$ equation were derived applying Darboux transformation [43]. In [44, 45], the authors gave two integrable extensions of $K(-2, -2)$ equation and studied their cuspon and kink wave solutions taking the bifurcation method of dynamical systems.

In this article, we construct the explicit Riemann theta function representations of solutions to the $K(-2, -2)$ hierarchy. Our article is organised as follows. In Section 2, we derive the hierarchy of the $K(-2, -2)$ equations based on the Lenard recursion equations and zero-curvature equation. In Section 3, we introduce the Baker–Akhiezer function and hyperelliptic curve $\mathcal{K}_{n+1}$ of arithmetic genus $n+1$. Then we deduce the associated meromorphic function and the Dubrovin-type differential equations. In Section 4, we present the explicit theta function representation of meromorphic function and, in particular, that of solutions for the entire $K(-2, -2)$ hierarchy.

***Corresponding author: Lihua Wu,** Department of Mathematics, Huaqiao University, 269 Chenghua North Road, Quanzhou, Fujian 362021, People's Republic of China, E-mail: wulihua@hqu.edu.cn
Xianguo Geng: School of Mathematics and Statistics, Zhengzhou University, 100 Kexue Road, ZhengZhou, Henan 450001, People's Republic of China, E-mail: xggeng@zzu.edu.cn

2 The $K(-2, -2)$ Hierarchy

In this section, we derive the $K(-2, -2)$ hierarchy associated with the spectral problem

$$\psi_x = U\psi,\quad \psi=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},\quad U=\begin{pmatrix}-1 & \lambda u\\ u & 1\end{pmatrix}, \tag{3}$$

where u is a potential and λ a constant spectral parameter. To this end, we introduce a set of Lenard recursion equations

$$Kg_{j-1}=Jg_j,\quad Jg_{-1}=0,\quad j\geq 0, \tag{4}$$

with starting points

$$g_{-1}=\frac{1}{u^2}\begin{pmatrix}1\\ 2u\end{pmatrix}, \tag{5}$$

and two operators are defined as

$$K=\begin{pmatrix}\partial^3-4\partial & 0\\ u\partial & -\partial\end{pmatrix},\quad J=\begin{pmatrix}0 & 4\partial u\\ u\partial & -\partial\end{pmatrix}.$$

Hence, g_j are uniquely determined by the recursive relation (4), which means to identify constants of integration as zero, for example, the second member reads as

$$g_0=\frac{1}{4u^6}\begin{pmatrix}5u_x^2-2uu_{xx}-3u^2\\ u(6u_x^2-2uu_{xx}-4u^2)\end{pmatrix},$$

In order to generate a hierarchy of nonlinear evolution equations associated with the spectral problem (3), we solve the stationary zero-curvature equation

$$V_x-[U, V]=0,\quad V=\begin{pmatrix}\lambda V_{11} & \lambda V_{12}\\ V_{21} & -\lambda V_{11}\end{pmatrix}, \tag{6}$$

which is equivalent to

$$\begin{aligned} &V_{11,x}+uV_{12}-uV_{21}=0,\\ &V_{12,x}+2V_{12}+2\lambda uV_{11}=0,\\ &V_{21,x}-2V_{21}-2\lambda uV_{11}=0, \end{aligned} \tag{7}$$

where each entry $V_{ij}=V_{ij}(a, b)$ is a Laurent expansion in λ:

$$V_{11}=4b,\quad V_{12}=a_{xx}-2a_x-4\lambda ub,\quad V_{21}=a_{xx}+2a_x-4\lambda ub. \tag{8}$$

A direct calculation shows that (7) and (8) imply the Lenard equations

$$\begin{aligned} &(\partial^3-4\partial)a-4\lambda\partial ub=0,\\ &u\partial a-\partial b=0. \end{aligned} \tag{9}$$

Substituting the expansions

$$a=\sum_{j\geq 0}a_{j-1}\lambda^{-j},\quad b=\sum_{j\geq 0}b_{j-1}\lambda^{-j} \tag{10}$$

into (9) and collecting terms with the same powers of λ, we arrive at the following recursion relation

$$KG_{j-1}=JG_j,\quad JG_{-1}=0, \tag{11}$$

where $G_j=(a_j, b_j)^T$. Noticing (4) and (5), then function G_j can be expressed as

$$G_j=\alpha_0 g_j+\cdots+\alpha_j g_0+\alpha_{j+1}g_{-1},\quad j\geq -1, \tag{12}$$

where α_j is arbitrary constants.

Let ψ satisfy the spectral problem (3) and an auxiliary problem

$$\psi_{t_m}=\tilde{V}^{(m)}\psi,\quad \tilde{V}^{(m)}=\begin{pmatrix}\lambda\tilde{V}_{11}^{(m)} & \lambda\tilde{V}_{12}^{(m)}\\ \tilde{V}_{21}^{(m)} & -\lambda\tilde{V}_{11}^{(m)}\end{pmatrix}, \tag{13}$$

with

$$\begin{aligned} &\tilde{V}_{11}^{(m)}=4\tilde{b}^{(m)},\quad \tilde{V}_{12}^{(m)}=\tilde{a}_{xx}^{(m)}-2\tilde{a}_x^{(m)}-4\lambda u\tilde{b}^{(m)},\\ &\tilde{V}_{21}^{(m)}=\tilde{a}_{xx}^{(m)}+2\tilde{a}_x^{(m)}-4\lambda u\tilde{b}^{(m)},\\ &\tilde{a}^{(m)}=\sum_{j=0}^{m}\tilde{a}_{j-1}\lambda^{m-j},\quad \tilde{b}^{(m)}=\sum_{j=0}^{m}\tilde{b}_{j-1}\lambda^{m-j}, \end{aligned} \tag{14}$$

and $\tilde{a}_j$, $\tilde{b}_j$ determined by

$$\tilde{G}_j=(\tilde{a}_j, \tilde{b}_j)^T=\tilde{\alpha}_0 g_j+\cdots+\tilde{\alpha}_j g_0+\tilde{\alpha}_{j+1}g_{-1},\quad j\geq -1.$$

The constants $\tilde{\alpha}_0, \ldots, \tilde{\alpha}_{j+1}$ are independently of $\alpha_0, \ldots, \alpha_{j+1}$. Then the compatibility condition of (3) and (13) yields the zero-curvature equation, $U_{t_m}-\tilde{V}_x^{(m)}+[U, \tilde{V}^{(m)}]=0$, which is equivalent to the hierarchy of nonlinear evolution equations

$$u_{t_m}=(\partial^3-4\partial)\tilde{a}_{m-1}=4\partial u\tilde{b}_m,\quad m\geq 0. \tag{15}$$

The first nontrivial member in the hierarchy (15) is

$$u_{t_0}+4\tilde{\alpha}_0(u^{-2})_x-\tilde{\alpha}_0(u^{-2})_{xxx}=0, \tag{16}$$

which is just the $K(-2, -2)$ equation (2) with $\epsilon=1$ as $\tilde{\alpha}_0=1$, $t_0=t$.

3 The Baker–Akhiezer Function and the Dubrovin-type Equations

In this section, we first introduce the Baker–Akhiezer function and Lax matrix for the $K(-2, -2)$ hierarchy, from which a hyperelliptic curve $\mathcal{K}_{n+1}$ and meromorphic function are defined. Then the hierarchy of $K(-2, -2)$ equations are decomposed into the system of solvable differential equations.

Now, we introduce the Baker–Akhiezer function $\psi(P, x, x_0, t_m, t_{0,m})$ by

$$\begin{aligned}&\psi_x(P, x, x_0, t_m, t_{0,m})=U(u(x, t_m);\\&\quad\lambda(P))\psi(P, x, x_0, t_m, t_{0,m}),\\&\psi_{t_m}(P, x, x_0, t_m, t_{0,m})=\tilde{V}^{(m)}(u(x, t_m);\\&\quad\lambda(P))\psi(P, x, x_0, t_m, t_{0,m}),\\&V^{(n)}(u(x, t_m); \lambda(P))\psi(P, x, x_0, t_m, t_{0,m})\\&\quad=y(P)\psi(P, x, x_0, t_m, t_{0,m}),\ x, t_m\in\mathbb{C},\end{aligned}\tag{17}$$

where

$$V^{(n)}=\begin{pmatrix}\lambda V_{11}^{(n)} & \lambda V_{12}^{(n)}\\ V_{21}^{(n)} & -\lambda V_{11}^{(n)}\end{pmatrix},\ V_{11}^{(n)}=4b^{(n)},\ V_{12}^{(n)}=a_{xx}^{(n)}-2a_x^{(n)}-4\lambda u b^{(n)},$$

$$V_{21}^{(n)}=a_{xx}^{(n)}+2a_x^{(n)}-4\lambda u b^{(n)},\ a^{(n)}=\sum_{j=0}^{n}a_{j-1}\lambda^{n-j},\ b^{(n)}=\sum_{j=0}^{n}b_{j-1}\lambda^{n-j}.$$

The compatibility conditions of the three expressions in (17) yield that

$$U_{t_m}-\tilde{V}_x^{(m)}+[U, \tilde{V}^{(m)}]=0,\tag{18}$$

$$-V_x^{(n)}+[U, V^{(n)}]=0,\tag{19}$$

$$-V_{t_m}^{(n)}+[\tilde{V}^{(m)}, V^{(n)}]=0.\tag{20}$$

A direct calculation shows that $yI-V^{(n)}$ satisfies the (19) and (20). Then the characteristic polynomial of the Lax matrix $V^{(n)}$, $\mathcal{F}_{2n+3}(\lambda, y)=\det(yI-V^{(n)})$, is an independent constant of the variables x and t_m with the expansion

$$\det(yI-V^{(n)})=y^2-\lambda R_{2n+2}(\lambda),$$

where $\lambda R_{2n+2}(\lambda)$ are polynomials with constant coefficients of λ, i.e.

$$\begin{aligned}\lambda R_{2n+2}(\lambda)&=\lambda[\lambda(V_{11}^{(n)})^2+V_{12}^{(n)}V_{21}^{(n)}]=\lambda[64\alpha_0^2\lambda^{2n+2}\\&\quad+128\alpha_0\alpha_1\lambda^{2n+1}+\cdots]\\&=\lambda\prod_{j=1}^{2n+2}(\lambda-\lambda_j).\end{aligned}\tag{21}$$

Hence, $\mathcal{F}_{2n+3}(\lambda, y)=0$ naturally leads to a hyperelliptic curve of degree $2n+3$

$$\mathcal{K}_{n+1}:\ \mathcal{F}_{2n+3}(\lambda, y)=y^2-\lambda R_{2n+2}(\lambda)=0.\tag{22}$$

For the convenience, we also denote the compactification of the hyperelliptic curve $\mathcal{K}_{n+1}$ by the same symbol $\mathcal{K}_{n+1}$. Assume that $\{\lambda_j\}_{j=1}^{2n+2}$ in (21) are mutually distinct and nonzero, then $\mathcal{K}_{n+1}$ becomes nonsingular.

Next we define the meromorphic function $\phi(P, x, t_m)$ on $\mathcal{K}_{n+1}$ as

$$\phi(P, x, t_m)=\frac{\psi_1(P, x, x_0, t_m, t_{0,m})}{\psi_2(P, x, x_0, t_m, t_{0,m})},\tag{23}$$

where $P=(\lambda, y)\in\mathcal{K}_{n+1}$, $x, x_0, t_m, t_{0,m}\in\mathbb{C}$. It infers from (23) and (17) that

$$\begin{aligned}\phi(P, x, t_m)&=\frac{\lambda V_{12}^{(n)}(\lambda, x, t_m)}{y(P)-\lambda V_{11}^{(n)}(\lambda, x, t_m)}\\&=\frac{y(P)+\lambda V_{11}^{(n)}(\lambda, x, t_m)}{V_{21}^{(n)}(\lambda, x, t_m)}.\end{aligned}\tag{24}$$

By observing (12) and (17), we know that $V_{12}^{(n)}$ and $V_{21}^{(n)}$ are polynomials with respect to λ of degree $n+1$, thereby they may be decomposed as

$$V_{12}^{(n)}(\lambda, x, t_m)=-8\alpha_0\prod_{j=1}^{n+1}(\lambda-\mu_j(x, t_m)),\tag{25}$$

$$V_{21}^{(n)}(\lambda, x, t_m)=-8\alpha_0\prod_{j=1}^{n+1}(\lambda-\nu_j(x, t_m)).\tag{26}$$

Defining

$$\begin{aligned}\hat{\mu}_j(x, t_m)&=(\mu_j(x, t_m), y(\hat{\mu}_j(x, t_m)))\\&=(\mu_j(x, t_m), \mu_j(x, t_m)V_{11}^{(n)}(\mu_j(x, t_m), x, t_m))\in\mathcal{K}_{n+1},\\\hat{\nu}_j(x, t_m)&=(\nu_j(x, t_m), y(\hat{\nu}_j(x, t_m)))\\&=(\nu_j(x, t_m), -\nu_j(x, t_m)V_{11}^{(n)}(\mu_j(x, t_m), x, t_m))\in\mathcal{K}_{n+1},\end{aligned}\tag{27}$$

and $P_0=(0, 0)$, then we have the following results.

Lemma 1. *Suppose that $\{\mu_j(x, t_m)\}_{j=1}^{n+1}$ and $\{\nu_j(x, t_m)\}_{j=1}^{n+1}$ remain distinct and nonzero for $(x, t_m)\in\Omega_\mu$ and $(x, t_m)\in\Omega_\nu$, respectively, where $\Omega_\mu, \Omega_\nu\subseteq\mathbb{C}^2$ are open and connected. Then they satisfy the system of Dubrovin-type differential equations*

$$\mu_{j,x}(x, t_m)=\frac{-u(x, t_m)y(\hat{\mu}_j(x, t_m))}{4\alpha_0\prod_{k=1, k\neq j}^{n+1}(\mu_j(x, t_m)-\mu_k(x, t_m))},\quad 1\le j\le n+1,\tag{28}$$

$$\mu_{j,t_m}(x, t_m)=\frac{-y(\hat{\mu}_j(x, t_m))\tilde{V}_{12}^{(m)}(\mu_j(x, t_m), x, t_m)}{4\alpha_0\prod_{k=1, k\neq j}^{n+1}(\mu_j(x, t_m)-\mu_k(x, t_m))},\quad 1\le j\le n+1,\tag{29}$$

$$\nu_{j,x}(x, t_m)=\frac{-u(x, t_m)y(\hat{\nu}_j(x, t_m))}{4\alpha_0\prod_{k=1, k\neq j}^{n+1}(\nu_j(x, t_m)-\nu_k(x, t_m))},\quad 1\le j\le n+1,\tag{30}$$

$$\nu_{j,t_m}(x,t_m)=\frac{-y(\hat{\nu}_j(x,t_m))\tilde{V}_{21}^{(m)}(\nu_j(x,t_m),x,t_m)}{4\alpha_0\prod_{\substack{k=1\\k\neq j}}^{n+1}(\nu_j(x,t_m)-\nu_k(x,t_m))},\quad 1\leq j\leq n+1. \tag{31}$$

Proof. Equations (19) and (20) imply that

$$\begin{aligned}
V_{12,x}^{(n)}(\lambda,x,t_m)|_{\lambda=\mu_j(x,t_m)}&=-2u(x,t_m)\mu_j(x,t_m)V_{11}^{(n)}(\mu_j(x,t_m),x,t_m),\\
V_{12,t_m}^{(n)}(\lambda,x,t_m)|_{\lambda=\mu_j(x,t_m)}&=-2\mu_j(x,t_m)\tilde{V}_{12}^{(m)}(\mu_j(x,t_m),x,t_m)\\
&\quad V_{11}^{(n)}(\mu_j(x,t_m),x,t_m).
\end{aligned} \tag{32}$$

On the other hand, differentiating (25) respect to x and t_m gives rise to

$$\begin{aligned}
V_{12,x}^{(n)}(\lambda,x,t_m)|_{\lambda=\mu_j(x,t_m)}&=8\alpha_0\mu_{j,x}(x,t_m)\prod_{\substack{k=1\\k\neq j}}^{n+1}(\mu_j(x,t_m)-\mu_k(x,t_m)),\\
V_{12,t_m}^{(n)}(\lambda,x,t_m)|_{\lambda=\mu_j(x,t_m)}&=8\alpha_0\mu_{j,t_m}(x,t_m)\prod_{\substack{k=1\\k\neq j}}^{n+1}(\mu_j(x,t_m)-\mu_k(x,t_m)).
\end{aligned} \tag{33}$$

Comparing (32) and (33), we obtain (28) and (29). Similarly, (30) and (31) can be proved. □

4 Quasi-periodic Solutions to the $K(-2,-2)$ Hierarchy

In this section, we derive explicit Riemann theta function representations for the meromorphic function $\phi(P,x,t_m)$, and in particular, that of potential u, for the entire $K(-2,-2)$ hierarchy.

With the aid of (3) and (23), we arrive at that the meromorphic function $\phi(P,x,t_m)$ satisfies the Riccati equation

$$\phi_x(P,x,t_m)+u(x,t_m)\phi^2(P,x,t_m)+2\phi(P,x,t_m)=\lambda u(x,t_m). \tag{34}$$

To investigate the property of $\phi(P,x,t_m)$ near $P_\infty\in\mathcal{K}_{n+1}$, we take the local coordinate $\zeta=\lambda^{-\frac{1}{2}}$, and obtain the Laurent series

$$\phi(P,x,t_m)\underset{\zeta\to 0}{=}\sum_{j=-1}^{\infty}\kappa_j(x,t_m)\zeta^j,\quad P\to P_\infty, \tag{35}$$

with

$$\begin{aligned}
&\kappa_{-1}=1,\quad \kappa_0=-\frac{1}{u},\quad \kappa_1=\frac{u-u_x}{2u^3},\\
&\kappa_{j+1}=-\frac{1}{2u}\left[\kappa_{j,x}+2\kappa_j+u\sum_{i=0}^{j}\kappa_i\kappa_{j-i}\right],\quad (j\geq -1).
\end{aligned}$$

Immediately, one obtains from (24) and (35) that the divisor $(\phi(P,x,t_m))$ of $\phi(P,x,t_m)$ is given by

$$(\phi(P,x,t_m))=\mathcal{D}_{P_0,\hat{\mu}_1(x,t_m),\ldots,\hat{\mu}_{n+1}(x,t_m)}(P)-\mathcal{D}_{P_\infty,\hat{\nu}_1(x,t_m),\ldots,\hat{\nu}_{n+1}(x,t_m)}(P), \tag{36}$$

which means $P_0,\hat{\mu}_1(x,t_m),\ldots,\hat{\mu}_{n+1}(x,t_m)$ are the $n+2$ zeros of $\phi(P,x,t_m)$ and $P_\infty,\hat{\nu}_1(x,t_m),\ldots,\hat{\nu}_{n+1}(x,t_m)$ its $n+2$ poles. In addition, direct computation gives the asymptotic property of $y(P)$ near P_∞

$$y(P)\underset{\zeta\to 0}{=}-8\alpha_0\zeta^{-2n-3}\left[1+\frac{\alpha_1}{\alpha_0}\zeta^2+O(\zeta^4)\right],\quad P\to P_\infty. \tag{37}$$

Equip the Riemann surface $\mathcal{K}_{n+1}$ with homology basis $\{\mathfrak{a}_j,\mathfrak{b}_j\}_{j=1}^{n+1}$, which are independent and have intersection numbers as follows

$$\mathfrak{a}_j\circ\mathfrak{b}_k=\delta_{j,k},\quad \mathfrak{a}_j\circ\mathfrak{a}_k=0,\quad \mathfrak{b}_j\circ\mathfrak{b}_k=0,\quad j,k=1,\ldots,n+1.$$

On $\mathcal{K}_{n+1}$, we introduce $n+1$ linearly independent holomorphic differentials

$$\varpi_l(P)=\frac{\lambda^{l-1}\mathrm{d}\lambda}{y(P)},\quad l=1,2,\ldots,n+1, \tag{38}$$

from which the period matrices A and B can be constructed from

$$A_{jk}=\int_{\mathfrak{a}_k}\varpi_j,\quad B_{jk}=\int_{\mathfrak{b}_k}\varpi_j. \tag{39}$$

It is possible to prove that the matrices A and B are invertible. Defining the matrix C and τ by $C=A^{-1}$, $\tau=A^{-1}B$, one can show that matrix τ is symmetric ($\tau_{jk}=\tau_{kj}$) and has a positive-definite imaginary part ($\mathrm{Im}\,\tau>0$) [46, 47]. Let us now normalize $\varpi_l(P)$ into new basis ω_j by

$$\omega_j=\sum_{l=1}^{n+1}C_{jl}\varpi_l,\quad j=1,\ldots,n+1, \tag{40}$$

which satisfy $\int_{\mathfrak{a}_k}\omega_j=\delta_{jk},\int_{\mathfrak{b}_k}\omega_j=\tau_{jk},\ j,k=1,\ldots,n+1.$ A straightforward Laurent expansion of (40) near P_∞ yields that

$$\begin{aligned}
&\omega=(\omega_1,\cdots,\omega_j,\cdots,\omega_{n+1}),\\
&\omega_j\underset{\zeta\to 0}{=}\frac{1}{4\alpha_0}(C_{j,n+1}+O(\zeta^2))\mathrm{d}\zeta,\quad j=1,\ldots,n+1.
\end{aligned} \tag{41}$$

Let $\omega_{P_\infty,2}^{(2)}(P)$ denote the normalised Abelian differential of the second kind defined by

$$\omega_{P_\infty,2}^{(2)}(P)=\frac{4\alpha_0}{y(P)}\prod_{j=1}^{n+1}(\lambda-\gamma_j)\mathrm{d}\lambda, \tag{42}$$

which is holomorphic on $\mathcal{K}_{n+1}\backslash\{P_\infty\}$ with a pole of order 2 at P_∞, and the constants $\{\gamma_j\}_{j=1}^{n+1}$ are determined by the normalization conditions

$$\int_{\mathfrak{a}_j}\omega^{(2)}_{P_\infty,2}(P)=0,\quad j=1,\ldots,n+1. \tag{43}$$

The b-periods of the differential $\omega^{(2)}_{P_\infty,2}$ are denoted by

$$U_2^{(2)}=(U^{(2)}_{2,1},\ldots,U^{(2)}_{2,n+1}),\quad U^{(2)}_{2,j}=\frac{1}{2\pi i}\int_{\mathfrak{b}_j}\omega^{(2)}_{P_\infty,2}(P)=\frac{C_{j,n+1}}{4\alpha_0},\quad j=1,\ldots,n+1. \tag{44}$$

Furthermore, let $\omega^{(3)}_{P_\infty,P_0}(P)$ denote the normalised Abelian differential of the third kind defined by

$$\omega^{(3)}_{P_\infty,P_0}(P)=-\frac{d\lambda}{2\lambda}+\frac{\eta_{n+1}}{y(P)}\prod_{j=1}^{n}(\lambda-\eta_j)d\lambda, \tag{45}$$

which is holomorphic on $\mathcal{K}_{n+1}\backslash\{P_\infty,P_0\}$ with simple poles at P_∞ and P_0 with residues ±1, respectively, and the constants $\{\eta_j\}_{j=1}^{n+1}$ are determined by the normalization conditions

$$\int_{\mathfrak{a}_j}\omega^{(3)}_{P_\infty,P_0}(P)=0,\quad j=1,\ldots,n+1. \tag{46}$$

A direct calculation shows that

$$\omega^{(3)}_{P_\infty,P_0}(P)\underset{\zeta\to0}{=}\begin{cases}(\zeta^{-1}+\frac{\eta_{n+1}}{4\alpha_0}+O(\zeta^2))d\zeta, & P\to P_\infty,\\ (-\zeta^{-1}+O(1))d\zeta, & P\to P_0,\end{cases} \tag{47}$$

which infers that

$$\int_{Q_0}^{P}\omega^{(3)}_{P_\infty,P_0}(P)=\begin{cases}\ln\zeta+e^{(3)}_\infty(Q_0)+\frac{\eta_{n+1}}{4\alpha_0}\zeta+O(\zeta^3), & P\to P_\infty,\\ -\ln\zeta+e^{(3)}_0(Q_0)+O(\zeta), & P\to P_0,\end{cases} \tag{48}$$

with Q_0 a chosen base point on $\mathcal{K}_{n+1}\backslash\{P_\infty,P_0\}$ and $e^{(3)}_\infty(Q_0)$, $e^{(3)}_0(Q_0)$ two integration constants.

Let $\mathcal{T}_{n+1}$ be the period lattice $\{\underline{\varrho}\in\mathbb{C}^{n+1}|\underline{\varrho}=\underline{N}+\tau\underline{L},\ \underline{N},\underline{L}\in\mathbb{Z}^{n+1}\}$. The complex torus $\mathcal{J}_{n+1}=\mathbb{C}^{n+1}/\mathcal{T}_{n+1}$ is called the Jacobian variety of $\mathcal{K}_{n+1}$. An Abel map $\mathcal{A}:\mathcal{K}_{n+1}\to\mathcal{J}_{n+1}$ is defined by

$$\mathcal{A}(P)=(\mathcal{A}_1(P),\cdots,\mathcal{A}_{n+1}(P))=\left(\int_{Q_0}^{P}\omega_1,\cdots,\int_{Q_0}^{P}\omega_{n+1}\right),\ (\mathrm{mod}\,\mathcal{T}_{n+1})$$

with the natural linear extension to the factor group $\mathrm{Div}(\mathcal{K}_{n+1})$

$$\mathcal{A}(\sum n_kP_k)=\sum n_k\mathcal{A}(P_k).$$

Considering the nonspecial divisor $\mathcal{D}_{\underline{\hat{\mu}}(x,t_m)}=\sum_{k=1}^{n+1}\hat{\mu}_k(x,t_m)$ and $\mathcal{D}_{\underline{\hat{\nu}}(x,t_m)}=\sum_{k=1}^{n+1}\hat{\nu}_k(x,t_m)$, we define

$$\mathcal{A}(\sum_{k=1}^{n+1}P_k^{(i)}(x,t_m))=\sum_{k=1}^{m-1}\mathcal{A}(P_k^{(i)}(x,t_m))=\sum_{k=1}^{n+1}\int_{Q_0}^{P_k^{(i)}(x,t_m)}\omega=\rho^{(i)}(x,t_m),\quad i=1,2, \tag{49}$$

whose components are $\sum_{k=1}^{n+1}\int_{Q_0}^{P_k^{(i)}(x,t_m)}\omega_j=\rho_j^{(i)}(x,t_m)$, $i=1,2,\ j=1,\ldots,n+1$, where $\underline{\hat{\mu}}(x,t_m)=(\hat{\mu}_1(x,t_m),\cdots,\hat{\mu}_{n+1}(x,t_m))$, $\underline{\hat{\nu}}(x,t_m)=(\hat{\nu}_1(x,t_m),\cdots,\hat{\nu}_{n+1}(x,t_m))$, $P_k^{(1)}(x,t_m)=\hat{\mu}_k(x,t_m)$, and $P_k^{(2)}(x,t_m)=\hat{\nu}_k(x,t_m)$.

The Riemann theta function [46, 47] associated with $\mathcal{K}_{n+1}$ is defined as

$$\theta(\underline{z}(P,\underline{Q}))=\theta(M-\mathcal{A}(P)+\mathcal{A}(\mathcal{D}_{\underline{Q}})),\quad P\in\mathcal{K}_{n+1}, \tag{50}$$

where $\underline{Q}=(Q_1,\ldots,Q_{n+1})$ and $M=(M_1,\ldots,M_{n+1})$ is Riemann constant vector. Then

$$\theta(\underline{z}(P,\underline{\hat{\mu}}(x,t_m)))=\theta(M^{(1)}-\mathcal{A}(P)+\rho^{(1)}(x,t_m)),\quad P\in\mathcal{K}_{n+1},$$
$$\theta(\underline{z}(P,\underline{\hat{\nu}}(x,t_m)))=\theta(M^{(2)}-\mathcal{A}(P)+\rho^{(2)}(x,t_m)),\quad P\in\mathcal{K}_{n+1}. \tag{51}$$

Hence, the theta function representation of $\phi(P,x,t_m)$ reads as follows.

Theorem 1. *Assume the curve $\mathcal{K}_{n+1}$ to be nonsingular and let $P=(\lambda,y)\in\mathcal{K}_{n+1}\backslash(\{P_\infty\},(x,t_m),(x_0,t_{0,m})\in\Omega_\mu$, where $\Omega_\mu\subseteq\mathbb{C}^2$ is open and connected. Suppose also that $\mathcal{D}_{\underline{\hat{\mu}}(x,t_m)}$, or equivalently, $D_{\underline{\hat{\nu}}(x,t_m)}$ is nonspecial for $(x,t_m)\in\Omega_\mu$. Then*

$$\phi(P,x,t_m)=\frac{\theta(\underline{z}(P,\underline{\hat{\mu}}(x,t_m)))\theta(\underline{z}(P_\infty,\underline{\hat{\nu}}(x,t_m)))}{\theta(\underline{z}(P_\infty,\underline{\hat{\mu}}(x,t_m)))\theta(\underline{z}(P,\underline{\hat{\nu}}(x,t_m)))}\exp\left(e^{(3)}_\infty(Q_0)-\int_{Q_0}^{P}\omega^{(3)}_{P_\infty,P_0}\right). \tag{52}$$

Proof. It is easily seen from (36) that ϕ has simple zeros at P_0 and $\underline{\hat{\mu}}(x,t_m)$, and simple poles at P_∞ and $\underline{\hat{\nu}}(x,t_m)$. Let Φ denote the right-hand side of (52). It follows from (48) that

$$\exp\left(e^{(3)}_\infty(Q_0)-\int_{Q_0}^{P}\omega^{(3)}_{P_\infty,P_0}\right)\underset{\zeta\to0}{=}\zeta^{-1}+O(1),\quad P\to P_\infty,$$
$$\exp\left(e^{(3)}_\infty(Q_0)-\int_{Q_0}^{P}\omega^{(3)}_{P_\infty,P_0}\right)\underset{\zeta\to0}{=}(\zeta+O(\zeta^2))\exp(e^{(3)}_\infty(Q_0)-e^{(3)}_0(Q_0)),\quad P\to P_0. \tag{53}$$

Applying the Riemann's vanishing theorem and the Riemann-Roch theorem, one concludes that Φ and ϕ share same simple poles and zeros, which leads to $\frac{\Phi}{\phi}=\gamma$, with γ a constant. On the other hand, (53) and (35) show

$$\frac{\Phi}{\phi} \underset{\zeta\to 0}{=} \frac{(1+O(\zeta))(\zeta^{-1}+O(1))}{(\zeta^{-1}+O(1))} \underset{\zeta\to 0}{=} 1+O(\zeta), \quad P\to P_\infty, \tag{54}$$

which implies $\gamma=1$ and completes the proof. □

With the above-mentioned results, we obtain the theta function representations of solutions for the entire $K(-2,-2)$ hierarchy.

Theorem 2. *Assume the curve $\mathcal{K}_{n+1}$ to be nonsingular and let $(x,t_m)\in\Omega_\mu$, where $\Omega_\mu\subseteq\mathbb{C}^2$ is open and connected. Suppose also that $\mathcal{D}_{\underline{\hat{\mu}}(x,t_m)}$, or equivalently, $\mathcal{D}_{\underline{\hat{\nu}}(x,t_m)}$ is nonspecial for $(x,t_m)\in\Omega_\mu$. Then the $K(-2,-2)$ hierarchy* (15) *admits quasi-periodic solutions*

$$u(x,t_m)=\frac{4\alpha_0}{\eta_{n+1}-4\alpha_0\partial_{U_2^{(2)}}\ln\dfrac{\theta(\underline{z}(P_\infty,\underline{\hat{\nu}}(x,t_m)))}{\theta(\underline{z}(P_\infty,\underline{\hat{\mu}}(x,t_m)))}}, \tag{55}$$

with η_{n+1} defined in (46).

Proof. From (41) and (51), we get

$$\begin{aligned}\underline{z}(P,\underline{\hat{\mu}}(x,t_m))&=M^{(1)}-\int_{Q_0}^{P}\omega+\sum_{k=1}^{n+1}\int_{Q_0}^{\hat{\mu}_k(x,t_m)}\omega\\&=\left(\ldots,M_j^{(1)}-\int_{Q_0}^{P}\omega_j+\sum_{k=1}^{n+1}\int_{Q_0}^{\hat{\mu}_k(x,t_m)}\omega_j,\ \ldots\right)\\&=\left(\ldots,M_j^{(1)}-\int_{Q_0}^{P_\infty}\omega_j+\sum_{k=1}^{n+1}\int_{Q_0}^{\hat{\mu}_k(x,t_m)}\omega_j-\int_{P_\infty}^{P}\omega_j,\ \ldots\right)\\&\underset{\zeta\to 0}{=}\left(\ldots,M_j^{(1)}-\int_{Q_0}^{P_\infty}\omega_j+\sum_{k=1}^{n+1}\int_{Q_0}^{\hat{\mu}_k(x,t_m)}\omega_j-U_{2,j}^{(2)}\zeta+O(\zeta^3),\ \ldots\right),\\&\quad P\to P_\infty\end{aligned}\tag{56}$$

then

$$\begin{aligned}&\frac{\theta(\underline{z}(P,\underline{\hat{\mu}}(x,t_m)))}{\theta(\underline{z}(P_\infty,\underline{\hat{\mu}}(x,t_m)))}\\&\underset{\zeta\to 0}{=}\frac{\theta\left(\ldots,M_j^{(1)}-\int_{Q_0}^{P_\infty}\omega_j+\sum_{k=1}^{n+1}\int_{Q_0}^{\hat{\mu}_k(x,t_m)}\omega_j-U_{2,j}^{(2)}\zeta+O(\zeta^3),\ \ldots\right)}{\theta(\underline{z}(P_\infty,\underline{\hat{\mu}}(x,t_m)))}\\&\underset{\zeta\to 0}{=}\frac{\theta\left(\underline{z}(P_\infty,\underline{\hat{\mu}}(x,t_m)))-\left[\sum_{j=1}^{n+1}U_{2,j}^{(2)}\frac{\partial}{\partial z_j}\theta(M^{(1)}-\mathcal{A}(P_\infty)+\rho^{(1)}(x,t_m)-U_2^{(2)}\zeta+O(\zeta^3))\right]\Big|_{\zeta=0}\zeta+O(\zeta^2\right)}{\theta(\underline{z}(P_\infty,\underline{\hat{\mu}}(x,t_m)))}\\&\underset{\zeta\to 0}{=}1-\frac{\partial_{U_2^{(2)}}\theta(\underline{z}(P_\infty,\underline{\hat{\mu}}(x,t_m)))}{\theta(\underline{z}(P_\infty,\underline{\hat{\mu}}(x,t_m)))}\zeta+O(\zeta^2)\\&\underset{\zeta\to 0}{=}1-[\partial_{U_2^{(2)}}\ln\theta(\underline{z}(P_\infty,\underline{\hat{\mu}}(x,t_m)))]\zeta+O(\zeta^2),\quad P\to P_\infty.\end{aligned}\tag{57}$$

where $\partial_{U_2^{(2)}}\theta(\underline{z}(P_\infty,\underline{\hat{\mu}}(x,t_m)))=[\partial_{U_2^{(2)}}\theta(M^{(1)}-\mathcal{A}(P_\infty)+\rho^{(1)}(x,t_m)-U_2^{(2)}\zeta+O(\zeta^3))]|_{\zeta=0}$, and $\partial_{U_2^{(2)}}=\sum_{j=1}^{n+1}U_{2,j}^{(2)}\frac{\partial}{\partial z_j}$. Similarly, we have

$$\frac{\theta(\underline{z}(P,\underline{\hat{\nu}}(x,t_m)))}{\theta(\underline{z}(P_\infty,\underline{\hat{\nu}}(x,t_m)))}\underset{\zeta\to 0}{=}1-[\partial_{U_2^{(2)}}\ln\theta(\underline{z}(P_\infty,\underline{\hat{\nu}}(x,t_m)))]\zeta+O(\zeta^2),\quad P\to P_\infty. \tag{58}$$

where $\partial_{U_2^{(2)}}\theta(\underline{z}(P_\infty,\underline{\hat{\nu}}(x,t_m)))=[\partial_{U_2^{(2)}}\theta(M^{(2)}-\mathcal{A}(P_\infty)+\rho^{(2)}(x,t_m)-U_2^{(2)}\zeta+O(\zeta^3))]|_{\zeta=0}$. With the help of Theoerm 1 and (48), we get

$$\begin{aligned}\phi(P,x,t_m)&\underset{\zeta\to 0}{=}\{1-[\partial_{U_2^{(2)}}\ln\theta(\underline{z}(P_\infty,\underline{\hat{\mu}}(x,t_m)))]\zeta+O(\zeta^2)\}\\&\quad\{1+[\partial_{U_2^{(2)}}\ln\theta(\underline{z}(P_\infty,\underline{\hat{\nu}}(x,t_m)))]\zeta\\&\quad+O(\zeta^2)\}\times\left[\zeta^{-1}-\frac{\eta_{n+1}}{4\alpha_0}+O(\zeta)\right],\\&\underset{\zeta\to 0}{=}\{1+\left[\partial_{U_2^{(2)}}\ln\frac{\theta(\underline{z}(P_\infty,\underline{\hat{\nu}}(x,t_m)))}{\theta(\underline{z}(P_\infty,\underline{\hat{\mu}}(x,t_m)))}\right]\zeta\\&\quad+O(\zeta^2)\}\left[\zeta^{-1}-\frac{\eta_{n+1}}{4\alpha_0}+O(\zeta)\right]\\&\underset{\zeta\to 0}{=}\zeta^{-1}+\partial_{U_2^{(2)}}\ln\frac{\theta(\underline{z}(P_\infty,\underline{\hat{\nu}}(x,t_m)))}{\theta(\underline{z}(P_\infty,\underline{\hat{\mu}}(x,t_m)))}-\frac{\eta_{n+1}}{4\alpha_0}\\&\quad+O(\zeta),\quad P\to P_\infty,\end{aligned}\tag{59}$$

which together with (35) shows (55). □

Acknowledgments: This work was supported by the National Natural Science Foundation of China (project nos. 11331008, 11326163, and 11401230), Cultivation Program for Outstanding Young Scientific talents of the Higher Education Institutions of Fujian Province in 2015, and Promotion Program for Young and Middle-aged Teacher in Science and Technology Research of Huaqiao University (project no. ZQN-PY301).

References

[1] S. P. Novikov, Funct. Anal. Appl. **8**, 236 (1974).
[2] I. M. Krichever, Dokl. Akad. Nauk SSSR **227**, 394 (1976).
[3] I. M. Krichever, Funct. Anal. Appl. **11**, 12 (1977).
[4] Y. C. Ma and M. J. Ablowitz, Stud. Appl. Math. **65**, 113 (1981).
[5] B. A. Dubrovin, Funct. Anal. Appl. **9**, 215 (1975).
[6] B. A. Dubrovin, Russ. Math. Surv. **36**, 11 (1981).
[7] P. D. Miller, N. M. Ercolani, I. M. Krichever, and C. D. Levermore, Commun. Pur. Appl. Math. **48**, 1369 (1995).
[8] V. B. Matveev and A. O. Smirnov, Lett. Math. Phys. **14**, 25 (1987).
[9] V. B. Matveev and A. O. Smirnov, T. Am. Math. Soc. **157**, 227 (1993).
[10] A. O. Smirnov, Theor. Math. Phys. **66**, 19 (1986).
[11] A. O. Smirnov, Math. Notes **58**, 735 (1995).
[12] E. Previato, Duke Math. J. **52**, 329 (1985).
[13] M. S. Alber and Y. N. Fedorov, Inverse Prob. **17**, 1017 (2001).
[14] C. W. Cao, Y. T. Wu, and X. G. Geng, J. Math. Phys. **40**, 3948 (1999).
[15] X. G. Geng and C. W. Cao, Nonlinearity **14**, 1433 (2001).
[16] X. G. Geng, Y. T. Wu, and C. W. Cao, J. Phys. A: Math. Gen. **32**, 3733 (1999).
[17] X. G. Geng, H. H. Dai, and C. W. Cao, J. Math. Phys. **44**, 4573 (2003).
[18] X. G. Geng, J. Phys. A: Math. Gen. **36**, 2289 (2003).
[19] R. G. Zhou and W. X. Ma, IL Nuovo Cimento B **115**, 1419 (2000).
[20] Z. J. Qiao, Rev. Math. Phys. **21**, 545 (2001).
[21] B. Xue, F. Li, and X. G. Geng, J. Nonlinear Math. Phy. **20**, 61 (2013).
[22] X. G. Geng, L. H. Wu, and G. L. He, Physica D **240**, 1262 (2011).
[23] F. Gesztesy and H. Holden, Soliton Equations and Their Algebro-Geometric Solutions, Cambridge University Press, Cambridge 2003.
[24] F. Gesztesy and H. Holden, Rev. Mat. Iberoamericana **19**, 73 (2003).
[25] R. Dickson, F. Gesztesy, and K. Unterkofler, Rev. Math. Phys. **11**, 823 (1999).
[26] P. Rosenau and J. M. Hyman, Phys. Rev. Lett. **70**, 564 (1993).
[27] P. J. Olver and P. Rosenau, Phys. Rev. E **53**, 1900 (1996).
[28] P. Rosenau, Phys. Rev. Lett. **73**, 1737 (1994).
[29] P. Rosenau, Phys. Lett. A **211**, 265 (1997).
[30] P. Rosenau, Physica D **230**, 535 (1998).
[31] M. S. Ismail and T. R. Taha, Math. Comput. Simul. **47**, 519 (1998).
[32] M. S. Ismail, Int. J. Comput. Math. **74**, 185 (2000).
[33] A. M. Wazwaz, Chaos Soliton. Fract. **13**, 321 (2002).
[34] A. M. Wazwaz, Chaos Soliton. Fract. **13**, 161 (2002).
[35] C. H. Xu and L. X. Tian, Chaos Soliton. Fract. **40**, 893 (2009).
[36] H. Bin and M. Qing, Appl. Math. Comput. **217**, 1697 (2010).
[37] A. Biswas, Phys. Lett. A **372**, 4601 (2008).
[38] A. Biswas, Comput. Math. Appl. **59**, 2536 (2010).
[39] C. W. Cao and X. G. Geng, J. Math. Phys. **32**, 2323 (1991).
[40] Z. J. Qiao, J. Math. Phys. **48**, 082701 (2007).
[41] X. G. Geng, Chaos Soliton. Fract. **16**, 749 (2003).
[42] S. Sakovich, J. Math. Phys. **52**, 023509 (2010).
[43] N. Zhaqilaoa, Chinese Phys. B **4**, 040201 (2013).
[44] C. H. Pan, L. M. Ling, and Z. R. Liu, Phys. Scr. **89**, 105207 (2014).
[45] C. H. Pan and L. M. Ling, Phys. Scr. **90**, 025208 (2015).
[46] P. Griffiths and J. Harris, Principles of Algebraic Geometry, Wiley, New York 1994.
[47] D. Mumford, Tata Lectures on Theta II, Birkhäuser, Boston 1984.