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Nonlocal Symmetry and its Applications in Perturbed mKdV Equation

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Abstract: Based on the modified direct method, the variable-coefficient perturbed mKdV equation is changed to the constant-coefficient perturbed mKdV equation. The truncated Painlevé method is applied to obtain the nonlocal symmetry of the constant-coefficient perturbed mKdV equation. By introducing one new dependent variable, the nonlocal symmetry can be localized to the Lie point symmetry. Thanks to the localization procedure, the finite symmetry transformation is presented by solving the initial value problem of the prolonged systems. Furthermore, many explicit interaction solutions among different types of solutions such as solitary waves, rational solutions, and Painlevé II solutions are obtained using the symmetry reduction method to the enlarged systems. Two special concrete soliton-cnoidal interaction solutions are studied in both analytical and graphical ways.

Keywords: Interaction Solution; Modified Direct Method; Nonlocal Symmetry; Perturbed mKdV Equation; Symmetry Reduction.

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1 Introduction

The mathematical theory of soliton collisions in the integrable models is a well developed field. Among these soliton collisions, multi-peakon, multi-soliton, multi-cuspon, and soliton-cuspon solutions have been widely investigated [1–6]. However, the interaction solutions among different types of nonlinear excitations are hardly studied. Recently, the localization procedure related with the nonlocal symmetry to find these types of interaction solutions has been proposed [7–10]. The method has been applied some constant-coefficient nonlinear systems [7–12]. In this article, we can use this method to variable-coefficient perturbed mKdV equation. The model can describe both the plasma-sheath transition layer and the sheath inner layer [13]. The investigations on the variable-coefficient perturbed mKdV equation is transformed to the constant-coefficient one by the modified direct method. The Painlevé method analysis is carried out on the constant-coefficient perturbed mKdV equation. The nonlocal symmetry of the equation is constructed. The finite symmetry transformation related with the nonlocal symmetry is obtained by solving the initial value problem of the Lie's first principle. The interaction solutions among solitons and other complicated waves including the Painlevé waves, rational waves, and periodic cnoidal waves of the perturbed mKdV equation are derived by the symmetry reduction method. Those interaction solutions are difficult to obtain with other traditional methods, such as inverse scattering method [14], Darboux and Bäklund transformations [15, 16], Hirota's bilinear method [17], and so on [18–22].

The structure of this article is as follows. In Section 2, based on the modified direct method, the variable-coefficient perturbed mKdV equation is changed to the the constant-coefficient perturbed mKdV equation. In Section 3, the nonlocal symmetry for the perturbed mKdV equation is obtained with the truncated Painlevé method. The finite symmetry transformation is given by localization of the nonlocal symmetry to the Lie point symmetry. In Section 4, the prolonged systems are investigated according to the Lie point symmetry theory. The interaction solutions are presented by the symmetry reductions. The last section is a short summary and discussion.

2 Modified Direct Method for Perturbed mKdV Equation

The modified direct method is a powerful and direct method to investigate the nonlinear equations [23, 24]. The expression of the finite transformations of the Lie groups is much simpler than those obtained via the standard approaches such as the classical Lie group approach [18], the nonclassical Lie group approach [25], and the Clarkson–Kruskal direct method [26]. In this section,

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we can perform the modified direct method to study the variable-coefficient perturbed mKdV equation. Via the modified direct method, the investigations on the variable-coefficient perturbed equation can be based on the constant-coefficient one.

The variable-coefficient perturbed mKdV equation reads

$$u'_{t'} - u'_{x'x'x'} + \beta^2 u'^2 u'_{x'} + 6u'u'_{x'} + h(t')u'_{x'} = 0, \tag{1}$$

where β is arbitrary constant and $h(t')$ is an analytic function. It has been used to consider attention in many different physical fields including ocean dynamics [27], fluid mechanics [28], and plasma physics [13]. The multi-solitonic solutions in terms of the double Wronskian of (1) is obtained by the reducing technique [29].

Based on the modified direct method [23], the solution of (1) has a following form

$$u' = \nu + \mu u(x, t), \tag{2}$$

where ν, μ, x, and t are functions of x' and t'. We assume that the field u satisfies the following constant-coefficient perturbed mKdV equation

$$u_t - u_{xxx} + \beta^2 u^2 u_x + 6uu_x + u_x = 0. \tag{3}$$

By substituting (2) into the variable-coefficient perturbed mKdV system (1) and collecting the coefficients of u and its derivatives, we obtain

$$\nu = \frac{9\delta_1\delta_2}{\beta^2\sqrt{9-\beta^2}} - \frac{3}{\beta^2}, \quad \mu = \frac{3\delta_1\delta_2}{\sqrt{9-\beta^2}}, \quad x = \frac{3\delta_1(x' - \int h(t')\mathrm{d}t')}{\sqrt{9-\beta^2}} + B_2,$$
$$t = B_1 + \frac{27\beta^3}{(9-\beta^2)^{\frac{3}{2}}}t', \quad \delta_1^2 = 1, \qquad \delta_2^2 = 1. \tag{4}$$

where B_1 and B_2 are arbitrary constants.

Remark Once we obtain the solution of constant-coefficient perturbed mKdV equation (3), then the solution for the variable-coefficient perturbed mKdV equation will be expressed as (2).

3 Nonlocal Symmetry and its Localization for Perturbed mKdV Equation

In this section, the Painlevé analysis is carried out the perturbed mKdV equation. The nonlocal symmetry and the finite symmetry transformation for the equation can be constructed by the Painlevé analysis.

According to the Painlevé test, one suppose u is [30]

$$u = \frac{u_0}{\phi} + u_1, \tag{5}$$

where the function ϕ is an arbitrary function defining the singular manifold by $\phi = 0$. By substituting of expansion (5) into (3) and vanishing the most dominant term, we obtain

$$u_0 = \pm\frac{\sqrt{6}}{\beta}\phi_x. \tag{6}$$

We proceed further and collect the coefficient of ϕ^{-3} to get

$$u_1 = \frac{\sqrt{6}}{2\beta}\frac{\phi_{xx}}{\phi_x} - \frac{3}{\beta^2}. \tag{7}$$

Substituting the expressions (5), (6), and (7) into (1), one find the following Schwarzian perturbed mKdV form

$$\frac{\phi_t}{\phi_x} + \{\phi; x\} + \frac{9-\beta^2}{\beta^2} = 0, \tag{8}$$

where $\{\phi; x\} = \frac{\partial}{\partial x}\left(\frac{\phi_{xx}}{\phi_x}\right) - \frac{1}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2$ is the Schwarzian derivative.

By the definition of residual symmetry (RS) [10], the nonlocal symmetry of the perturbed mKdV equation (1) reads out from the truncated Painlevé analysis (5)

$$\sigma^u = \pm\frac{\sqrt{6}}{\beta}\phi_x. \tag{9}$$

The nonlocal symmetry (9) can also be given using (8) and (7) [11, 12]. The Schwarzian form (8) is invariant under the Möbious group [30]

$$\phi \to \frac{a\phi + b}{c\phi + d}, \quad ac \neq bd. \tag{10}$$

It means that (8) possesses the symmetry

$$\sigma^\phi = b + (a+b)\phi + a\phi^2, \tag{11}$$

where the constants are $d = 1$, $c = -\epsilon$ in (10). The nonlocal symmetry (9) will be obtained with substituting the Möbious transformation symmetry (11) into the symmetry equation of (7).

According to the Lie's first principle, the initial value problem related with the nonlocal symmetry (9) reads

$$\frac{\mathrm{d}\bar{u}}{\mathrm{d}\epsilon}=\pm\frac{\sqrt{6}}{\beta}\phi_x,\quad \bar{u}|_{\epsilon=0}=u. \tag{12}$$

To solve the initial value problem (12), one can localize the nonlocal symmetry to the local Lie point symmetry for the prolonged systems (10). To eliminate the space derivative of field ϕ, the potential field is introduced

$$\phi_x=g. \tag{13}$$

With the help of (5), we obtain an auto-Bäklund transformation of (3). u_1 is also a solution of (3). It is easily verified that the solution of the symmetry equation for the prolonged systems (3), (7) and (13) gives

$$\sigma^u=\frac{\sqrt{6}g}{\beta},\quad \sigma^\phi=\phi^2,\quad \sigma^g=2\phi g. \tag{14}$$

The initial value problem (12) is correspondingly changed

$$\frac{\mathrm{d}\bar{u}}{\mathrm{d}\epsilon}=\frac{\sqrt{6}g}{\beta},\qquad \bar{u}|_{\epsilon=0}=u, \tag{15a}$$

$$\frac{\mathrm{d}\bar{\phi}}{\mathrm{d}\epsilon}=\phi^2,\qquad \bar{\phi}|_{\epsilon=0}=\phi, \tag{15b}$$

$$\frac{\mathrm{d}\bar{g}}{\mathrm{d}\epsilon}=2\phi g,\qquad \bar{g}|_{\epsilon=0}=g. \tag{15c}$$

By solving the above initial value problem (3) for the enlarged perturbed mKdV systems, we get the following BT theorem.

Theorem 1 If u, ϕ and g are solutions of the enlarged perturbed mKdV systems (3), (7), and (13), then $\bar{u}$, $\bar{\phi}$, and $\bar{g}$ are also solutions of the enlarged perturbed mKdV systems

$$\bar{u}=u-\frac{\sqrt{6}\epsilon g}{\beta(\epsilon\phi-1)},\quad \bar{\phi}=\frac{\phi}{1-\epsilon\phi},\quad \bar{g}=\frac{g}{(1-\epsilon\phi)^2}, \tag{16}$$

where ϵ is an arbitrary group parameter.

4 Similarity Reductions Related with Nonlocal Symmetry

Thanks to the localization process, the nonlocal symmetry becomes the usual Lie point symmetry in the prolonged systems. The symmetry reduction related with the nonlocal symmetry can be performed by the Lie point symmetry method (18). These similar reduction solutions can not be obtained within the framework of the direct Lie's symmetry method.

Based on the symmetry definition, the prolonged systems are invariance under the transformation

$$u\to u+\epsilon\sigma^u,\quad \phi\to\phi+\epsilon\sigma^\phi,\quad g\to g+\epsilon\sigma^g, \tag{17}$$

with the infinitesimal parameter ϵ. The corresponding Lie point symmetries σ^k $(k=u,\ \phi,\ g)$ are the solutions of the linearized prolonged systems (1), (7), and (13), (18)

$$\sigma^u_t-\sigma^u_{xxx}+6(u\sigma^u)_x+\beta^2(u^2\sigma^u)_x+\sigma^u_x=0, \tag{18a}$$

$$\sigma^u-\frac{\sqrt{6}}{2\beta}\frac{\sigma^\phi_{xx}}{\phi_x}+\frac{\sqrt{6}}{2\beta}\frac{\sigma^\phi_x\phi_{xx}}{\phi_x^2}=0, \tag{18b}$$

$$\sigma^\phi_x-\sigma^g=0. \tag{18c}$$

The symmetry components σ^k $(k=u,\ \phi,\ g)$ are supposed to have the forms

$$\sigma^u=Xu_x+Tu_t-U, \tag{19a}$$

$$\sigma^\phi=X\phi_x+T\phi_t-\Phi, \tag{19b}$$

$$\sigma^g=Xg_x+Tg_t-G, \tag{19c}$$

where X, T, U, Φ, and G are functions of x, t, u, ϕ, and g. Substituting (19) into the symmetry equation (18) and requiring u, ϕ, and g to satisfy the prolonged systems, we get the over-determined equations with collecting the coefficients of u, ϕ, g, and its derivatives. Solving the over-determining equations leads to the infinitesimals X, T, U, Φ, and G as

$$X=\frac{C_1}{3}x-\frac{6C_1}{\beta^2}t+\frac{2C_1}{3}t+C_4,\quad T=C_1t+C_2,\quad \Phi=\frac{\beta C_3}{\sqrt{6}}\phi^2+C_5\phi-C_6,$$
$$G=\frac{\sqrt{6}C_3\beta}{3}\phi g-\frac{C_1}{3}g+C_5g,\quad U=-\frac{C_1}{3}u+C_3g-\frac{C_1}{\beta^2}, \tag{20}$$

where C_i $(i=1\cdots 6)$ are arbitrary constants. The symmetry will degenerate to the usual Lie point symmetry of the original (3) with $C_3=0$. The similarity solutions associated with the infinitesimal symmetries (20) can be given with the symmetry constraint condition $\sigma^k=0$ defined by (19). It is equivalent to solve the related characteristic equation

$$\frac{\mathrm{d}t}{T}=\frac{\mathrm{d}x}{X}=\frac{\mathrm{d}u}{U}=\frac{\mathrm{d}\phi}{\Phi}=\frac{\mathrm{d}g}{G}, \tag{21}$$

where X, T, U Φ, and G satisfy (20). Three subcases are distinguished concerning the solutions in the following.

Case I $C_1\neq 0,\ C_2\neq 0$. We take the parameter $\Delta=\sqrt{6\sqrt{6}\beta C_3C_6+9C_5^2}$ for simplicity. Two situations with $\Delta\neq 0$ and $\Delta=0$ are discussed, respectively.

When $\Delta\neq 0$, the similarity solutions are the following forms after solving out the characteristic equation (21)

$$\phi=-\frac{\Delta}{\sqrt{6}C_3\beta}\tanh\left(\frac{\Delta}{6C_1}(C_1\Phi+\ln(C_1t+C_2)\right)-\frac{C_5}{2\beta C_3},\tag{22a}$$

$$g=-\frac{G}{(C_1t+C_2)^{\frac{1}{3}}\cosh^2\left(\frac{\Delta}{6C_1}(C_1\Phi+\ln(C_1t+C_2)\right)},\tag{22b}$$

$$u=\frac{U}{(C_1t+C_2)^{\frac{1}{3}}}-\frac{6C_3}{\Delta(C_1t+C_2)^{\frac{1}{3}}}G\tanh\left(\frac{\Delta}{6C_1}(C_1\Phi+\ln(C_1t+C_2)\right)-\frac{3(2C_1t-C_2)}{8\beta^2(C_1t+C_2)},\tag{22c}$$

where three group invariant functions $\Phi=\Phi(\xi)$, $G=G(\xi)$, and $U=U(\xi)$ and the similarity variable

$$\xi=\frac{9(C_1t+3C_2)}{C_1\beta^2(C_1t+C_2)^{\frac{1}{3}}}+\frac{C_1x+3C_4-C_1t-3C_2}{C_1(C_1t+C_2)^{\frac{1}{3}}}.\tag{23}$$

It is obvious that C_1 cannot equal to zero for (23). The situation with $C_1=0$ will be studied in case II. Substituting (22) into (7), (8), and (13), the invariant functions G, U, and Φ satisfy the reduction systems

$$G=-\frac{\sqrt{6}\Delta^2}{36C_3\beta}\Phi_\xi,\tag{24a}$$

$$U=\frac{\sqrt{6}}{2\beta}\frac{\Phi_{\xi\xi}}{\Phi_\xi}-\frac{9(2C_1t+3C_2)}{8\beta^2(C_1t+C_2)^{\frac{2}{3}}},\tag{24b}$$

where Φ satisfies a three-order ordinary differential equation (ODE)

$$18\Phi_\xi\Phi_{\xi\xi\xi}-\Delta^2\Phi_\xi^4+6C_1\xi\Phi_\xi^2-27\Phi_{\xi\xi}^2-18\Phi_\xi=0,\tag{25}$$

which is just the second Painlevé (P_{II}) equation. Once one get the solution of Φ from (25), the explicit solution of (3) would be immediately obtained through (24) and (22). The following reduction theorem for the perturbed mKdV equation (1) can be obtained through above calculations.

Theorem 2 If Φ is a solution of the P_{II} (25), then u given by

$$u=-\frac{\Delta}{\sqrt{6}\beta(C_1t+C_2)^{\frac{1}{3}}}\Phi_\xi\tanh\left(\frac{\Delta}{6C_1}(C_1\Phi+\ln(C_1t+C_2)\right)-\frac{\sqrt{6}}{2\beta(C_1t+C_2)^{\frac{1}{3}}}\frac{\Phi_{\xi\xi}}{\Phi_\xi}-\frac{3}{\beta^2},\tag{26}$$

is also a solution of the perturbed mKdV equation with ξ determined by (23). It is obvious that (26) can be considered as the interaction solution between the explicit solitary wave and the general Painlevé II wave. The generic solutions of P_{II} (25) are transcendental. We can get the rational solutions and Bessel and Airy function solutions of P_{II} at special values of the parameters in (25) (7). The rich interaction solutions for the perturbed mKdV equation (3) can be thus obtained through (26).

For the rational solution of (25), it has a solution

$$\Phi=\ln(a\xi^3),\tag{27}$$

where the constants in (25) should satisfy $C_1=1$, $\Delta^2=1$. Then a rational solution of (3) is given using (26)

$$u=-\frac{\sqrt{6}(2aX\xi^3-1)}{\beta X\xi(aX\xi^3+1)}-\frac{3}{\beta^2},\quad X=(C_2+3t)^{\frac{1}{3}}.\tag{28}$$

When $\Delta=0$, following the similar steps of the above case $\Delta\neq 0$, the similarity solutions are

$$\phi=-\frac{\sqrt{6}C_5}{2\beta C_3}-\frac{\sqrt{6}C_1}{\beta C_3(C_1\Phi+\ln(C_1t+C_2))},\tag{29a}$$

$$g=\frac{G}{(C_1t+C_2)^{\frac{1}{3}}(C_1\Phi+\ln(C_1t+C_2))^2},\tag{29b}$$

$$u=\frac{U}{(C_1t+C_2)^{\frac{1}{3}}}-\frac{C_3G}{C_1(C_1+C_2)^{\frac{1}{3}}(C_1\Phi+\ln(C_1t+C_2))}-\frac{3}{3\beta^2},\tag{29c}$$

where $\xi=\frac{9(C_1t+3C_2)}{C_1\beta^2(C_1t+C_2)^{\frac{1}{3}}}-\frac{C_1(t-x)+3(C_2-C_4)}{C_1(C_1t+C_2)^{\frac{1}{3}}}$. Substituting (29) into (7), (8), and (13), the invariant functions Φ, G, and U satisfy the reduction systems

$$G=\frac{\sqrt{6}C_1^2}{\beta C_3}\Phi_\xi\tag{30a}$$

$$U=\frac{\sqrt{6}}{2\beta}\frac{\Phi_{\xi\xi}}{\Phi_\xi},\tag{30b}$$

where Φ satisfies an ODE

$$6\Phi_\xi-2C_1\xi\Phi_\xi^2-6\Phi_\xi\Phi_{\xi\xi\xi}+9\Phi_{\xi\xi}^2=0. \quad (31)$$

In the same way, we can get the reduction theorem.

Theorem 3 If Φ satisfies a special P_{II} (31), the explicit solution of perturbed mKdV equation is provided as

$$u=\frac{\sqrt{6}}{2\beta(C_1t+C_2)^{\frac{1}{3}}}\frac{\Phi_{\xi\xi}}{\Phi_\xi}-\frac{\sqrt{6}C_1}{\beta(C_1t+C_2)^{\frac{1}{3}}}\frac{\Phi_\xi}{(C_1\Phi+\ln(C_1t+C_2))}-\frac{3}{\beta^2}. \quad (32)$$

It is obvious that the solution (32) of the perturbed mKdV equation represents the interaction between a logarithm of singularity $\ln(C_1t+C_2)^{\frac{1}{3}}$ and a Painlevé II wave.

Case II $C_1=0$, $C_2\neq 0$. We also redefine the parameter $\Delta=\sqrt{6\sqrt{6}\beta C_3C_6+9C_5^2}$ for facilitating the later computation. Two situations with $\Delta\neq 0$ and $\Delta=0$ are given, respectively.

When $\Delta\neq 0$, similar procedure as case I, the similarity solutions are given after solving out the characteristic equations (21)

$$\phi=-\frac{\Delta}{\sqrt{6}\beta C_3}\tanh\left(\frac{\Delta}{6C_2}(\Phi'+t)\right)-\frac{3C_5}{\sqrt{6}\beta C_3}, \quad (33a)$$

$$g=-\frac{G'}{\cosh^2\left(\frac{\Delta}{6C_2}(\Phi'+t)\right)}, \quad (33b)$$

$$u=U'-\frac{6C_3}{\Delta}G'\tanh\left(\frac{\Delta}{6C_2}(\Phi'+t)\right). \quad (33c)$$

where $\xi=x-\frac{C_4}{C_2}t$. Similar as above case, we will discuss the situation with $C_1=C_2=0$ in case III. The reduction systems present as

$$G'=-\frac{\sqrt{6}\Delta^2}{36\beta C_2C_3}\Phi'_\xi \quad (34a)$$

$$U'=\frac{\sqrt{6}}{2\beta}\frac{\Phi'_{\xi\xi}}{\Phi'_\xi}-\frac{3}{\beta^2}, \quad (34b)$$

where Φ' satisfies the following ODE

$$\Phi'_\xi+\left(1-\frac{9}{\beta^2}-\frac{C_4}{C_2}\right)\Phi'^2_\xi-\Phi'_\xi\Phi'_{\xi\xi\xi}+\frac{3}{2}\Phi'^{\,2}_{\xi\xi}+\frac{\Delta^2}{18C_2^2}\Phi'^4_\xi=0. \quad (35)$$

The general solution of (35) can be solved out in terms of Jacobi elliptic functions (11).

Theorem 4 Once one get the solution of Φ' for (35), the explicit solution of the perturbed mKdV equation will be read as

$$u=\frac{\sqrt{6}}{2\beta}\frac{\Phi'_{\xi\xi}}{\Phi'_\xi}-\frac{\Delta}{\sqrt{6}\beta C_2}\Phi'_\xi\tanh\left(\frac{\Delta}{6C_2}(\Phi'+t)\right)-\frac{3}{\beta^2}. \quad (36)$$

This type of the solution (36) is just the explicit interactions between one soliton and cnoidal periodic waves. To show more clearly of this kind of solution, we give two special cases. Type 1. The special solution for the reduction (35) is

$$\Phi'=k_1\xi+a_1E_\pi(\mathrm{sn}(k\xi,\ m),\ n,\ m), \quad (37)$$

where k_1, a_1, k, n, and m are constants and E_π is the third incomplete elliptic integral. Using the theorem (4), the special soliton-cnoidal wave interaction solution u for (3) can be expressed by Jacobi elliptic functions

$$u=\frac{\Delta(nk_1S^2-a_1k-k_1)}{\sqrt{6}C_2\beta(1-nS^2)}T+\frac{\sqrt{6}na_1k^2SCD}{\beta(nS^2-1)(nk_1S^2-a_1k-k_1)}-\frac{3}{\beta^2} \quad (38)$$

where $S=\mathrm{sn}(k\xi,\ m)$, $C=\mathrm{cn}(k\xi,\ m)$, $D=\mathrm{dn}(k\xi,\ m)$, $T=\tanh\left(\frac{a_1\Delta}{6C_2}E_\pi(S,\ n,\ m)+\frac{\Delta k_1}{6C_2}\xi\right)$. Substituting (37) into (35) and vanishing all the coefficients of different powers of sn, the nontrivial solution for constants are

$$a_1=\frac{k_1(n-1)}{k},\quad C_4=C_2+2C_2k^2-\frac{9C_2}{\beta^2}+\frac{C_2}{2k_1}+\frac{C_2}{nk_1},$$
$$m=\frac{\sqrt{k_1^3n^3(4k^2k_1n-n+1)}}{2kk_1^2n^2},\quad \Delta=\frac{3\sqrt{k_1\beta^2C_2^2(4k^2nk_1+1)}}{n\beta k_1^2}. \quad (39)$$

Finally, the solution of the variable-coefficient perturbed mKdV equation (1) can be obtained using the symmetry group theorem (2). Figure 1 exhibits the special type of one kink soliton in the periodic wave background for the variable-coefficient perturbed mKdV equation (1). The parameters are $n=0.5$, $k_1=-1$, $k=1$, $C_2=2$, $\beta=2$, $\delta_1=-1$, $\delta_2=-1$, $B_1=1$, $B_2=2$, $h(t')=\cos(t')$. Figure 1a plots the solution (38) with a time-sliced view at $t=0$. Figure 1b is the corresponding three-dimensional image.

Type 2. Another special solution for the reduction equation (35) is

$$\Phi'=a_0\xi+a_1\ln(\mathrm{dn}(k\xi,\ m)-m\mathrm{cn}(k\xi,\ m)), \quad (40)$$

where a_0, a_1, k, and m are constants. The corresponding special soliton-cnoidal wave interaction solution u for (3) is given

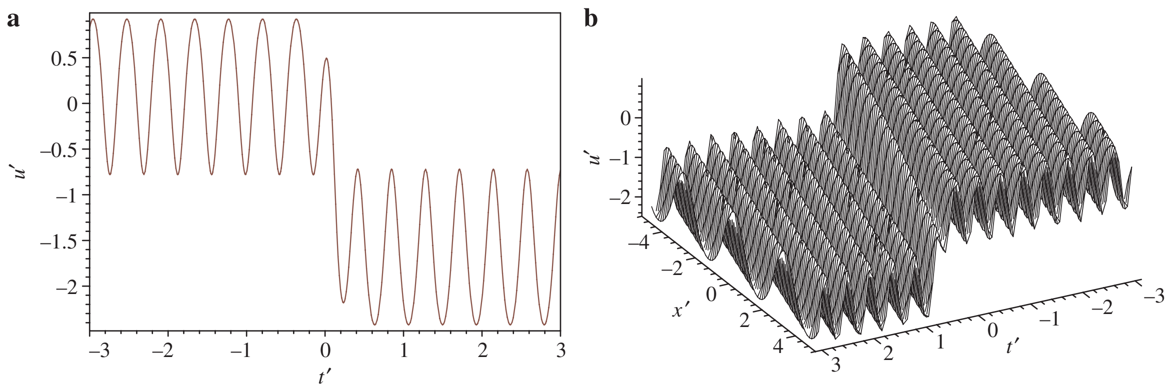

Figure 1: Plot of one kink soliton in the periodic wave background. The parameters are $n=0.5$, $k_1=-1$, $k=1$, $C_2=2$, $\beta=2$, $\delta_1=-1$, $\delta_2=-1$, $B_1=1$, $B_2=2$, $h(t')=\cos(t')$. (a) One-dimensional image at $t=0$. (b) The corresponding three-dimensional view.

$$u=\frac{\Delta(kma_1S-a_0)}{2\beta C_2}T-\frac{a_1mk^2CD}{2\beta(kma_1S+a_0)}+\frac{\alpha}{12\beta^2} \tag{41}$$

where $S=\mathrm{sn}(k\xi,\ m)$, $C=\mathrm{cn}(k\xi,\ m)$, $D=\mathrm{dn}(k\xi,\ m)$, $T=\tanh\left(\frac{a_1\Delta}{2C_2}\ln(D-mC)+\frac{\Delta a_0}{6C_2}\xi+\frac{\Delta}{6C_2}t\right)$. Substituting (40) into (35), vanishing all the coefficients of different powers of sn leads to the nontrivial solution for constants

$$\begin{aligned}
a_1&=\frac{a_0\sqrt{2a_0(m^2-1)}}{m},\\
C_2&=\frac{4a_0C_4\beta^2(m^2-1)}{4a_0\beta^2m^2+5\beta^2m^2-4a_0\beta^2-36a_0m^2-\beta^2+36a_0},\\
k&=\frac{1}{\sqrt{2a_0(m^2-1)}},\\
\Delta&=\frac{6C_4\beta^2m\sqrt{2(m^2-1)}}{\sqrt{a_0}\,(4a_0\beta^2m^2+5\beta^2m^2-4a_0\beta^2-36a_0m^2-\beta^2+36a_0)}.
\end{aligned} \tag{42}$$

When $\Delta=0$, the interaction solution of perturbed mKdV equation can be given similar as $\Delta\neq 0$. The following reduction theorem is obtained by omitting the tedious calculations.

Theorem 5 If the solution Φ' satisfies a elliptic function equation

$$2\Phi'_{\xi\xi\xi}\Phi'_{\xi}-2\Phi'_{\xi}-3\Phi'^2_{\xi\xi}+\left(\frac{2C_4}{C_2}+\frac{18}{\beta^2}-2\right)\Phi'^2_{\xi}=0. \tag{43}$$

The explicit solution of perturbed mKdV equation expresses as

$$u=-\frac{\sqrt{6}}{\beta}\frac{\Phi'_{\xi}}{\Phi'+t}+\frac{\sqrt{6}}{2\beta}\frac{\Phi'_{\xi\xi}}{\Phi'_{\xi}}-\frac{3}{\beta^2}, \tag{44}$$

where $\xi=x-\frac{C_4}{C_2}t$. Hence, this type of the solution (44) is just interactions among cnoidal periodic waves and rational waves for the perturbed mKdV equation. A nontrivial solution by Jacobi elliptic functions reads as

$$\Phi'=k_1\xi+a_1E_{\pi}(\mathrm{sn}(k\xi,\ m),\ n,\ m), \tag{45}$$

where k_1, a_1, k, n, and m are constants and E_{π} is the third incomplete elliptic integrals. The constants are

$$k_1=\frac{1}{4k^2m^2},\quad a_1=-\frac{1}{4k^3m^2},\quad C_2=\frac{C_4\beta^2}{4\beta^2k^2m^2-2\beta^2k^2+\beta^2-9},\quad n=m^2, \tag{46}$$

then (45) is a solution of (43).

Case III $C_1=C_2=0$. The similarity solutions are given

$$\phi=-\frac{\sqrt{4\sqrt{6}\beta C_3C_6+6C_5^2}}{2\beta C_3}\tanh\left(\frac{\sqrt{6\sqrt{6}\beta C_3C_6+9C_5^2}}{6C_4}(\Phi''(t)+x)\right)-\frac{\sqrt{6}C_5}{2\beta C_3}, \tag{47a}$$

$$g=-\frac{G''(t)}{\cosh^2\left(\frac{\sqrt{6\sqrt{6}\beta C_3C_6+9C_5^2}}{6C_4}(\Phi''(t)+x)\right)}, \tag{47b}$$

$$u=U''(t)-\frac{6C_3}{\sqrt{6\sqrt{6}\beta C_3C_6+9C_5^2}}G''(t)\tanh\left(\frac{\sqrt{6\sqrt{6}\beta C_3C_6+9C_5^2}}{6C_4}(\Phi''(t)+x)\right). \tag{47c}$$

Substituting (47) into (7), (8), and (13), the invariant functions $\Phi''(t)$, $G''(t)$, and $U''(t)$ satisfy the reduction systems

$$G''=\frac{C_6}{C_4}+\frac{\sqrt{6}C_5^2}{4\beta C_3 C_4} \tag{48a}$$

$$U''=-\frac{3}{\beta^2}, \tag{48b}$$

where Φ'' satisfies

$$\Phi_t''=\frac{9}{\beta^2}-\frac{C_5^2}{2C_4^2}-\frac{\sqrt{6}\beta C_3 C_6}{3C_4^2}-1, \tag{49}$$

From above the detail calculations, one get the following reduction theorem.

Theorem 6 If Φ'' is a solution of the reduction equation (49), then the explicit solution for the perturbed mKdV equation reads

$$u=\frac{\sqrt{4\sqrt{6}\beta C_3 C_6+6C_5^2}}{2\beta^2 C_4}\tanh\left(\frac{\sqrt{6\sqrt{6}\beta C_3 C_6+9C_5^2}}{6C_4}(\Phi''(t)+x)\right)-\frac{3}{\beta^2}. \tag{50}$$

Remark Though we study the constant-coefficient perturbed mKdV equation (3) by the reduction method, the interaction solutions of the variable-coefficient perturbed mKdV equation (1) can be obtained through the reduction theorem and symmetry group theorem (2).

5 Conclusions

We have changed the variable-coefficient perturbed mKdV equation to the constant-coefficient perturbed mKdV equation by the modified direct method. We obtain the nonlocal symmetry of the constant-coefficient perturbed mKdV equation by the truncated Painlevé analysis or the Möbious invariant form. To solve the first Lie's principle related by nonlocal symmetry, the nonlocal symmetry is localized the local Lie point symmetries for the prolonged system. We have also carried out a detailed invariance analysis of the prolonged systems. A variety of exact explicit interaction solutions among solitons and the rational solution hierarchy, Painlevé II waves, and cnoidal waves are obtained based on the nonlocal symmetry. The most interesting and meaningful solution among them is the interaction between soliton and cnoidal wave solution. We study this type interaction solution both in analytical and graphical ways.

Furthermore, the interaction solutions among different types of excitations for the perturbed mKdV equation can be explored with a consistent Riccati expansion method [31, 32]. For the perturbed mKdV equation (1), we may take the term $h(t')u'_{x'}$ as the perturbation form. One can develop the approximate symmetry reduction approach [33] to explore the perturbed mKdV equation (1). Recently, the integrable continuous and discrete nonlocal nonlinear Schrödinger equations with PT (parity-time) symmetry invariant are proposed [34–36]. The details on the interaction solutions for these nonlocal models are worthy of further study.

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References

[1] R. Beals, D. H. Sattinger, and J. Szmigielski, Inverse Probl. **15**, L1 (1999).
[2] H. H. Dai and Y. Li, J. Phys. A **38**, L685 (2005).
[3] H. H. Dai, Y. Li, and T. Su, J. Phys. A **42**, 055203 (2009).
[4] M. C. Ferreira, R. A. Kraenkel, and A. I. Zenchuk, J. Phys. A **32**, 8665 (1999).
[5] Y. Matsuno, J. Phys. Soc. Jpn. **74**, 1983 (2005).
[6] A. Parker, Chaos Soliton Fract. **41**, 1531 (2009).
[7] X. R. Hu, S. Y. Lou, and Y. Chen, Phys. Rev. E **85**, 056607 (2012).
[8] X. P. Chen, S. Y. Lou, C. L. Chen, and X. Y. Tang, Phys. Rev. E **89**, 043202 (2014).
[9] S. Y. Lou, X. R. Hu, and Y. Chen, J. Phys. A: Math. Theor. **45**, 155209 (2012).
[10] X. N. Gao, S. Y. Lou, and X. Y. Tang, J. High Energy Phys. **05**, 029 (2013).
[11] B. Ren, Phys. Scr. **90**, 065206 (2015).
[12] W. G. Cheng, B. Li, and Y. Chen, Commun. Nonlinear Sci. Numer. Simulat. **29**, 198 (2015).
[13] H. L. Liu and M. Slemrod, Appl. Math. Lett. **17**, 401 (2004).
[14] C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, Phys. Rev. Lett. **19**, 1095 (1967).
[15] V. B. Matveev and M. A. Salle, Darboux Transformations and Solitons, Springer, Berlin 1991.
[16] C. Rogers and W. K. Schief, Bäcklund and Darboux Transformations Geometry and Modern Applications in Soliton Theory, Cambridge texts in applied mathematics, Cambridge University Press, Cambridge 2002.
[17] R. Hirota, The direct Method in Soliton Theory, Cambridge University Press, Cambridge 2004.
[18] P. J. Olver, Application of Lie Group to Differential Equation, Springer, Berlin 1986.

[19] G. W. Bluman, A. F. Cheviakov, and S. C. Anco, Applications of Symmetry Methods to Partial Differential Equations, Springer, New York 2010.
[20] X. Y. Tang, S. Y. Lou, and Y. Zhang, Phys. Rev. E **66**, 046601 (2002).
[21] X. Lü, Commun. Nonlinear Sci. Numer. Simulat. **19**, 3969 (2014).
[22] A. M. Wazwaz, Appl. Math. Comput. **184**, 1002 (2007).
[23] S. Y. Lou and H. C. Ma, J. Phys. A: Math. Gen. **38**, L129 (2005).
[24] B. Ren, X. J. Xu, and J. Lin, J. Math. Phys. **50**, 123505 (2009).
[25] G. W. Bluman and J. D. Cole, J. Math. Mech. **18**, 1025 (1969).
[26] P. A. Clarkson and M. Kruskal, J. Math. Phys. **30**, 2201 (1989).
[27] V. Djordjevic and L. Redekopp, J. Phys. Oceanogr. **8**, 1016 (1978).
[28] B. A. Malomed and V. I. Shrira, Physica D **53**, 1 (1991).
[29] L. Wang, Y. T. Gao, and F. H. Qi Ann. Phys. **327**, 1974 (2012).
[30] J. Weiss, M. Tabor, and G. Carnevale, J. Math. Phys. **24**, 522 (1983).
[31] S. Y. Lou, Stud. Appl. Math. **134**, 372 (2015).
[32] B. Ren and J. Lin, Z. Naturforsch. **70**a, 539 (2015).
[33] X. Y. Jiao, R. X. Yao, and S. Y. Lou, J. Math. Phys. **49**, 093505 (2008).
[34] M. J. Ablowitz and Z. H. Musslimani, Phys. Rev. Lett. **110**, 064105 (2013).
[35] M. J. Ablowitz and Z. H. Musslimani, Phys. Rev. E **90**, 032912 (2014).
[36] M. J. Ablowitz and Z. H. Musslimani, Nonlinearity **29**, 915 (2016).