Eduard Marušić-Paloka, Igor Pažanin* and Marko Radulović

Flow of a Micropolar Fluid Through a Channel with Small Boundary Perturbation

DOI 10.1515/zna-2016-0085
Received March 3, 2016; accepted April 12, 2016; previously published online May 13, 2016

Abstract: The aim of this paper is to investigate the effects of small boundary perturbations on the flow of an incompressible micropolar fluid. The fluid domain is described as follows: we start from a simple rectangular domain and then perturb part of its boundary by the product of a small parameter ϵ and some smooth function h. Using formal asymptotic analysis with respect to ϵ, we derive the effective model in the form of the explicit formulae for the velocity, pressure and microrotation. The asymptotic solution clearly acknowledges the effects of the boundary perturbation and the micropolar nature of the fluid. The obtained results are illustrated by some numerical examples confirming that the considered perturbation has a nonlocal impact on the solution.

Keywords: Asymptotic Analysis; Boundary Perturbation; Micropolar Fluid; Nonlocal Effects.

1 Introduction

It is well known that a limited number of fluid flow problems can be solved (approximately) by the solutions in the explicit form. To accomplish that, one usually has to start from very simplified mathematical models and to consider idealised geometries of the flow domains. However, in real-life situations, the boundary of the fluid domain can have some complicated shape containing small rugosities, dents or some other irregularities. Such problems are very difficult to be handled analytically and, in most situations, are only amenable to numerical simulations. Introducing a small parameter as the perturbation quantity in the domain boundary makes analytical treatments very complicated because of the tedious change of variable that needs to be performed. In view of that, not many analytical results on the subject can be found in the engineering literature. In particular, the perturbation of the boundary has remained a rather neglected mathematical topic as well (see monograph [1] and the references therein).

The purpose of this paper is to analyse the effects of small boundary perturbations on the micropolar fluid flow. In case of numerous real fluids (e.g. polymeric suspensions, liquid crystals, muddy fluids, blood, even water in models with small scales), fluid particles can exhibit some microscopical effects such as rotation and shrinking. Such phenomena cannot be captured by the classical Navier–Stokes equations, and, thus, other (non-Newtonian) models need to be employed. One of the best-established models describing the local structure and micro-motions of the fluid elements in such a situation is the micropolar fluid model proposed by Eringen [2] in the 1960s. Physically, micropolar fluids consist of rigid, spherical particles suspended in a viscous medium where the deformation of the particles is ignored. The individual particles may rotate (independently of the movement of the fluid), and, thus, a new vector field, the angular velocity field of rotation of particles (called microrotation), is introduced. Consequently, one new vector equation is added to the Navier–Stokes system resulting from the conservation of the angular momentum. As a result, a non-Newtonian model is obtained in the form of a complex (coupled) system of partial differential equations (PDEs) satisfied by the fluid velocity, pressure and microrotation. We refer to [3] providing a detailed derivation of the micropolar fluid model from the general conservation laws. Micropolar fluid flow has been extensively studied due to its practical importance in various applications (see, e.g. [4–7] for some recent results).

In this work, we consider an incompressible micropolar fluid flowing through a two-dimensional domain:

$$\Omega^{\epsilon}=\{(x,\ y)\in\mathbf{R}^2:0<x<1,\ 0<y<1-\epsilon h(x)\}. \tag{1}$$

As you can see, a small perturbation of magnitude $\epsilon\ll 1$ is applied on the upper part of the boundary of the fluid domain (see Fig. 1). The function h is assumed to be smooth enough and given in advance. The classical approach in the asymptotic analysis would be to introduce a suitable

***Corresponding author: Igor Pažanin,** Faculty of Science, Department of Mathematics, University of Zagreb, Bijenička 30, 10000, Zagreb, Croatia, Tel.: +385-14605887, E-mail: pazanin@math.hr
Eduard Marušić-Paloka and Marko Radulović: Faculty of Science, Department of Mathematics, University of Zagreb, Croatia

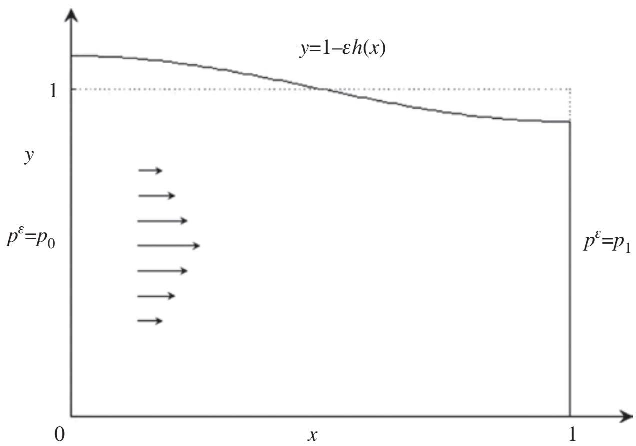

Figure 1: The domain.

change of variable and to study the governing problem in the ϵ-independent domain $\Omega=(0, 1)^2$. However, such procedure is more complicated for practical computations as it requires changing the variables (before asymptotic analysis) and bringing back the original variables after the analysis. Thus, we employ a different (and more elegant) approach[1] by expanding the unknowns in the Taylor series with respect to y (near the upper boundary) and using the asymptotic expansion technique (in powers of ϵ). Although the boundary-value problems deduced for the expansion terms in such a manner are still complicated (due to the strong coupling of the micropolar equations), we manage to solve them analytically and derive explicit expressions for the first two terms in the asymptotic expansion. The first-order approximation does not feel the effects of the boundary perturbation and that was to be expected. The effects we seek for can be clearly observed from the derived correctors suggesting that the small perturbation of the boundary affects the solution not only in some boundary layer near the upper boundary, but in the whole domain. This represents our main contribution confirmed also by providing some numerical examples in the last section.

To conclude the Introduction, we provide some bibliographic remarks. As indicated above, one cannot find too many theoretical results on the boundary perturbations in the existing literature. In the context of the elasticity, useful applications of the asymptotic methods can be found in [9–11]. In fluid mechanics, boundary perturbations are mostly investigated in the context of periodically corrugated boundaries, see, for example, [12–18]. Most recently, the first author of this work in [8] considered the flow of a classical Newtonian fluid in the domain given by (1). Starting from a simple Stokes system, the asymptotic approximation has been built clearly showing the effects of the boundary perturbation. The goal of this paper is to extend the analysis from [8] to a more general (non-Newtonian) micropolar case described by the coupled system of PDEs [see (2)–(4)]. We choose to work in 2D setting in order to be able to clearly detect the effects of boundary distortion on the asymptotic solution (derived in the explicit form). Although 2D setting (in which the microrotation is a scalar function) is, of course, only the first step to more realistic 3D situation, it has often been employed, for example, in blood motion modelling. Thus, we believe that the result presented here could be instrumental for improving the known engineering practice.

1 Following exactly the same arguments as presented in [8] for a classical Newtonian case, it can be proved that both procedures are, at the end, equivalent and lead to the same result.

2 Micropolar Equations

A micropolar fluid flow is described by the (coupled) system of PDEs expressing the balance of momentum, mass and angular momentum which in stationary regime reads (see [3])

$$-(\nu+\nu_r)\Delta\mathbf{u}^\epsilon+\nabla p^\epsilon=2\nu_r \text{ Rot } w^\epsilon, \tag{2}$$

$$\text{div }\mathbf{u}^\epsilon=0, \quad \text{in } \Omega^\epsilon, \tag{3}$$

$$-(c_a+c_d)\Delta w^\epsilon+4\nu_r w^\epsilon=2\nu_r \text{ rot } \mathbf{u}^\epsilon. \tag{4}$$

The unknowns in the above system are $\mathbf{u}^\epsilon=(u_x^\epsilon, u_y^\epsilon)$, w^ϵ and p^ϵ, which stand for the velocity, the microrotation and the pressure of the fluid, respectively. Note that we assume a small Reynolds number and omit the inertial terms in momentum equations. Here and in the following we use the notation

$$\text{Rot } w^\epsilon=\left(-\frac{\partial w^\epsilon}{\partial y}, \frac{\partial w^\epsilon}{\partial x}\right), \quad \text{rot } u^\epsilon=\frac{\partial u_x^\epsilon}{\partial y}-\frac{\partial u_y^\epsilon}{\partial x}.$$

In the micropolar equations, four new viscosities are introduced resulting from the non-symmetric properties of the stress tensor. Along with the usual Newtonian viscosity ν, we have ν_r as the microrotation viscosity, and c_0, c_d as the coefficients of angular viscosities. All the coefficients are assumed to be positive constants. Observe that if we put $\nu_r=0$, the system becomes decoupled, and (2)–(3) reduces to a classical Stokes system.

The system (2)–(4) needs to be endowed with the appropriate boundary conditions. We use standard no-slip boundary conditions for the velocity on the lower and upper boundary, while on the lateral boundary of the channel we prescribe constant pressures p_0, p_1 ($p_1<p_0$). This means that flow is governed by the prescribed pressure drop $\delta_p=p_1-p_0$. For the microrotation we employ a commonly used zero boundary condition. In view of that, the system (2)–(4) is completed by

$$\mathbf{u}^\epsilon=\mathbf{0},\quad \text{for}\quad y=0,\ y=1-\epsilon h, \tag{5}$$

$$\mathbf{u}^\epsilon\times\mathbf{e}_1=\mathbf{0},\quad \text{for}\quad x=0,\ 1, \tag{6}$$

$$p^\epsilon=p_k,\quad \text{for}\quad x=k\in\{0,\ 1\}, \tag{7}$$

$$w^\epsilon=0,\quad \text{on}\quad \partial\Omega^\epsilon. \tag{8}$$

The aim of this paper is to investigate the asymptotic behaviour of the flow governed by (2)–(8) with respect to the small parameter ϵ. We will have to deal with two major challenges in the process. The first one is, of course, concerned with the perturbation of the upper boundary. As explained in the Introduction, we will address this issue directly (using the Taylor series approach and the expansion technique), that is, without introducing the change of variable $z=\dfrac{y}{1-\epsilon h}$. As a consequence, no additional terms will appear in the governing equations. Nevertheless, the second difficulty (arising from the micropolar nature of the fluid) persists, since the system (2)–(4) remains to be strongly coupled even at the main order term.

3 Analysis

In the following, we assume that $h<0$, implying $\Omega=\{(x,\ y)\in\mathbf{R}^2{:}0<x<1,\ 0<y<1\}\subset\Omega^\epsilon$. Consequently, the solution ($\mathbf{u}^\epsilon$, p_ϵ, w_ϵ) of the governing problem is defined on Ω and we are in a position to directly expand velocity and microrotation in the Taylor series with respect to y around $y=1$ (near the upper boundary). Note that the assumption $h<0$ is taken for the sake of notational simplicity, because, otherwise, we would have to extend the solution to Ω polluting the notation. We formally expand

$$\mathbf{u}^\epsilon(x,\ y)=\sum_{k=0}^{\infty}\frac{1}{k!}\frac{\partial^k\mathbf{u}^\epsilon}{\partial y^k}(x,\ 1)(y-1)^k, \tag{9}$$

$$w^\epsilon(x,\ y)=\sum_{k=0}^{\infty}\frac{1}{k!}\frac{\partial^k w^\varepsilon}{\partial y^k}(x,\ 1)(y-1)^k. \tag{10}$$

From the boundary conditions (5) and (8), we deduce

$$\mathbf{0}=\mathbf{u}^\epsilon(x,\ 1-\epsilon h)=\mathbf{u}^\epsilon(x,\ 1)-\epsilon\frac{\partial\mathbf{u}^\epsilon}{\partial y}(x,\ 1)h+\frac{\epsilon^2}{2}\frac{\partial^2\mathbf{u}^\epsilon}{\partial y^2}(x,\ 1)h^2-\cdots, \tag{11}$$

$$0=w^\epsilon(x,\ 1-\epsilon h)=w^\epsilon(x,\ 1)-\epsilon\frac{\partial w^\epsilon}{\partial y}(x,\ 1)h+\frac{\epsilon^2}{2}\frac{\partial^2 w^\epsilon}{\partial y^2}(x,\ 1)h^2-\cdots. \tag{12}$$

On the other hand, we expand the solution in the asymptotic expansion in powers of a small parameter ϵ as follows:

$$\mathbf{u}^\epsilon(x,\ y)=\mathbf{V}^0(x,\ y)+\epsilon\mathbf{V}^1(x,\ y)+\epsilon^2\mathbf{V}^2(x,\ y)+\cdots, \tag{13}$$

$$w^\epsilon(x,\ y)=W^0(x,\ y)+\epsilon W^1(x,\ y)+\epsilon^2 W^2(x,\ y)+\cdots, \tag{14}$$

$$p^\epsilon(x,\ y)=Q^0(x,\ y)+\epsilon Q^1(x,\ y)+\epsilon^2 Q^2(x,\ y)+\cdots. \tag{15}$$

Plugging the expansions (13)–(14) into (11)–(12), we have

$$\begin{aligned}\mathbf{0}=&\mathbf{V}^0(x,\ 1)+\epsilon\Big(\mathbf{V}^1(x,\ 1)-h\frac{\partial\mathbf{V}^0}{\partial y}(x,\ 1)\Big)\\&+\epsilon^2\Big(\mathbf{V}^2(x,\ 1)-h\frac{\partial\mathbf{V}^1}{\partial y}(x,\ 1)+\frac{h^2}{2}\frac{\partial^2\mathbf{V}^0}{\partial y^2}(x,\ 1)\Big)+\cdots,\end{aligned} \tag{16}$$

$$\begin{aligned}0=&W^0(x,\ 1)+\epsilon\Big(W^1(x,\ 1)-h\frac{\partial W^0}{\partial y}(x,\ 1)\Big)\\&+\epsilon^2\Big(W^2(x,\ 1)-h\frac{\partial W^1}{\partial y}(x,\ 1)+\frac{h^2}{2}\frac{\partial^2 W^0}{\partial y^2}(x,\ 1)\Big)+\cdots,\end{aligned} \tag{17}$$

implying

$$\mathbf{V}^0(x,\ 1)=\mathbf{0},\quad W^0(x,\ 1)=0, \tag{18}$$

$$\mathbf{V}^1(x,\ 1)=h\frac{\partial\mathbf{V}^0}{\partial y}(x,\ 1),\quad W^1(x,\ 1)=h\frac{\partial W^0}{\partial y}(x,\ 1). \tag{19}$$

Since $\mathbf{V}^0(x,\ 1)=\mathbf{0}$, it follows that $\dfrac{\partial V_x^0}{\partial x}(x,\ 1)=0$, so $\dfrac{\partial V_y^0}{\partial y}(x,\ 1)=0$, due to the divergence condition (3). Now, from (19), we conclude

$$V_y^1(x,\ 1)=0,\quad V_x^1(x,\ 1)=h\frac{\partial V_x^0}{\partial y}(x,\ 1). \tag{20}$$

This completes the derivation of the boundary conditions (at $y=1$) satisfied by the first two terms from the asymptotic expansions (13) and (14).

3.1 First-Order Approximation

Substituting the expansions (13)–(15) into (2)–(4) and taking into account the boundary conditions, we arrive at

$$-(\nu+\nu_r)\Delta\mathbf{V}^0+\nabla Q^0=2\nu_r \text{ Rot } W^0, \quad \text{in } \Omega, \tag{21}$$

$$\text{div } \mathbf{V}^0=0, \quad \text{in } \Omega, \tag{22}$$

$$-(c_a+c_d)\Delta W^0+4\nu_r W^0=2\nu_r \text{ rot } V^0, \quad \text{in } \Omega, \tag{23}$$

$$\mathbf{V}^0(x,0)=\mathbf{0},\ \mathbf{V}^0(x,1)=\mathbf{0},\ W^0(x,0)=0,\ W^0(x,1)=0, \tag{24}$$

$$V_y^0(0,y)=0, \quad V_y^0(1,y)=0, \tag{25}$$

$$Q^0(0,y)=p_0, \quad Q^0(1,y)=p_1. \tag{26}$$

Observe that the problem satisfied by our asymptotic approximation remains strongly coupled even at the main order, as oppose to the thin-domain framework (without boundary perturbation) leading to a standard solution of the Poiseuille problem for a micropolar fluid (see [16, 18–20]). Nevertheless, the solution of the problem (21)–(26) can be computed explicitly, following the idea from [21] (see also the approach from [22] dealing with 3D setting). We assume that the solution has the form

$$\mathbf{V}^0=\mathbf{V}^0(y)=V_x^0(y)\mathbf{e}_1, \quad Q^0=Q^0(x), \quad W^0=W^0(y), \tag{27}$$

with $\dfrac{\partial Q^0}{\partial x}$ being constant. From the pressure boundary conditions (26) we immediately deduce

$$Q^0(x)=p_0+\delta p\cdot x, \quad \delta p=p_1-p_0. \tag{28}$$

Integrating (21) with respect to y, we obtain

$$(\nu+\nu_r)\frac{\partial V_x^0}{\partial y}=\delta p\cdot y+2\nu_r W^0+K_1, \tag{29}$$

where K_1 is an unknown constant. Putting (29) into (23) we get

$$-(c_a+c_d)\frac{\partial^2 W^0}{\partial y^2}+\frac{4\nu_r\nu}{\nu+\nu_r}W^0=\frac{2\nu_r}{\nu+\nu_r}\delta p y+\frac{2\nu_r}{\nu+\nu_r}K_1, \tag{30}$$

implying

$$W^0(y)=C_1e^{My}+C_2e^{-My}+\frac{\delta p}{2\nu}y+\frac{K_1}{2\nu}, \quad M=2\sqrt{\frac{\nu_r\nu}{(c_a+c_d)(\nu+\nu_r)}}. \tag{31}$$

In view of $(24)_3$, $(24)_4$, we compute constants C_1, C_2 as

$$C_1=\frac{-\delta p-K_1+K_1e^{-M}}{2\nu(e^M-e^{-M})}, \quad C_2=\frac{\delta p+K_1-K_1e^M}{2\nu(e^M-e^{-M})}. \tag{32}$$

Substituting (31) into (29), we arrive at

$$\frac{\partial V_x^0}{\partial y}=\frac{\delta p}{\nu}y+\frac{2\nu_r}{\nu+\nu_r}(C_1e^{My}+C_2e^{-My})+\frac{K_1}{\nu}. \tag{33}$$

Integrating (33) with respect to y yields

$$V_x^0=\frac{2\nu_r}{(\nu+\nu_r)M}(A\sinh(My)+B\cosh(My))+\frac{\delta p}{\nu}\frac{y^2}{2}+\frac{K_1}{\nu}y+K_2, \tag{34}$$

where we put $A=C_1+C_2$, $B=C_1-C_2$. In view of that, W^0 from (31) can be rewritten in terms of the hyperbolic functions as well:

$$W^0=A\sinh(My)+B\cosh(My)+\frac{\delta p}{2\nu}y+\frac{K_1}{2\nu},$$
$$A=-\frac{K_1}{2\nu}, \quad B=\frac{-\delta p-K_1+K_1\cosh(M)}{2\nu\sinh(M)}. \tag{35}$$

Finally, using velocity boundary conditions $(24)_1$, $(24)_2$ and the expressions (32), from (34) we determine the constants K_1 and K_2 as

$$K_1=\frac{\delta p(2\nu_r(1-\cosh(M))+(\nu+\nu_r)M\sinh(M))}{4\nu_r\cosh(M)-2M(\nu+\nu_r)\sinh(M)-4\nu_r}, \tag{36}$$

$$K_2=\frac{\nu_r(\delta p+K_1-K_1\cosh(M))}{\nu(\nu+\nu_r)M\sinh(M)}. \tag{37}$$

To conclude, our first-order approximation ($\mathbf{V}^0$, Q^0, W^0) reads

$$V^0=V_x^0(y)\mathbf{e}_1, \quad Q^0(x)=p_0+(p_1-p_0)x, \quad W^0=W^0(y), \tag{38}$$

where V_x^0 and W^0 are given by (34)–(37).

3.2 Correctors

At this stage, it is important to emphasise that no effects of the boundary perturbation can be observed from (38). Thus, we need to compute the lower order terms from the asymptotic expansions (13)–(15). The problem satisfied by the corrector ($\mathbf{V}^1$, Q^1, W^1) is the following one [see (19) and (20)]:

$$-(\nu+\nu_r)\Delta\mathbf{V}^1+\nabla Q^1=2\nu_r \text{ Rot } W^1, \quad \text{in } \Omega, \tag{39}$$

$$\operatorname{div} \mathbf{V}^1 = 0, \quad \text{in } \Omega, \tag{40}$$

$$-(c_a + c_d)\Delta W^1 + 4\nu_r W^1 = 2\nu_r \operatorname{rot} \mathbf{V}^1, \quad \text{in } \Omega, \tag{41}$$

$$\mathbf{V}^1(x, 1) = h(x)\frac{\partial \mathbf{V}^0}{\partial y}(x, 1) = h(x)\frac{\partial V_x^0}{\partial y}(1)\mathbf{e}_1, \quad \mathbf{V}^1(x, 0) = \mathbf{0}, \tag{42}$$

$$V_y^1(0, y) = 0, \quad V_y^1(1, y) = 0, \quad Q^1(0, y) = 0, \quad Q^1(1, y) = 0, \tag{43}$$

$$W^1(x, 1) = h(x)\frac{\partial W^0}{\partial y}(1), \quad W^1(x, 0) = 0. \tag{44}$$

The problem is again strongly coupled, now with (non-constant) function $h(x)$ appearing in the boundary condition at $y = 1$. This makes the above problem significantly more complicated to solve than the one from Section 3.1. Nevertheless, it can be done by employing the Fourier series and the separation of variables, as shown in the following. First, we expand the function h as follows:

$$h(x) = \frac{h_0}{2} + \sum_{k=1}^{\infty} h_k \cos(k\pi x), \quad h_k = 2\int_0^1 h(t)\cos(k\pi t)\,dt. \tag{45}$$

The idea is to seek for the solution ($\mathbf{V}^1$, Q^1, W^1) in the form of the trigonometric Fourier series:

$$V_x^1 = \sum_{k=0}^{\infty} a_k(y)\cos(k\pi x), \tag{46}$$

$$V_y^1 = \sum_{k=1}^{\infty} b_k(y)\sin(k\pi x), \tag{47}$$

$$Q^1 = \sum_{k=1}^{\infty} d_k(y)\sin(k\pi x), \tag{48}$$

$$W^1 = \sum_{k=0}^{\infty} f_k(y)\cos(k\pi x), \tag{49}$$

where $\mathbf{V}^1 = (V_x^1, V_y^1)$. Taking into account the boundary conditions (42) and (44), we deduce

$$V_x^1(x, 0) = 0 \Rightarrow a_k(0) = 0, \quad k \geq 0, \tag{50}$$

$$V_y^1(x, 0) = 0 \Rightarrow b_k(0) = 0, \quad k \geq 1, \tag{51}$$

$$V_y^1(x, 1) = 0 \Rightarrow b_k(1) = 0, \quad k \geq 1, \tag{52}$$

$$V_x^1(x, 1) = h(x)\frac{\partial V_x^0}{\partial y}(1)$$

$$\Rightarrow \sum_{k=0}^{\infty} a_k(1)\cos(k\pi x) = \left(\frac{h_0}{2} + \sum_{k=1}^{\infty} h_k \cos(k\pi x)\right)\frac{\partial V_x^0}{\partial y}(1)$$

$$\Rightarrow a_0(1) = \frac{h_0}{2}\frac{\partial V_x^0}{\partial y}(1), \quad a_k(1) = \frac{\partial V_x^0}{\partial y}(1)h_k, \quad k \geq 1, \tag{53}$$

$$W^1(x, 0) = 0 \Rightarrow f_k(0) = 0, \quad k \geq 0, \tag{54}$$

$$W^1(x, 1) = h(x)\frac{\partial W^0}{\partial y}(1)$$

$$\Rightarrow \sum_{k=0}^{\infty} f_k(1)\cos(k\pi x) = \left(\frac{h_0}{2} + \sum_{k=1}^{\infty} h_k \cos(k\pi x)\right)\frac{\partial W^0}{\partial y}(1)$$

$$\Rightarrow f_0(1) = \frac{h_0}{2}\frac{\partial W^0}{\partial y}(1), \quad f_k(1) = \frac{\partial W^0}{\partial y}(1)h_k, \quad k \geq 1. \tag{55}$$

Observe that the boundary conditions (43) are automatically fulfilled, namely

$$V_y^1(0, y) = V_y^1(1, y) = Q^1(0, y) = Q^1(1, y) = 0. \tag{56}$$

Now, applying (46)–(49) into (39)–(41), we obtain

$$-(\nu + \nu_r)a_0'' = -2\nu_r f_0', \tag{57}$$

$$-(\nu + \nu_r)a_k'' + (\nu + \nu_r)(k\pi)^2 a_k + (k\pi)d_k = -2\nu_r f_k', \quad k \geq 1, \tag{58}$$

$$-(\nu + \nu_r)b_k'' + (\nu + \nu_r)(k\pi)^2 b_k + d_k' = -2\nu_r (k\pi) f_k, \quad k \geq 1, \tag{59}$$

$$-(k\pi)a_k + b_k' = 0, \quad k \geq 1, \tag{60}$$

$$-(c_a + c_d)f_0'' + 4\nu_r f_0 = 2\nu_r a_0', \tag{61}$$

$$-(c_a + c_d)f_k'' + (c_a + c_d)(k\pi)^2 f_k + 4\nu_r f_k = 2\nu_r a_k' - 2\nu_r (k\pi) b_k, \quad k \geq 1. \tag{62}$$

Let us first solve the system given by (57) and (61). Integrating (57) with respect to y gives

$$-(\nu + \nu_r)a_0' = -2\nu_r f_0 + E_1, \tag{63}$$

where E_1 is a constant. Plugging (63) into (61) yields

$$-(c_a + c_d)f_0'' + 4\nu_r f_0 = 2\nu_r\left(\frac{2\nu_r}{\nu + \nu_r} f_0 - \frac{E_1}{\nu + \nu_r}\right),$$

implying

$$-(c_a + c_d)f_0'' + \frac{4\nu\nu_r}{\nu + \nu_r} f_0 = -\frac{2\nu_r}{\nu + \nu_r} E_1.$$

Consequently,

$$f_0(y)=F_0^1\sinh(My)+F_0^2\cosh(My)-\frac{E_1}{2\nu}, \tag{64}$$

where F_0^1 and F_0^2 are constants to be determined later, while $M=2\sqrt{\frac{\nu_r\nu}{(c_a+c_d)(\nu+\nu_r)}}$. From (63) we deduce

$$a_0(y)=\frac{2\nu_r}{(\nu+\nu_r)M}F_0^1\cosh(My)+\frac{2\nu_r}{(\nu+\nu_r)M}F_0^2\sinh(My) -\frac{E_1}{\nu}y-\frac{E_2}{\nu+\nu_r}, \tag{65}$$

where E_1 and E_2 are new unknown constants. Taking into account that $a_0(0)=f_0(0)=0$ [see (50) and (54)] and $a_0(1)=\frac{h_0}{2}\frac{\partial V_x^0}{\partial y}(0)$, $f_0(1)=\frac{h_0}{2}\frac{\partial W^0}{\partial y}(1)$ [see $(53)_1$ and $(55)_1$], we derive the system of equations satisfied by the constants F_0^1, F_0^2, E_1, and E_2:

$$\begin{aligned}&2\nu_rF_0^1-ME_2=0,\\&2\nu F_0^2-E_1=0,\\&2\nu_r\nu F_0^1\cosh(M)+2\nu_r\nu F_0^2\sinh(M)-E_1(\nu+\nu_r)M\\&\quad -E_2\nu M=\nu(\nu+\nu_r)M\frac{h_0}{2}\frac{\partial V_x^0}{\partial y}(1),\\&2\nu F_0^1\sinh(M)+2\nu F_0^2\cosh(M)-E_1=\nu h_0\frac{\partial W^0}{\partial y}(1).\end{aligned}$$

Solving the above equations, we get

$$F_0^1=\frac{1}{\sinh(M)}F_0^2(1-\cosh(M))+\frac{1}{\sinh(M)}\frac{h_0}{2}\frac{\partial W^0}{\partial y}(1), \tag{66}$$

$$F_0^2=\frac{-\frac{1}{\sinh(M)}\frac{h_0}{2}\frac{\partial W^0}{\partial y}(1)\cdot(2\nu_r\cosh(M)-2\nu_r)+(\nu+\nu_r)M\frac{h_0}{2}\frac{\partial V_x^0}{\partial y}(1)}{-\frac{2\nu_r}{\sinh(M)}(1-\cosh(M))^2+2\nu_r\sinh(M)-2(\nu+\nu_r)M}, \tag{67}$$

$$E_1=2\nu F_0^2, \tag{68}$$

$$E_2=\frac{2\nu_r}{M}F_0^1. \tag{69}$$

Now we focus on the system given by (58)–(60) and (62). From (60) we get

$$a_k=\frac{1}{k\pi}b_k',\quad k\geq 1, \tag{70}$$

implying

$$-\frac{\nu+\nu_r}{k\pi}b_k'''+(\nu+\nu_r)(k\pi)b_k'+(k\pi)d_k=-2\nu_rf_k',\quad k\geq 1, \tag{71}$$

$$-(\nu+\nu_r)b_k''+(\nu+\nu_r)(k\pi)^2b_k+d_k'=-2\nu_r(k\pi)f_k,\quad k\geq 1, \tag{72}$$

$$-(c_a+c_d)f_k''+(c_a+c_d)(k\pi)^2f_k+4\nu_rf_k=\frac{2\nu_r}{k\pi}b_k''-2\nu_r(k\pi)b_k,\ k\geq 1. \tag{73}$$

From (71) we deduce

$$d_k=\frac{\nu+\nu_r}{(k\pi)^2}b_k'''-(\nu+\nu_r)b_k'-\frac{2\nu_r}{k\pi}f_k', \tag{74}$$

so the system (71)–(73) reduces to

$$\frac{\nu+\nu_r}{(k\pi)^2}b_k^{(4)}-2(\nu+\nu_r)b_k''+(\nu+\nu_r)(k\pi)^2b_k=\frac{2\nu_r}{k\pi}f_k''-2\nu_r(k\pi)f_k, \tag{75}$$

$$-(c_a+c_d)f_k''+[(c_a+c_d)(k\pi)^2+4\nu_r]f_k=\frac{2\nu_r}{k\pi}b_k''-2\nu_r(k\pi)b_k. \tag{76}$$

To solve (75) and (76) we proceed as follows. From (76) we first deduce

$$2\nu_rb_k''-2\nu_r(k\pi)^2b_k=-(c_a+c_d)(k\pi)f_k'' +[(c_a+c_d)(k\pi)^2+4\nu_r](k\pi)f_k. \tag{77}$$

Now we differentiate twice the above equation to obtain

$$2\nu_rb_k^{(4)}-2\nu_r(k\pi)^2b_k''=-(c_a+c_d)(k\pi)f_k^{(4)} +[(c_a+c_d)(k\pi)^2+4\nu_r](k\pi)f_k''. \tag{78}$$

Multiplying (78) by $\frac{\nu+\nu_r}{2\nu_r}\cdot\frac{1}{(k\pi)^2}$, (77) by $\frac{\nu+\nu_r}{2\nu_r}$ and then subtracting the obtained equations, we obtain

$$\begin{aligned}&\frac{\nu+\nu_r}{(k\pi)^2}b_k^{(4)}-2(\nu+\nu_r)b_k''+(\nu+\nu_r)(k\pi)^2b_k\\&=-\frac{2\nu}{M^2}\frac{1}{k\pi}f_k^{(4)}+\frac{4\nu}{M^2}(k\pi)f_k''+\frac{2(\nu+\nu_r)}{k\pi}f_k''\\&-\frac{2\nu}{M^2}(k\pi)^3f_k-2(\nu+\nu_r)(k\pi)f_k.\end{aligned} \tag{79}$$

Note that the left-hand sides in (75) and (79) are exactly the same providing us the fourth-order ordinary differential equation (ODE) for f_k:

$$\frac{1}{M^2}f_k^{(4)}-\left(1+\frac{2}{M^2}(k\pi)^2\right)f_k''+(k\pi)^2\left(1+\frac{(k\pi)^2}{M^2}\right)f_k=0. \qquad (80)$$

It can be easily verified that

$$f_k(y)=G_k^1\sinh(\lambda_k y)+G_k^2\cosh(\lambda_k y)+G_k^3\sinh(\mu_k y)$$
$$+G_k^4\cosh(\mu_k y),\quad k\geq 1, \qquad (81)$$

where $\lambda_k=\sqrt{(k\pi)^2+M^2}$, $\mu_k=k\pi$ and G_k^i constants to be determined in the following. Taking into account the boundary conditions (54) and $(55)_2$, namely

$$f_k(0)=0,\quad f_k(1)=\frac{\partial W^0}{\partial y}(1)h_k,$$

we deduce

$$G_k^2=-G_k^4,\quad G_k^1\sinh(\lambda_k)+G_k^2\cosh(\lambda_k)+G_k^3\sinh(\mu_k)$$
$$+G_k^4\cosh(\mu_k)=\frac{\partial W^0}{\partial y}(1)h_k. \qquad (82)$$

Thus, $f_k=f_k(y;\,G_k^3,\,G_k^4)$ is given by [see (81) and (82)]

$$f_k=\left(\frac{\frac{\partial W^0}{\partial y}(1)}{\sinh(\lambda_k)}h_k-\frac{\sinh(\mu_k)}{\sinh(\lambda_k)}G_k^3-\frac{\cosh(\mu_k)-\cosh(\lambda_k)}{\sinh(\lambda_k)}G_k^4\right)$$
$$\sinh(\lambda_k y)+G_k^3\sinh(\mu_k y)-G_k^4\cosh(\lambda_k y)+G_k^4\cosh(\mu_k y). \qquad (83)$$

Now, we go back to (76) leading to

$$\frac{1}{k\pi}b_k''-k\pi b_k=\left(2A_k-\frac{A_k(c_a+c_d)M^2}{2\nu_r}\right)\sinh(\lambda_k y)$$
$$-\left(2G_k^4-\frac{G_k^4(c_a+c_d)M^2}{2\nu_r}\right)\cosh(\lambda_k y)$$
$$+2G_k^3\sinh(\mu_k y)+2G_k^4\cosh(\mu_k y),$$

where

$$A_k=\frac{\frac{\partial W^0}{\partial y}(1)}{\sinh(\lambda_k)}h_k-\frac{\sinh(\mu_k)}{\sinh(\lambda_k)}G_k^3-\frac{\cosh(\mu_k)-\cosh(\lambda_k)}{\sinh(\lambda_k)}G_k^4.$$

It follows that

$$b_k=B_k^1\sinh(\mu_k y)+B_k^2\cosh(\mu_k y)$$
$$+\left(2A_k-\frac{A_k(c_a+c_d)M^2}{2\nu_r}\right)\cdot\frac{\mu_k}{M^2}\sinh(\lambda_k y)$$
$$-\left(2G_k^4-\frac{G_k^4(c_a+c_d)M^2}{2\nu_r}\right)\cdot\frac{\mu_k}{M^2}\cosh(\lambda_k y)+G_k^3 y\cosh(\mu_k y)$$
$$+G_k^4 y\sinh(\mu_k y). \qquad (84)$$

Using (70), we deduce

$$a_k=B_k^1\cosh(\mu_k y)+B_k^2\sinh(\mu_k y)$$
$$+\left(2A_k-\frac{A_k(c_a+c_d)M^2}{2\nu_r}\right)\cdot\frac{\lambda_k}{M^2}\cosh(\lambda_k y)$$
$$-\left(2G_k^4-\frac{G_k^4(c_a+c_d)M^2}{2\nu_r}\right)\cdot\frac{\lambda_k}{M^2}\sinh(\lambda_k y)$$
$$+\frac{G_k^3}{\mu_k}\cosh(\mu_k y)+G_k^3 y\sinh(\mu_k y)$$
$$+\frac{G_k^4}{\mu_k}\sinh(\mu_k y)+G_k^4 y\cosh(\mu_k y). \qquad (85)$$

It remains to determine the constants B_k^1, B_k^2, G_k^3 and G_k^4. We use boundary conditions (51), (52), (50) and $(53)_2$, namely

$$b_k(0)=0,\quad b_k(1)=0,\quad a_k(0)=0,\quad a_k(1)=\frac{\partial V_x^0}{\partial y}(1)h_k,\quad k\geq 1$$

to obtain

$$B_k^2-\frac{2\mu_k G_k^4}{M^2}+\frac{G_k^4(c_a+c_d)\mu_k}{2\nu_r}=0,$$

$$B_k^1\sinh(\mu_k)+B_k^2\cosh(\mu_k)+\frac{2A_k\mu_k}{M^2}\sinh(\lambda_k)$$
$$-\frac{A_k\mu_k(c_a+c_d)}{2\nu_r}\sinh(\lambda_k)-\frac{2G_k^4\mu_k}{M^2}\cosh(\lambda_k)$$
$$+\frac{G_k^4\mu_k(c_a+c_d)}{2\nu_r}\cosh(\lambda_k)+G_k^3\cosh(\mu_k)+G_k^4\sinh(\mu_k)=0,$$

$$B_k^1+\frac{2A_k\lambda_k}{M^2}-\frac{A_k(c_a+c_d)\lambda_k}{2\nu_r}+\frac{G_k^3}{\mu_k}=0,$$

$$B_k^1\cosh(\mu_k)+B_k^2\sinh(\mu_k)+\frac{2A_k\lambda_k}{M^2}\cosh(\lambda_k)$$
$$-\frac{\lambda_k A_k(c_a+c_d)}{2\nu_r}\cosh(\lambda_k)-\frac{2G_k^4\lambda_k}{M^2}\sinh(\lambda_k)$$
$$+\frac{G_k^4(c_a+c_d)\lambda_k}{2\nu_r}\sinh(\lambda_k)+\frac{G_k^3}{\mu_k}\cosh(\mu_k)$$
$$+G_k^3\sinh(\mu_k)+\frac{G_k^4}{\mu_k}\sinh(\mu_k)+G_k^4\cosh(\mu_k)=\frac{\partial V_x^0}{\partial y}(1)h_k.$$

The above system gives

$$B_k^1=-A_k B_k-\frac{G_k^3}{\mu_k},\quad B_k^2=C_k G_k^4, \qquad (86)$$

$$G_k^3=\frac{-H_k G_k^4+I_k}{G_k},\quad G_k^4=\frac{N_k}{M_k}, \qquad (87)$$

with

$$A_k = \frac{\frac{\partial W^0}{\partial y}(1)}{\sinh(\lambda_k)} h_k - \frac{\sinh(\mu_k)}{\sinh(\lambda_k)} G_k^3 - \frac{\cosh(\mu_k)-\cosh(\lambda_k)}{\sinh(\lambda_k)} G_k^4,$$
$$B_k = \frac{4\lambda_k \nu_r - M^2 \lambda_k (c_a + c_d)}{2M^2 \nu_r}, \tag{88}$$

$$C_k = \frac{4\mu_k \nu_r - M^2 \mu_k (c_a + c_d)}{2M^2 \nu_r}, \quad D_k = \frac{\mu_k \cosh(\mu_k) - \sinh(\mu_k)}{\mu_k}, \tag{89}$$

$$E_k = C_k(\cosh(\mu_k) - \cosh(\lambda_k)) + \sinh(\mu_k),$$
$$F_k = B_k \sinh(\mu_k) - C_k \sinh(\lambda_k), \tag{90}$$

$$G_k = D_k + F_k \frac{\sinh(\mu_k)}{\sinh(\lambda_k)}, \quad H_k = E_k + F_k \frac{\cosh(\mu_k)-\cosh(\lambda_k)}{\sinh(\lambda_k)}, \tag{91}$$

$$I_k = F_k \frac{\frac{\partial W^0}{\partial y}(1)}{\sinh(\lambda_k)} h_k, \quad J_k = \sinh(\mu_k), \tag{92}$$

$$K_k = C_k \sinh(\mu_k) - B_k \sinh(\lambda_k) + \frac{\sinh(\mu_k)}{\mu_k} + \cosh(\mu_k),$$
$$L_k = B_k(\cosh(\mu_k) - \cosh(\lambda_k)), \tag{93}$$

$$M_k = K_k - \frac{H_k J_k}{G_k} - \frac{L_k H_k \sinh(\mu_k)}{G_k \sinh(\lambda_k)} + L_k \frac{\cosh(\mu_k)-\cosh(\lambda_k)}{\sinh(\lambda_k)}, \tag{94}$$

$$N_k = \frac{\partial V_x^0}{\partial y}(1) h_k - \frac{I_k J_k}{G_k} + \frac{L_k h_k \frac{\partial W^0}{\partial y}(1)}{\sinh(\lambda_k)} - \frac{L_k I_k \sinh(\mu_k)}{G_k \sinh(\lambda_k)}. \tag{95}$$

Finally, d_k is deduced from (74)

$$d_k = \left(\frac{\lambda_k}{\mu_k}(\nu + \nu_r) \left(2A_k - \frac{A_k(c_a + c_d)M^2}{2\nu_r} \right) - 2\nu_r \frac{\lambda_k}{\mu_k} A_k \right) \cosh(\lambda_k y)$$
$$- \left(-2\nu_r G_k^4 \frac{\lambda_k}{\mu_k} + \frac{\lambda_k}{\mu_k}(\nu + \nu_r) \left(2G_k^4 - \frac{G_k^4 (c_a + c_d) M^2}{2\nu_r} \right) \right) \sinh(\lambda_k y)$$
$$+ 2\nu G_k^3 \cosh(\mu_k y) + 2\nu G_k^4 \sinh(\mu_k y) \tag{96}$$

and this completes the derivation of the correctors given by (46)–(49).

3.3 Asymptotic Solution

Our asymptotic solution has the following form:

$$\mathbf{u}^\epsilon_{\text{approx}}(x, y) = V_x^0(y)\mathbf{e}_1 + \epsilon \mathbf{V}^1(x, y), \quad \mathbf{V}^1 = (V_x^1, V_y^1), \tag{97}$$

$$q^\epsilon_{\text{approx}}(x, y) = Q^0(x) + \epsilon Q^1(x, y) \tag{98}$$

$$w^\epsilon_{\text{approx}}(x, y) = W^0(y) + \epsilon W^1(x, y), \tag{99}$$

where the functions V_x^0, $\mathbf{V}^1$, Q^0, Q^1, W^0, W^1 are provided in Sections 3.1 and 3.2. Being in the explicit form, it enables us to clearly observe the effects of the perturbed channel boundary and the micropolar properties of the fluid. The boundary perturbation has strong impact on the behaviour of the correctors through the appearance of the Fourier coefficients h_k of the perturbation function h. Thus, we conclude that the influence of the boundary perturbation on the effective flow is not just local (in the vicinity of the upper boundary), especially if ϵ (the magnitude of the perturbation) is not too small (e.g. $\epsilon = 10^{-1}$). This will be confirmed numerically in the following section.

Remark 1 *The asymptotic approximation for the microrotation given by* (99) *was computed to satisfy the governing equations* (2)–(4) *and the zero boundary condition* (8) *at* $y = 0, 1$. *Note that the boundary condition at* $x = 0, 1$ *could not be taken into account in the process. As a consequence,* $w^\epsilon_{\text{approx}}(i, y) \neq 0$, *for* $i = 0, 1$. *This essentially means that the boundary layer phenomena for the microrotation take place. We can fix that in a standard manner by introducing the appropriate boundary layer correctors in the vicinity of the lateral ends of the channel, see, for example,* [19, 20]. *However, it can be proved that those correctors would have the exponential decay towards zero, that is, it would not affect the approximation outside the boundary layers. In fact, it would only serve for the convergence proof, namely to derive satisfactory error estimates in* L^2 *and* H^1 *norm* (*being out of the scope of the present paper*). *Thus, there is no reason to formally correct the approximation* (99) *in the vicinity of* $x = 0, 1$ *since the effects of such correction would not effectively contribute to the derived asymptotic model.*

4 Numerical Example

In this section, we aim to visually present our asymptotic solution in order to confirm the effects of the perturbed boundary on the fluid flow. We employ the boundary perturbation function

$$h(x) = -\cos\left(\frac{\pi x}{2}\right).$$

We take the pressure drop $\delta p = -1$, while for the viscosity constants we use the following values (see [23]):

$$\nu=2.9\times10^{-3},\quad \nu_r=2.32\times10^{-4},\quad c_a+c_d=10^{-6}.$$

We compute the corrector approximations up to $k=10$ in the Fourier series (46)–(49), since increasing k leads to no significant improvements. We do the same for the function h in (45), where we computed the coefficients h_k using the numerical integration in MATLAB.

In Section 4.1, we first depict the correctors computed in Section 3.2. We present the 2D velocity profiles for fixed values of x and y along with 3D figures. We clearly observe that the correctors acknowledge the effects of the boundary perturbation and that those effects are not just local (near the upper boundary). This is particularly noticeable for the x-component of the velocity corrector and for the microrotation corrector (see Figs. 2–5 and 6). The y-component of the velocity corrector and the pressure corrector also feel those effects but to a significantly smaller extent (see Figs. 7–9).

In Section 4.2, we visually present the whole asymptotic solution (97)–(99) for different magnitudes of small parameter ϵ (namely, $\epsilon\in\{0.1, 0.01\}$). The pressure approximation and the y-component of the velocity approximation are omitted since the perturbation effects have negligible impact on the corresponding correctors. However, the x-component of the velocity approximation feels those effects, if ϵ is not too small ($\epsilon=0.1$). This can be visually observed from the 2D profiles shown in Figures 10 and 11 (see also Fig. 12 for 3D representation). For ϵ of smaller magnitude ($\epsilon=0.01$), the perturbation effect becomes negligible (see Fig. 13). On the other hand, the microrotation approximation is affected by the boundary perturbation in both cases (see Figs. 14 and 15).

Remark 2 *Though we present the numerical example for the function h satisfying* $h<0$ *on* $(0, 1)$, *it must be emphasised that the results from Section 3 are valid for a general function h. That is due to the fact that it can be proved that our approximation (constructed directly without the change of variables) is asymptotically the same as the one that could be built if we have first passed to the* ϵ*-independent domain* $\Omega=(0, 1)^2$ *by introducing the change of variable* $z=\dfrac{y}{1-\epsilon h}$. *Though tedious, this part is straightforward and can be done following the same lines as in the classical Newtonian case (see* [8] *for details).*

Remark 3 *Note that the velocity approximation is particularly affected by the boundary perturbation if* ϵ *is not too small. A rigorous way to confirm that the asymptotic solution* (97) *is good (even for moderately small values of* ϵ *such as* $\epsilon=0.1$) *is to prove the satisfactory error estimates in the appropriate rescaled norm. To accomplish that, we have to consider the problem in the* ϵ*-independent domain* Ω *and to formally link the corresponding asymptotic solution with the one we derived, as explained in Remark 2. After that, the proof of the error estimates follows the standard arguments, see, for example,* [19, 20].

4.1 Correctors

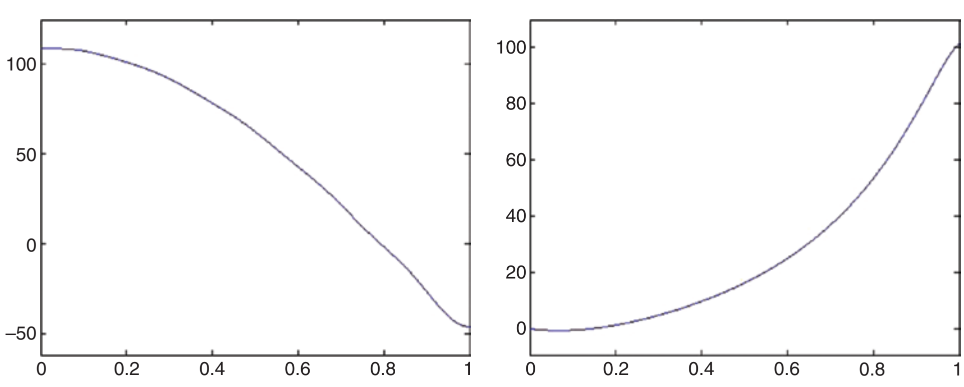

Figure 2: Velocity corrector V_x^1 profile for fixed $y=1$ (left) and for fixed $x=0.2$ (right).

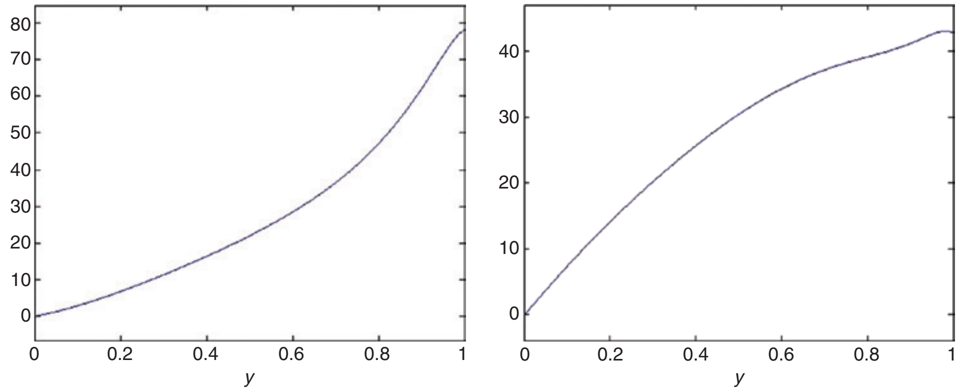

Figure 3: Velocity corrector V_x^1 profile for fixed $x = 0.4$ (left) and $x = 0.6$ (right).

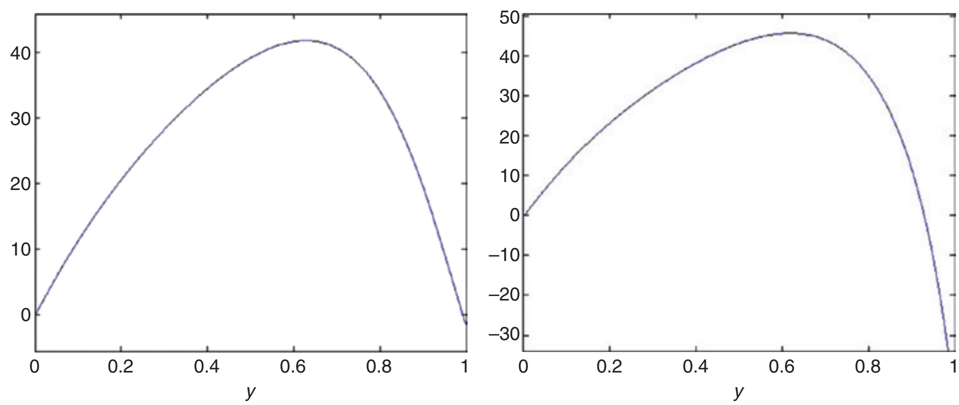

Figure 4: Velocity corrector V_x^1 profile for fixed $x = 0.8$ (left) and $x = 1$ (right).

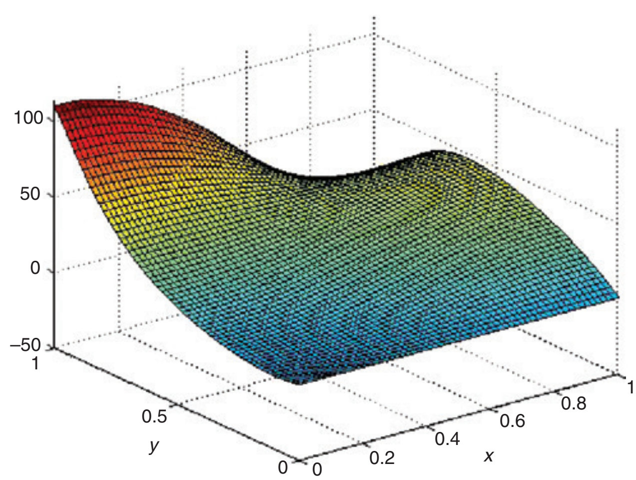

Figure 5: Velocity corrector V_x^1.

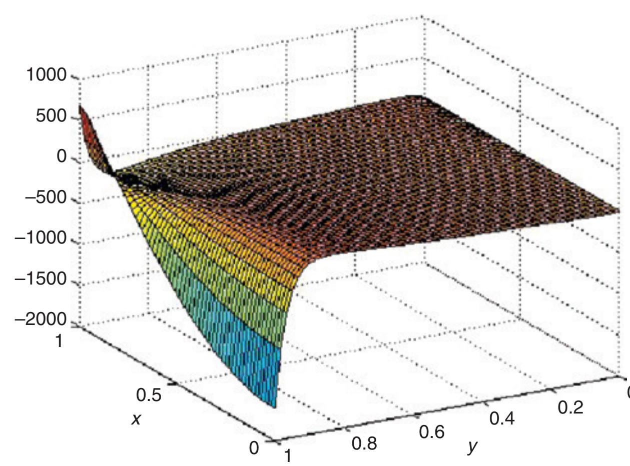

Figure 6: Microrotation corrector W^1.

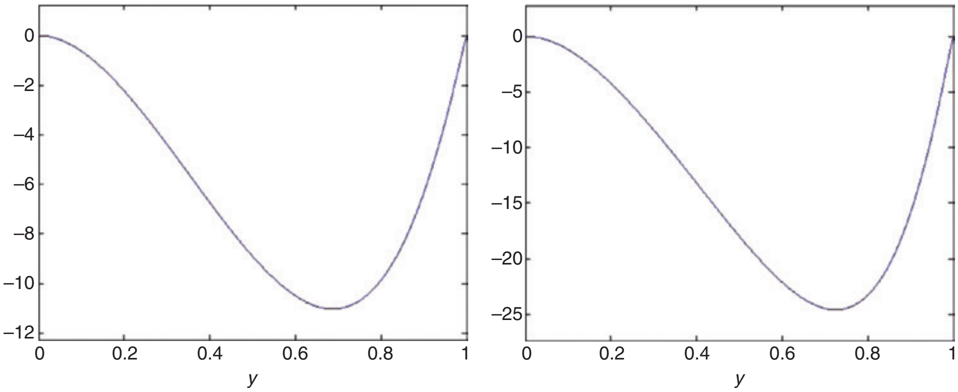

Figure 7: Velocity corrector V_y^1 profile for fixed $x = 0.2$ (left) and $x = 0.6$ (right).

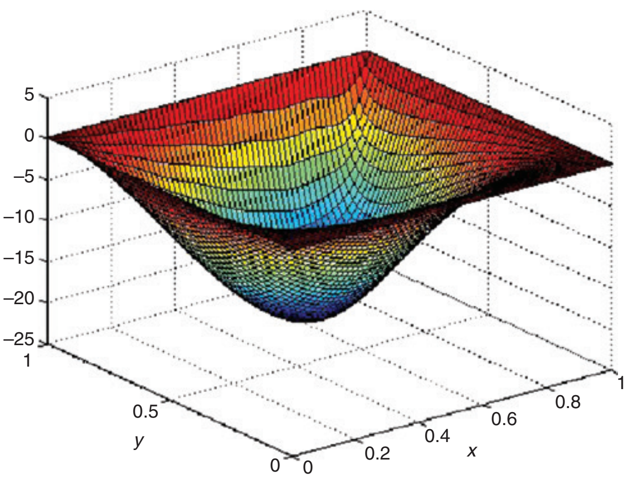

Figure 8: Velocity corrector V_y^1.

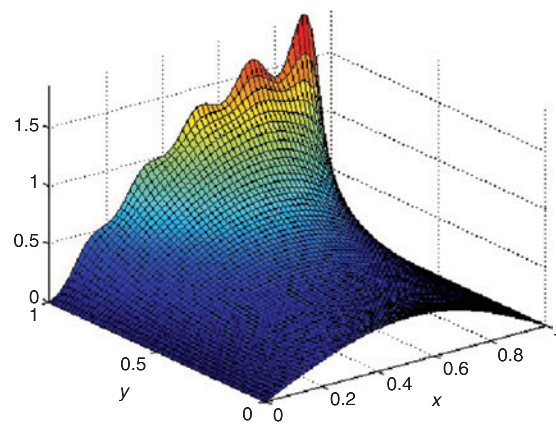

Figure 9: Pressure corrector Q^1.

4.2 Asymptotic Solution

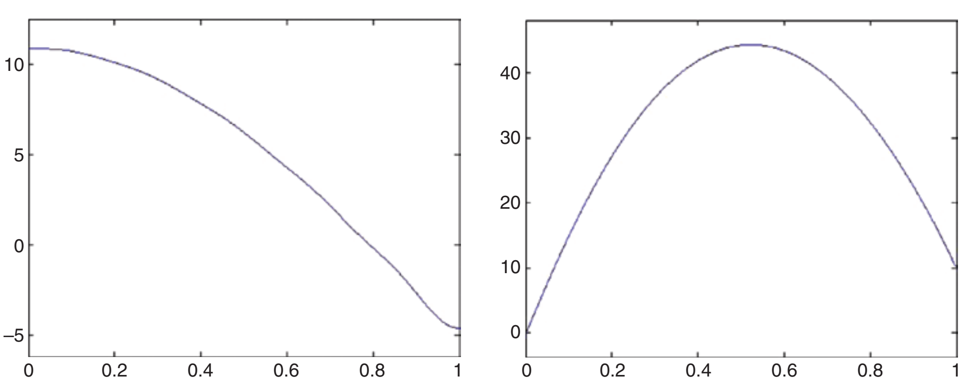

Figure 10: Velocity approximation (x-component) profile for fixed $y = 1$ (left) and for fixed $x = 0.2$ (right) ($\epsilon = 0.1$).

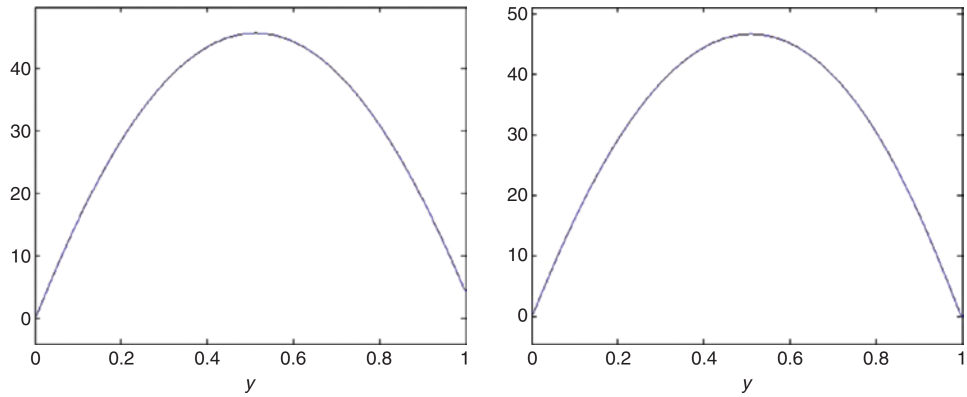

Figure 11: Velocity approximation (x-component) profile for fixed $x=0.6$ (left) and $x=0.8$ (right) ($\epsilon=0.1$).

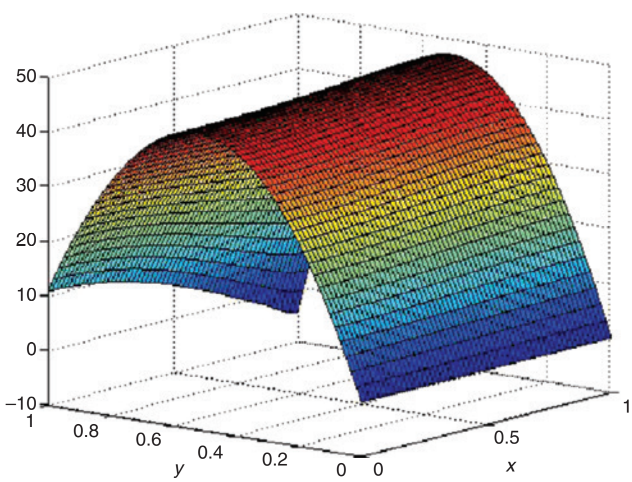

Figure 12: Velocity approximation (x-component) for $\epsilon=0.1$.

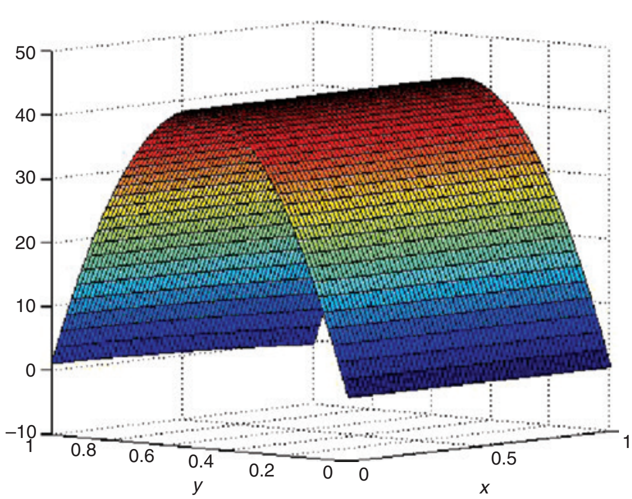

Figure 13: Velocity approximation (x-component) for $\epsilon=0.01$.

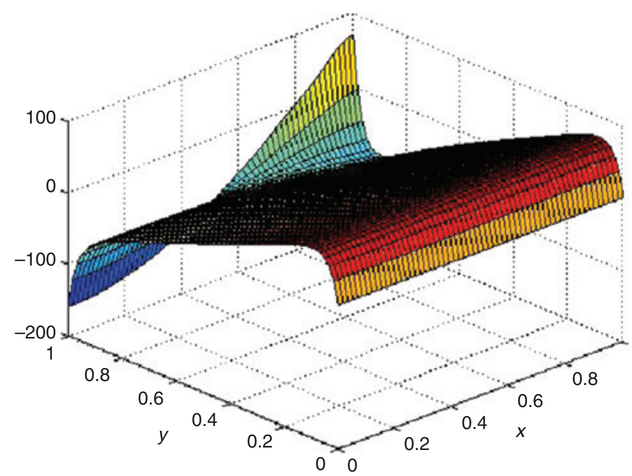

Figure 14: Microrotation approximation for $\epsilon=0.1$.

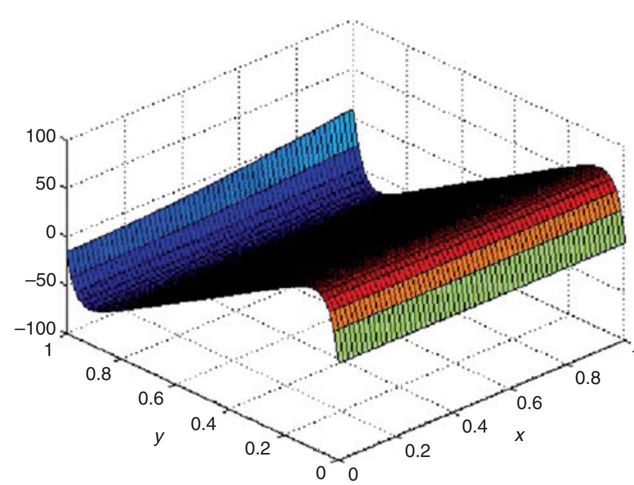

Figure 15: Microrotation approximation for $\epsilon=0.01$.

Acknowledgments: The authors of this work have been supported by the Croatian Science Foundation (scientific project 3955: *Mathematical modeling and numerical simulations of processes in thin or porous domains*).The authors would like to thank the referees for their helpful comments and suggestions that helped to improve the paper.

References

[1] D. Henry, Perturbation of the Boundary in Boundary-Value Problems of Partial Differential Equations, London Mathematical Society Lecture Note Series 318, Cambridge University Press, New York 2005.
[2] A.C. Eringen, J. Math. Mech. **16**, 1 (1966).
[3] G. Lukaszewicz, Micropolar Fluids: Theory and Applications, Birkhäuser, Boston 1999.
[4] I. Abdullah and N. Amin, Math. Meth. Appl. Sci. **33**, 1910 (2010).
[5] A. Ishaka, Y.Y. Lokb, and I. Pop, Chem. Eng. Comm. **197**, 1417 (2010).
[6] M. Sheikholeslamia, M. Hatami, and D.D. Ganji, J. Mol. Liq. **194**, 30 (2014).
[7] A. R. Haghighi and M. S. Asl, Int. J. Biomath. **08**, 1550056 (2015).
[8] E. Marušić-Paloka, J. Appl. Math. Mech. (2016). DOI: 10.1002/zamm.201500195.
[9] E. Beretta and E. Francini, SIAM J. Math. Anal. **38**, 1249 (2006).
[10] T.C. Ekneligoda and R.W. Zimmerman, J. Appl. Mech. **75**, 011015 (2008).
[11] H. Ammari, H. Kang, H. Lee, and J. Lim, J. Elasticity **113**, 75 (2013).
[12] Y. Achdou, O. Pironneau, and F. Valentin, J. Comput. Phys. **147**, 187 (1998).
[13] W. Jäger and A. Mikelić, J. Differ. Equations **170**, 96 (2001).
[14] S. Sisavath, X. Jing, and R.W. Zimmerman, Phys. Fluids **12**, 2762 (2001).
[15] D. Bresch, C. Choquet, L. Chupin, T. Colin, and M. Gisclon, Multiscale Model. Simul. **8**, 997 (2010).
[16] M. Boukrouche and L. Paoli, J. Math. Anal. **44**, 1211 (2012).
[17] I. Pažanin and F. J. Suárez-Grau, C. R. Mecanique **341**, 646 (2013).
[18] I. Pažanin and F. J. Suárez-Grau, Comput. Math. Appl. **68**, 1915 (2014).
[19] D. Dupuy, G. Panasenko, and R. Stavre, Math. Mod. Meth. Appl. Sci. **14**, 735 (2004).
[20] I. Pažanin, Acta Appl. Math. **116**, 1 (2011).
[21] G. Bayada, N. Benhaboucha, and M. Chambat, Math. Mod. Meth. Appl. Sci. **15**, 343 (2005).
[22] M. Beneš and I. Pažanin, Acta Appl. Math. 1 (2016). DOI: 10.1007/s10440-015-0026-1.
[23] I. Papautsky, J. Brazzle, T. Ameel, and A.B. Frazier, Sensors Actuators A Phys. **73**, 101 (1999).