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The Integrability of an Extended Fifth-Order KdV Equation in 2+1 Dimensions: Painlevé Property, Lax Pair, Conservation Laws, and Soliton Interactions

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Abstract: In this article, we apply the singularity structure analysis to test an extended 2+1-dimensional fifth-order KdV equation for integrability. It is proven that the generalized equation passes the Painlevé test for integrability only in three distinct cases. Two of those cases are in agreement with the known results, and a new integrable equation is first given. Then, for the new integrable equation, we employ the Bell polynomial method to construct its bilinear forms, bilinear Bäcklund transformation, Lax pair, and infinite conversation laws systematically. The N-soliton solutions of this new integrable equation are derived, and the propagations and collisions of multiple solitons are shown by graphs.

Keywords: Conservation Laws; Integrability Test; Lax Pair; Soliton Solutions.

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1 Introduction

In the past decades, many powerful methods to solve integrable systems, such as the inverse scattering transformations (ISTs), Darboux transformations, Bäcklund transformations (BTs), symmetry reduction, Hirota bilinear method, and Painlevé analysis method, have been developed [1]. The usual real physical nonlinear systems can be treated as perturbations of the related integrable models. It is an interesting task to generate new integrable equations in the soliton theory.

Painlevé analysis method has been identified as one of the most effective tools in studying nonlinear evolution equations (NLEEs) [2]. This analysis not only can verify whether a given equation possesses the Painlevé property, but it also can search for all possible integrable cases of a NLEEs with general forms [3–9]. Moreover, the remarkable feature of Painlevé analysis is that a natural connection exists in relation to many integrable properties such as Hirota bilinear forms, Lax pairs, BTs, and various types of exact solutions [10–15]. Hirota bilinear method is a direct method to derive the multiple-soliton solutions, quasi-periodic wave solutions, bilinear BT, and other properties of a given NLEEs [16–19]. The crucial step of this method relies on a particular skill by choosing suitable variable transformations, but there is no general rule to find the transformations. Using the links between the Bell polynomials and Hirota D-operators, Lambert and his coworkers have established a direct method to derive bilinear forms, bilinear BT and Lax pairs of soliton equations systematically [20, 21]. The Bell polynomials approach has been extended to the variable-coefficient, supersymmetric, and discrete NLEEs [22–39].

In this work, we will employ the Painlevé analysis and Bell polynomial method to study an extended fifth-order Korteweg–de Vries (KdV) equation

$$\begin{aligned} & u_t+\lambda_1 u_{xxxxx}+\lambda_2 uu_{xxx}+\lambda_3 u_x u_{xx}+\lambda_4 u^2 u_x+\lambda_5 u_{xxy}+\lambda_6 \partial_x^{-1} u_{yy} \\ & \quad +\lambda_7 uu_y+\lambda_8 u_x \partial_x^{-1} u_y=0, \end{aligned} \tag{1}$$

which is an important higher-order extension of the famous KdV equation in fluid dynamics. The KdV-type equations have been widely recognized as ubiquitous mathematical models for describing some nonlinear phenomena in many branches o f physics. The higher-order dispersion and nonlinear terms must be taken into account in some complicated situations such as the surface and internal waves, gravity–capillary waves, longitudinal waves in microstructured solids, and so on. In (1), $u=u(x, y, t)$ and $\lambda_j (j=1, \ldots, 8)$ are real parameters. Much research has been done for its particular cases. However, the integrability of the generalized equation (1) has not been investigated.

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Equation (1) includes a number of important NLEEs as its special cases. Konopelchenko and colleagues [40, 41] constructed and studied two integrable equations:

$$u_t = u_{xxxxx} + 5uu_{xxx} + 5u_x u_{xx} + 5u^2 u_x + 5u_{xxy} - 5\partial_x^{-1} u_{yy} + 5uu_y + 5u_x \partial_x^{-1} u_y, \tag{2}$$

$$u_t = u_{xxxxx} + 5uu_{xxx} + \frac{25}{2} u_x u_{xx} + 5u^2 u_x + 5u_{xxy} - 5\partial_x^{-1} u_{yy} + 5uu_y + 5u_x \partial_x^{-1} u_y, \tag{3}$$

which are the (2+1)-dimensional integrable generalization of the Sawada–Kotera (SK) equation and the Kaup–Kuperschmidt (KK) equation, respectively. The Lax pair, conservation laws, symmetry structure, and multiple-soliton solutions of (2) and (3) have been investigated in [42–44].

When $\lambda_1 = 1/36$, $\lambda_2 = \lambda_3 = 5/12$, $\lambda_4 = 5/4$, $\lambda_5 = \lambda_6 = -5/36$, $\lambda_7 = \lambda_8 = -5/12$, (1) reduces to the (2+1)-dimensional Caudrey–Dodd–Gibbon–Kotera–Sawada equation:

$$36u_t = -u_{xxxxx} - 15(uu_{xx})_x - 45u^2 u_x + 5u_{xxy} + 5\partial_x^{-1} u_{yy} + 15uu_y + 15u_x \partial_x^{-1} u_y. \tag{4}$$

Its symmetry constraints and quasi-periodic solutions have been constructed [45, 46].

When $\lambda_1 = 1/9$, $\lambda_2 = 5/3$, $\lambda_3 = 25/6$, $\lambda_4 = 5$, $\lambda_5 = -\lambda_6 = -5\sigma/9$, $\lambda_7 = \lambda_8 = 5\sigma/3$, (1) reduces to the (2+1)-dimensional KK equation with another form:

$$9u_t + u_{xxxxx} + 15uu_{xxx} + \frac{75}{2} u_x u_{xx} + 45u^2 u_x + 5\sigma u_{xxy} - 5\sigma(\partial_x^{-1} u_{yy}) + 15\sigma uu_y + 15\sigma u_x (\partial_x^{-1} u_y) = 0. \tag{5}$$

Its multiple-soliton solutions and symmetry algebras have been presented [47].

Using gauge transformation, He and Li [48] studied two other particular cases of (1):

$$u_t = \frac{5}{9}\left(\partial_x^{-1} u_{yy} + 3u_x \partial_x^{-1} u_y - \frac{1}{5} u_{xxxxx} - 3uu_{xxx} - 3u_x u_{xx} - 9u^2 u_x + u_{xxy} + 3uu_y \right), \tag{6}$$

$$u_t = \frac{5}{9}\left(\partial_x^{-1} u_{yy} + 3u_x \partial_x^{-1} u_y - \frac{1}{5} u_{xxxxx} - 3uu_{xxx} - \frac{15}{2} u_x u_{xx} - 9u^2 u_x + u_{xxy} + 3uu_y \right). \tag{7}$$

Up to now, we have already known that (2)–(7) are integrable. The natural and important question is under what constraints on parameters λ_j ($j = 1, \ldots, 8$) is (1) integrable, i.e. does another new integrable subcase of (1) exist? In the following section, we will perform the Painlevé test for the extended equation (1) to derive all possible integrable subcases. In Section 3, by employing the Bell polynomials method, we systematically construct the bilinear forms, bilinear BT and Lax pair for the new integrable equation. In Section 4, the N-soliton solutions and their collisions are discussed. Some conclusions are given in the final section.

2 Integrability Test

To verify whether a given NLEEs possesses the Painlevé property, one may use different methods, such as the Weiss–Tabor–Carnevale (WTC) method, Kruskal's simplification for the WTC method, Conte invariant method, Lou's extended method, and so on. Here we apply the WTC–Kruskal's approach to carry out the Painlevé integrability test for (1). Taking $u_y = v_x$, (1) becomes the following coupled system:

$$\begin{aligned} &u_t + \lambda_1 u_{xxxxx} + \lambda_2 uu_{xxx} + \lambda_3 u_x u_{xx} + \lambda_4 u^2 u_x + \lambda_5 u_{xxy} + \lambda_6 v_y \\ &\quad + \lambda_7 uu_y + \lambda_8 u_x v = 0, \\ &u_y = v_x. \end{aligned} \tag{8}$$

To simplify the calculations, the Kruskal's ansatz $\phi(x, y, t) = x + \psi(y, t)$ is adopted here.

Inserting $u(x, y, t) = u_0 \phi^{\alpha_1}$, $v(x, y, t) = v_0 \phi^{\alpha_2}$ into the leading terms of (8), we have

$$\begin{aligned} &\alpha_1 = \alpha_2 = -2, \quad \lambda_4 u_0^2 + (6\lambda_3 + 12\lambda_2) u_0 + 360\lambda_1 = 0, \\ &v_0 - u_0 \psi_y = 0. \end{aligned} \tag{9}$$

To determine the resonant points, we set

$$\begin{aligned} &u(x, y, t) = u_0 \phi^{-2} + u_j \phi^{j-2}, \; v(x, y, t) = v_0 \phi^{-2} + v_j \phi^{j-2}, \\ &u_j = u_j(y, t), \; v_j = v_j(y, t). \end{aligned}$$

Substituting the truncated expansions into the dominant terms of (8) yields a polynomial equation in j,

$$\begin{aligned} &(j+1)(j-2)(j-6)[\lambda_1 j^3 - 15\lambda_1 j^2 + (\lambda_2 u_0 + 86\lambda_1) j \\ &\quad - 2\lambda_3 u_0 - 4\lambda_2 u_0 - 240\lambda_1] u_j v_j = 0, \end{aligned} \tag{10}$$

where u_0 is given by (9). From (10), we know that three resonant points occur at –1, 2, 6. Other three undetermined resonances, say j_1, j_2, j_3, must be integers, and this is possible if

$$\begin{aligned} &j_1 + j_2 + j_3 = 15, \quad \lambda_1 (j_1 j_2 + j_1 j_3 + j_2 j_3) = 86\lambda_1 + \lambda_2 u_0, \\ &\lambda_1 j_1 j_2 j_3 = 4\lambda_2 u_0 + 2\lambda_3 u_0 + 240\lambda_1. \end{aligned} \tag{11}$$

Solving the first equation of (11) yields $j_3 = 15 - j_1 - j_2$. The considered branch should be generic, i.e. five resonant points lie in positive integer positions. Then the positive integer conditions for j_1, j_2, j_3 result in 16 different subcases: (1) [1, 1, 13], (2) [1, 2, 12], (3) [1, 3, 11], (4) [1, 4, 10], (5) [1, 5, 9], (6) [1, 6, 8] (7) [1, 7, 7], (8) [2, 3, 10], (9) [2, 4, 9], (10) [2, 5, 8], (11) [2, 6, 7], (12) [3, 3, 9], (13) [3, 4, 8], (14) [3, 5, 7], (15) [4, 4, 7], (16) [4, 5, 6].

Testing all the resonant conditions for subcases (1)–(16) yields possible Painlevé integrable models under some constraints on parameters λ_j. For instance, for case (10), solving (9) and (11) with respect to λ_1, λ_2, λ_3, u_0, and v_0 will lead to

$$\lambda_2=\frac{\lambda_3}{2},\quad \lambda_4=\frac{3\lambda_3^2}{40\lambda_1},\quad u_0=-\frac{40\lambda_1}{\lambda_3},\quad v_0=-\frac{40\lambda_1\psi_y}{\lambda_3}. \tag{12}$$

Six resonant points occur at –1, 2, 2, 5, 6, 8. To test the resonant conditions, one may take

$$u(x,y,t)=\sum_{j=0}^{8}u_j\phi^{j-2},\; v(x,y,t)=\sum_{j=0}^{8}v_j\phi^{j-2}. \tag{13}$$

Inserting expansions (13) into (8) and setting the coefficients of $\{\phi^{-6}, \phi^{-2}\}$ to zero, we get $u_1=v_1=0$.

For $j=2$, collecting the coefficients of $\{\phi^{-5}, \phi^{-1}\}$ will lead to

$$\lambda_1(10\lambda_1\lambda_7+10\lambda_1\lambda_8-3\lambda_3\lambda_5)\psi_y=0, \tag{14}$$

where parameter λ_1 cannot be taken as zero, thus condition (14) is satisfied if and only if

$$\lambda_8=\frac{3\lambda_3\lambda_5-10\lambda_1\lambda_7}{10\lambda_1}. \tag{15}$$

For $j=3$ and $j=4$, setting the coefficients of $\{\phi^{-4}, \phi^{0}\}$ and $\{\phi^{-3}, \phi^{1}\}$ to zero, we get

$$u_3=0,\; v_3=u_{2,y},$$
$$u_4=\frac{40\lambda_1(\lambda_6\psi_y^2+\lambda_7u_2\psi_y+\psi_t-\lambda_7v_2)+3\lambda_3(\lambda_3u_2^2+4\lambda_5)}{40\lambda_1\lambda_3},$$
$$v_4=u_4\psi_y.$$

For $j=5$, the corresponding resonant condition is obtained as

$$3u_{2,y}(\lambda_3\lambda_5-5\lambda_1\lambda_7)-5\lambda_1\lambda_6\psi_{yy}=0, \tag{16}$$

where parameter λ_1 cannot be taken as zero; thus, condition (16) is consistent if and only if

$$\lambda_6=0,\; \lambda_7=\frac{\lambda_3\lambda_5}{5\lambda_1}. \tag{17}$$

Together with conditions (12), (15), and (17), the final parametric constraints read

$$\lambda_2=\frac{\lambda_3}{2},\; \lambda_4=\frac{3\lambda_3^2}{40\lambda_1},\; \lambda_6=0,\; \lambda_7=\frac{\lambda_3\lambda_5}{5\lambda_1},\; \lambda_8=\frac{\lambda_3\lambda_5}{10\lambda_1}. \tag{18}$$

Under constraints (18), the coefficients u_2, v_2, u_5, u_6, u_8, ψ in (13) are arbitrary functions of $\{y, t\}$, which indicate that the conditions at all non-negative resonant points are verified and the remaining coefficients in (13) are

$$u_1=v_1=u_3=0,\; v_3=u_{2,y},$$
$$u_4=\frac{3\lambda_3^2u_2^2+8\lambda_3\lambda_5u_2\psi_y+40\lambda_1\psi_t+4\lambda_3\lambda_5v_2}{40\lambda_1\lambda_3},\; v_4=u_4\psi_y,$$
$$u_7=\frac{\lambda_3\lambda_5u_2u_{2,y}+5\lambda_1(u_{2,t}-14\lambda_5u_5\psi_y-3\lambda_3u_2u_5-6\lambda_5u_{4,y}+8\lambda_5v_5)}{400\lambda_1^2},$$
$$v_5=u_5\psi_y+\frac{u_{4,y}}{3},\quad v_6=u_6\psi_y+\frac{u_{5,y}}{4},\quad v_7=\frac{u_{6,y}}{5}+u_7\psi_y,$$
$$v_8=\frac{u_{7,y}}{6}+u_8\psi_y.$$

Under parameter constraints (18), there is another secondary branch for (8). For the secondary branch, the leading order analysis leads to $\alpha_1=\alpha_2=-2$, $u_0=-40\lambda_1/\lambda_3$, $v_0=-40\lambda_1\psi_y/\lambda_3$. Six resonant points occur at –3, –1, 2, 6, 8, and 10. To verify the resonant conditions, we suppose

$$u(x,y,t)=\sum_{j=0}^{10}u_j\phi^{j-2},\quad v(x,y,t)=\sum_{j=0}^{10}v_j\phi^{j-2}. \tag{19}$$

Substituting it into (8) and collecting the different powers of ϕ will lead to

$$u_1=v_1=u_3=u_5=0,\; u_2=-\frac{\lambda_5\psi_y}{2\lambda_3},\; v_3=u_{2,y},\; u_4=\frac{10\lambda_1\psi_t-\lambda_5^2\psi_y^2+\lambda_3\lambda_5v_2}{70\lambda_1\lambda_3},\; v_4=u_4\psi_y,\quad v_5=\frac{u_{4,y}}{3},\quad v_6=u_6\psi_y,$$
$$u_7=\frac{\lambda_5(20\lambda_1\psi_{yt}+\lambda_3\lambda_5v_{2,y}-6\lambda_5^2\psi_y\psi_{yy})}{1200\lambda_1^2\lambda_3},\quad v_7=\frac{u_{6,y}}{5}+u_7\psi_y,\quad v_8=u_8\psi_y+\frac{u_{7,y}}{6},$$
$$u_9=\frac{5\lambda_1(u_{4,t}-24\lambda_3u_2u_7)+\lambda_3\lambda_5(v_3u_4+u_2u_{4,y}+u_4u_{2,y})-60\lambda_1\lambda_5(u_{6,y}+6u_7\psi_y-2v_7)}{1800\lambda_1^2},\quad v_9=\frac{u_{8,y}}{7}+u_9\psi_y,\quad v_{10}=\frac{u_{9,y}}{8}+u_{10}\psi_y,$$

where the coefficients v_2, u_6, u_8, u_{10} in (19) are arbitrary functions of $\{y, t\}$.

The above analysis for case (10) yields the first type of integrable model.

Integrable model I (new integrable model):

$$u_t+\lambda_1 u_{xxxxx}+\frac{\lambda_3}{2}uu_{xxx}+\lambda_3 u_x u_{xx}+\frac{3\lambda_3^2}{40\lambda_1}u^2u_x+\lambda_5 u_{xxy}$$
$$+\frac{\lambda_3\lambda_5}{5\lambda_1}uu_y+\frac{\lambda_3\lambda_5}{10\lambda_1}u_x\partial_x^{-1}u_y=0, \tag{20}$$

where parameters λ_1 and λ_3 are arbitrary nonzero constants.

Performing the similar analysis for other cases, one can obtain other two integrable subcases of (1). For simplicity, the detailed computations are omitted.

Integrable model II (2+1-dimensional generalized SK equation):

$$u_t+\lambda_1 u_{xxxxx}+\lambda_3 uu_{xxx}+\lambda_3 u_x u_{xx}+\frac{\lambda_3^2}{5\lambda_1}u^2u_x+\lambda_5 u_{xxy}-\frac{\lambda_5^2}{5\lambda_1}(\partial_x^{-1}u_{yy})$$
$$+\frac{\lambda_3\lambda_5}{5\lambda_1}(uu_y+u_x\partial_x^{-1}u_y)=0, \tag{21}$$

where parameters λ_1 and λ_3 are arbitrary nonzero constants. For model (21), there is a primary branch with the resonances located at $\{-1, 2, 2, 3, 6, 10\}$ and a secondary branch with the resonances located at $\{-2, -1, 2, 5, 6, 12\}$, and all the resonant conditions are consistent.

Integrable model III (2+1-dimensional generalized KK equation):

$$u_t+\lambda_1 u_{xxxxx}+\frac{2\lambda_3}{5}uu_{xxx}+\lambda_3 u_x u_{xx}+\frac{4\lambda_3^2}{125\lambda_1}u^2u_x+\lambda_5 u_{xxy}$$
$$-\frac{\lambda_5^2}{5\lambda_1}(\partial_x^{-1}u_{yy})+\frac{2\lambda_3\lambda_5}{25\lambda_1}(uu_y+u_x\partial_x^{-1}u_y)=0, \tag{22}$$

where parameters λ_1 and λ_3 are arbitrary nonzero constants. For model (22), there is a primary branch with the resonances located at $\{-1, 2, 3, 6, 7\}$ and a secondary branch with the resonances located at $\{-7, -1, 2, 6, 10, 12\}$, and all the resonant conditions are consistent.

If we take appropriate values for λ_j $(j=1, 3, 5)$, (21) and (22) can be reduced to the particular cases (2)–(7), which are exactly the 2+1-dimensional integrable generalizations of the SK and KK equations, respectively. Equation (20) is a new fifth-order integrable model in 2+1 dimensions and has not been reported in literature.

3 Bell Polynomials Method for (20)

In this section, we will employ the Bell polynomials method to study the new integrable equation (20).

3.1 The Bilinear Representation and N-Soliton Solutions

The standard Painlevé truncated expansion of (20) reads as

$$u=\frac{40\lambda_1}{\lambda_3}(\ln\phi)_{xx}+u_2, \tag{23}$$

where $\phi=\phi(x, y, t)$ and u_2 is the seed solution. First, we set $q=2\ln\phi$ and $u_2=0$. Then applying transformation (23) into (20) and integrating once with respect to x, one gets

$$\begin{aligned}&q_{xt}+\lambda_1(q_{6x}+10q_{xx}q_{4x}+5q_{3x}^2+10q_{xx}^3)\\&\quad+\lambda_5(q_{3x,y}+2q_{xx}q_{xy}+\partial_x^{-1}\partial_y q_{xx}^2)=0,\end{aligned} \tag{24}$$

which can be rewritten as

$$q_{xt}+\frac{\lambda_1}{6}(q_{6x}+15q_{xx}q_{4x}+15q_{xx}^3)+\frac{5\lambda_1}{6}(\partial_x^2+3q_{xx})(q_{4x}+3q_{xx}^2)$$
$$+\frac{2\lambda_5}{3}(q_{3x,y}+3q_{xx}q_{xy})+\frac{\lambda_5}{3}\partial_x^{-1}\partial_y(q_{4x}+3q_{xx}^2)=0. \tag{25}$$

To write (25) as the combination of $\mathscr{P}$-polynomial expression, one may introduce an auxiliary variable z and impose a subsidiary constraint condition $q_{4x}+3q_{xx}^2+q_{xz}=0$. Then the $\mathscr{P}$-polynomial expression can be obtained

$$\begin{aligned}&\mathscr{P}_{4x}(q)+\mathscr{P}_{xz}(q)=0,\\&\mathscr{P}_{xt}(q)+\frac{\lambda_1}{6}\mathscr{P}_{6x}(q)-\frac{5\lambda_1}{6}\mathscr{P}_{3x,z}(q)+\frac{2\lambda_5}{3}\mathscr{P}_{3x,y}(q)-\frac{\lambda_5}{3}\mathscr{P}_{yz}(q)=0.\end{aligned} \tag{26}$$

Via the transformation $q=2\ \ln\phi$, the $\mathscr{P}$-polynomial system (26) produces the bilinear forms

$$\begin{aligned}&[6D_xD_t+\lambda_1D_x^6-5\lambda_1D_x^3D_z+4\lambda_5D_x^3D_y-2\lambda_5D_yD_z]\phi\cdot\phi=0,\\&(D_x^4+D_xD_z)\phi\cdot\phi=0.\end{aligned} \tag{27}$$

Based on the bilinear representation (27), the N-soliton solutions of (20) can be expressed as

$$\begin{aligned}&u=\frac{40\lambda_1}{\lambda_3}\left[\ln\left(\sum_{\nu=0,1}\exp\left(\sum_{j=1}^{N}\nu_j\xi_j+\sum_{1\le i<j}^{N}\nu_i\nu_jA_{ij}\right)\right)\right]_{xx},\\&\xi_j=k_jx+l_jy-k_j^2(\lambda_1k_j^3+\lambda_5l_j)t+\xi_{j0},\\&\exp(A_{ij})=\left(\frac{k_i-k_j}{k_i+k_j}\right)^2,\ i<j,\ i,\ j=1,\ 2,\ \cdots,\end{aligned} \tag{28}$$

where k_j, l_j, ξ_j^0 $(j=1, \cdots N)$ are arbitrary real constants, $\sum_{\nu=0,1}$ is the summation over all possible combinations of $\nu_1=0, 1$, $\nu_2=0, 1, \cdots, \nu_N=0, 1$, and $\sum_{1\le i<j}^{N}$ is the summation over all possible pairs (i, j) chosen from the set $(1, 2, \cdots, N)$, with the condition that $1\le i<j$.

3.2 BT and Lax Pair

Following the procedure in [11], one can find the bilinear BT and Lax pair for (20). To this end, let $\bar{q}=2\ln G$ and $q=2\ln F$ be two different solutions of (24), respectively, and we then have

$$\begin{aligned}E(\bar{q})-E(q)&=(\bar{q}-q)_{xt}+\lambda_1[(\bar{q}-q)_{6x}+10(\bar{q}_{xx}\bar{q}_{4x}-q_{xx}q_{4x})\\&+5(\bar{q}_{3x}^2-q_{3x}^2)+10(\bar{q}_{xx}^3-q_{xx}^3)]+\lambda_5[(\bar{q}_{3x,y}-q_{3x,y})\\&+2(\bar{q}_{xx}\bar{q}_{xy}-q_{xx}q_{xy})+\partial_x^{-1}\partial_y(\bar{q}_{xx}^2-q_{xx}^2)],\end{aligned}\tag{29}$$

which can be regarded as an ansatz for a bilinear BT under some constraints. To find such constraint, by letting $(\bar{q}-q)/2=v$, $(\bar{q}+q)/2=w$, we obtain

$$\begin{aligned}\frac{E(\bar{q})-E(q)}{2}&=v_{xt}+\lambda_1[v_{6x}+10v_{xx}w_{4x}+10w_{xx}v_{4x}+10v_{3x}w_{3x}\\&+10v_{xx}(3w_{xx}^2+v_{xx}^2)]+\lambda_5[v_{3x,y}+2v_{xy}w_{xx}+2w_{xy}v_{xx}\\&+2\partial_x^{-1}\partial y(v_{xx}w_{xx})]\\&=\frac{\partial}{\partial x}(\mathscr{Y}_t(v)+\lambda_1\mathscr{Y}_{5x}(v,w)+\lambda_5\mathscr{Y}_{xxy}(v,w))+5R(v,w),\end{aligned}\tag{30}$$

where

$$\begin{aligned}R(v,w)&=\lambda_1(v_{xx}w_{4x}+3v_{xx}w_{xx}^2-v_xw_{5x}-2v_x^2v_{4x}-2v_x^3w_{3x}-v_x^4v_{xx}\\&+2v_{xx}^3-4v_xv_{xx}v_{3x}-6v_xw_{xx}w_{3x}-6v_x^2v_{xx}w_{xx})\\&+\frac{\lambda_5}{5}(v_{xy}w_{xx}-v_{xx}v_{xy}-v_yw_{3x}-2v_xw_{xxy}-2v_xv_yv_{xx})\\&+\frac{2\lambda_5}{5}\partial_x^{-1}\partial y(v_{xx}w_{xx}).\end{aligned}$$

If we take $v_x^2+w_{xx}=\mu$, $R(v,w)$ is simplified as

$$R(v,w)=\frac{\partial}{\partial x}\left[3\mu^2\lambda_1\mathscr{Y}_x(v)+\frac{3\mu\lambda_5}{5}\mathscr{Y}_y(v)\right],$$

one can obtain a linear system in term of $\mathscr{Y}$-polynomials

$$\begin{aligned}&\mathscr{Y}_{xx}(v,w)-\mu=0,\\&\frac{\partial}{\partial x}[\mathscr{Y}_t(v)+\lambda_1\mathscr{Y}_{5x}(v,w)+\lambda_5\mathscr{Y}_{xxy}(v,w)\\&\quad+15\mu^2\lambda_1\mathscr{Y}_x(v)+3\mu\lambda_5\mathscr{Y}_y(v)]=0.\end{aligned}\tag{31}$$

Then, a new BT of (20) is obtained as follows

$$\begin{aligned}&(D_x^2-\mu)F\cdot G=0,\\&(D_t+\lambda_1D_x^5+\lambda_5D_x^2D_y+15\mu^2\lambda_1D_x+3\mu\lambda_5D_y-\sigma)F\cdot G=0.\end{aligned}\tag{32}$$

By introducing the transformations $v=\ln\psi$ and $w=v+q$, (31) can be reduced to the linear system,

$$\begin{aligned}&\psi_{xx}+q_{xx}\psi-\mu\psi=0,\\&\psi_t+\lambda_1\psi_{5x}+10\lambda_1q_{xx}\psi_{3x}+\lambda_5\psi_{xxy}+[15\mu^2\lambda_1+2\lambda_5q_{xy}\\&\quad+5\lambda_1(q_{4x}+3q_{xx}^2)]\psi_x+\lambda_5(q_{xx}+3\mu)\psi_y-\sigma\psi=0,\end{aligned}\tag{33}$$

whose integrability condition gives (20) by replacing q_{xx} by $\lambda_3u/20\lambda_1$, which indicates that system (33) can be regarded as the Lax pair of (20).

3.2.1 Infinite Conservation Laws

By introducing a new potential function $\eta=(\bar{q}_x-q_x)/2$, one can get the relations

$$v_x=\eta,\quad w_x=q_x+\eta.\tag{34}$$

Substituting (34) into the system (31) will lead to a Riccati-type equation,

$$\eta_x+\eta^2+q_{xx}=\mu,\tag{35}$$

and a divergence-type equation,

$$\begin{aligned}&\eta_t+\partial x\{\lambda_1[\eta_{4x}+6\eta\eta_x^2+5\eta\eta_{3x}+4(2\mu+3q_{xx})\eta\eta_x\\&+(5q_{4x}+6q_{2x}^2+8\mu q_{2x}^2+16\mu^2)\eta+10\mu\eta_{xx}]+2\lambda_5q_{xy}\eta\}\\&+\partial y[\lambda_5(2\eta\eta_x+\eta_{xx}+4\mu\eta)]=0,\end{aligned}\tag{36}$$

where we have used (35) to obtain (36).

Next, we take $\eta=\bar{\eta}+\epsilon$ and $\mu=\epsilon^2$, then (35) and (36) reduce to

$$\bar{\eta}_x+\bar{\eta}^2+2\epsilon\bar{\eta}+q_{xx}=0,\tag{37}$$

$$\begin{aligned}&\bar{\eta}_t+\partial x\{\lambda_1[\bar{\eta}_{4x}+6\bar{\eta}\bar{\eta}_x^2+6\epsilon\bar{\eta}_x^2+5\bar{\eta}\bar{\eta}_{3x}+5\epsilon\bar{\eta}_{3x}+8\epsilon^2\bar{\eta}\bar{\eta}_x+8\epsilon^3\bar{\eta}_x\\&+12q_{xx}\bar{\eta}\bar{\eta}_x+12\epsilon q_{xx}\bar{\eta}_x+(5q_{4x}+6q_{2x}^2+8\epsilon^2q_{2x}^2+16\epsilon^4)\bar{\eta}\\&+(5\epsilon q_{4x}+6\epsilon q_{2x}^2+8\epsilon^3q_{2x}^2)+10\epsilon^2\bar{\eta}_{xx}]+2\lambda_5(q_{xy}\bar{\eta}+\epsilon\bar{\eta}_y+\epsilon q_{xy})\}\\&+\partial y[\lambda_5(2\bar{\eta}\bar{\eta}_x+\bar{\eta}_{xx}+4\epsilon^2\bar{\eta})]=0.\end{aligned}\tag{38}$$

Inserting the expansion

$$\bar{\eta}=\sum_{n=1}^{\infty}\mathscr{I}_n(q,q_x,\cdots)\epsilon^{-n}\tag{39}$$

into (37), and equating the coefficients for power of ϵ, we obtain the recursion formulae for $\mathscr{I}_n$

$$\mathscr{I}_{n+1}=-\frac{1}{2}\left(\mathscr{I}_{n,x}+\sum_{k=1}^{n}\mathscr{I}_{k}\mathscr{I}_{n-k}\right),\ n=1,\ 2,\ \cdots.$$

The first and second conserved densities turn out to be

$$\mathscr{I}_1=-\frac{1}{2}q_{xx}=-\frac{\lambda_3}{40\lambda_1}u,\quad \mathscr{I}_2=-\frac{1}{2}\mathscr{I}_{1,x}=\frac{\lambda_3}{80\lambda_1}u_x.$$

Subsequently, inserting expansion (39) into (38) yields

$$\mathscr{I}_{n,t}+\mathscr{F}_{n,x}+\mathscr{G}_{n,y}=0,\ n=1,\ 2,\ 3,\ \cdots, \tag{40}$$

where the first fluxes $\mathscr{F}_n$ read as

$$\mathscr{F}_1=-\frac{\lambda_3[40\lambda_1^2u_{4x}+10\lambda_1\lambda_3(u_x^2+2uu_{xx})+\lambda_3^2u^3+4\lambda_3\lambda_5u\partial_x^{-1}u_y-40\lambda_1\lambda_5u_{xy}]}{1600\lambda_1^2},$$

$$\begin{aligned}\mathscr{F}_n=\lambda_1[&\mathscr{I}_{n,4x}+6\sum_{j+k+s=n}\mathscr{I}_j\mathscr{I}_{k,x}\mathscr{I}_{s,x}+6\sum_{j+k=n+1}\mathscr{I}_{j,x}\mathscr{I}_{k,x}+5\sum_{j+k=n}\mathscr{I}_j\mathscr{I}_{k,3x}+5\mathscr{I}_{n+1,3x}+8\sum_{j+k=n+2}\mathscr{I}_j\mathscr{I}_{k,x}+8\mathscr{I}_{n+3,x}\\&+12q_{xx}\sum_{j+k=n}\mathscr{I}_j\mathscr{I}_{k,x}+12q_{xx}\mathscr{I}_{n+1,x}+(5q_{4x}+6q_{2x}^2)\mathscr{I}_n+8q_{2x}\mathscr{I}_{n+2}+16\mathscr{I}_{n+4}+10\mathscr{I}_{n+2,xx}]+2\lambda_5(q_{xy}\mathscr{I}_n+\mathscr{I}_{n+1,y}),\quad n=2,\ 3,\ \cdots,\end{aligned}$$

and the second fluxes $\mathscr{G}_n$ are given by

$$\mathscr{G}_1=\frac{\lambda_3\lambda_5(40\lambda_1u_{xx}+\lambda_3u^2)}{800\lambda_1^2},\quad \mathscr{G}_n=\lambda_5\left(2\sum_{j+k=n}\mathscr{I}_j\mathscr{I}_{k,x}+\mathscr{I}_{n,xx}+4\mathscr{I}_{n+2}\right),$$
$$n=2,\ 3,\ \cdots.$$

The first equation of conservation law (40) is just the new integrable model (20).

4 Interactions of Multiple Solitons

From (28), the one-soliton solution reads

$$u=\frac{40\lambda_1k_1^2}{\lambda_3}\operatorname{sech}^2\left(\frac{k_1x}{2}+\frac{l_1y}{2}-\frac{k_1^2(\lambda_1k_1^3+\lambda_5l_1)t}{2}+\frac{\xi_{10}}{2}\right), \tag{41}$$

where parameters k_1, l_1, and ξ_0 are arbitrary constants. The profile of the one-soliton via solution (41) is shown in Figure 1. It can be seen that the soliton amplitude increases with the increase of k_1 while other parameters are fixed.

The two-soliton solution can be written as

$$u=\frac{40\lambda_1}{\lambda_3}\left\{\ln\left[1+e^{\xi_1}+e^{\xi_2}+\frac{(k_1-k_2)^2}{(k_1+k_2)^2}e^{\xi_1+\xi_2}\right]\right\}_{xx}, \tag{42}$$
$$\xi_j=k_jx+l_jy-k_j^2(\lambda_1k_j^3+\lambda_5l_j)t+\xi_{j0},\ j=1,\ 2.$$

By selecting appropriate parameter values via solution (42), four different types of collisions of two solitary waves are shown in Figures 2–5. Elastic overtaking collisions of two solitary waves are illustrated in shown in

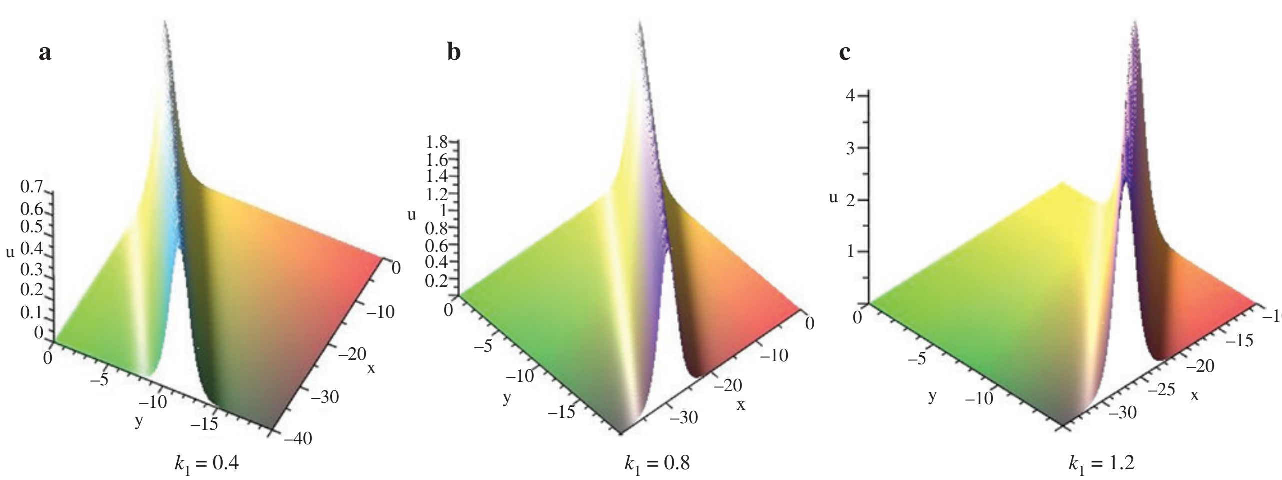

Figure 1: One soliton given by expression (41) with $\lambda_1=2.2$, $\lambda_3=7.6$, $\lambda_5=2$, $l_1=0.8$, $\xi_{10}=0$.

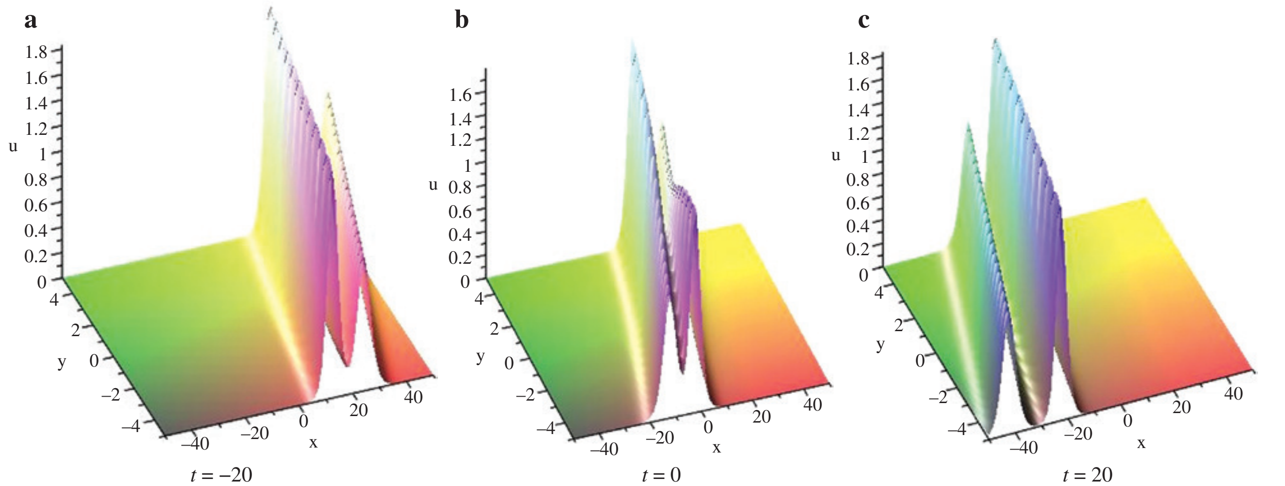

Figure 2: The overtaking collision of two solitary waves given by (42), where $\lambda_1=2.2$, $\lambda_3=7.6$, $\lambda_5=2$, $k_1=0.6$, $k_2=0.8$, $l_1=-1.5$, $l_2=-1.2$, $\xi_{10}=\xi_{20}=0$.

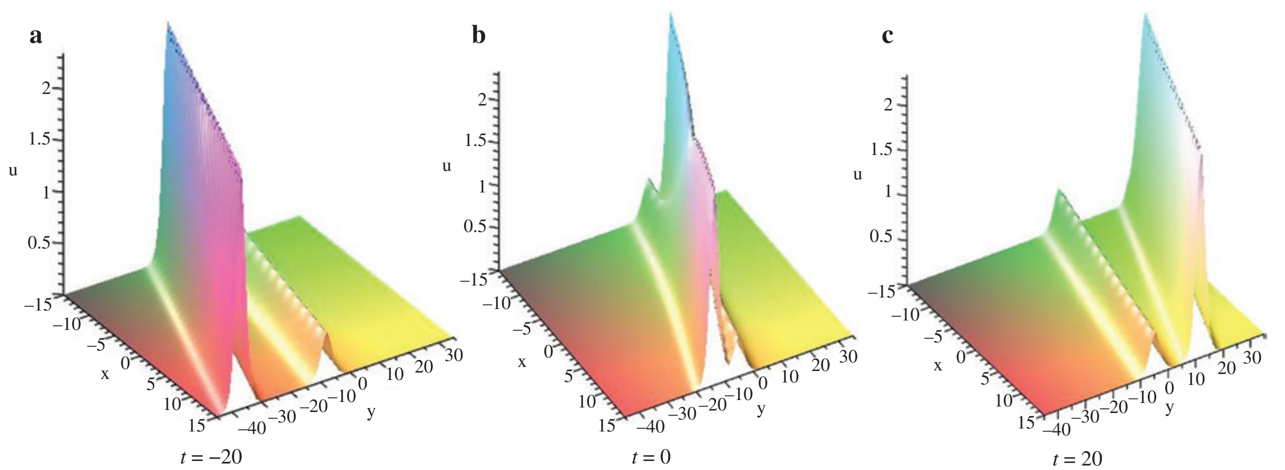

Figure 3: The overtaking collision of two solitary waves given by (42), where $\lambda_1=2.2$, $\lambda_3=7.6$, $\lambda_5=2$, $k_1=0.6$, $k_2=0.9$, $l_1=0.8$, $l_2=1.0$, $\xi_{10}=\xi_{20}=0$.

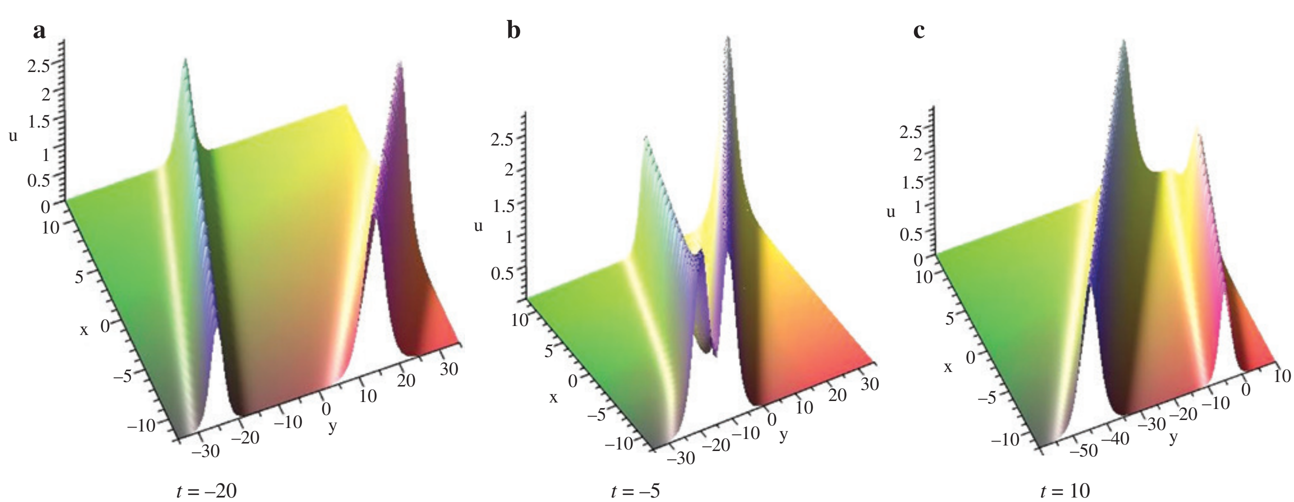

Figure 4: The headon collision of two solitary waves given by (42), where $\lambda_1=2.2$, $\lambda_3=7.6$, $\lambda_5=2$, $k_1=1.0$, $k_2=0.6$, $l_1=-0.5$, $l_2=-1.5$, $\xi_{10}=\xi_{20}=0$.

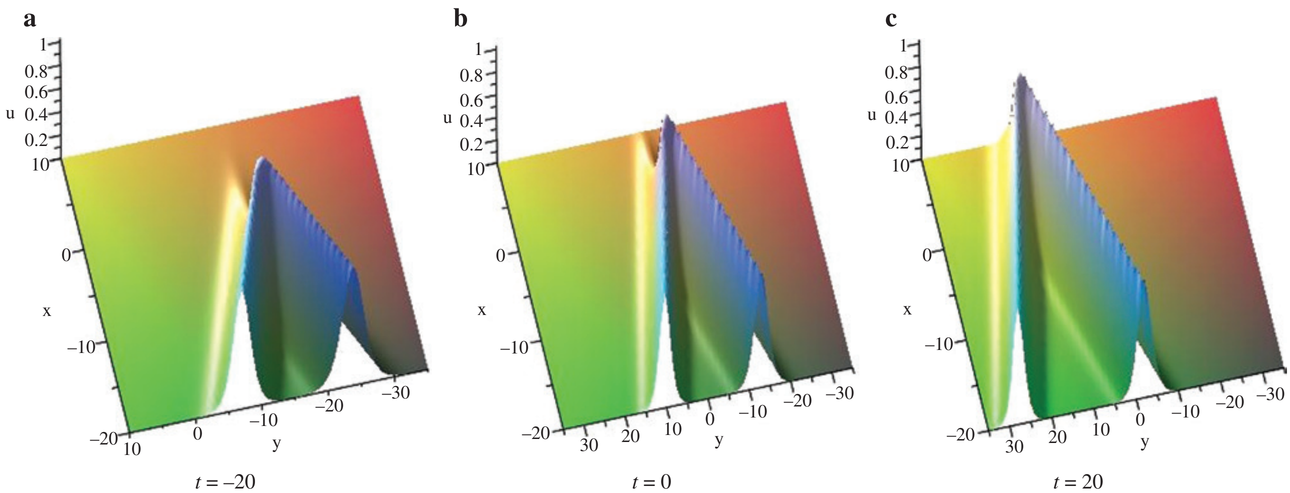

Figure 5: The V-type collision of two solitary waves given by (42), where $\lambda_1 = 2.2$, $\lambda_3 = 7.6$, $\lambda_5 = 2$, $k_1 = k_2 = 0.6$, $l_1 = -1.15$, $l_2 = -0.9$, $\xi_{10} = \xi_{20} = 0$.

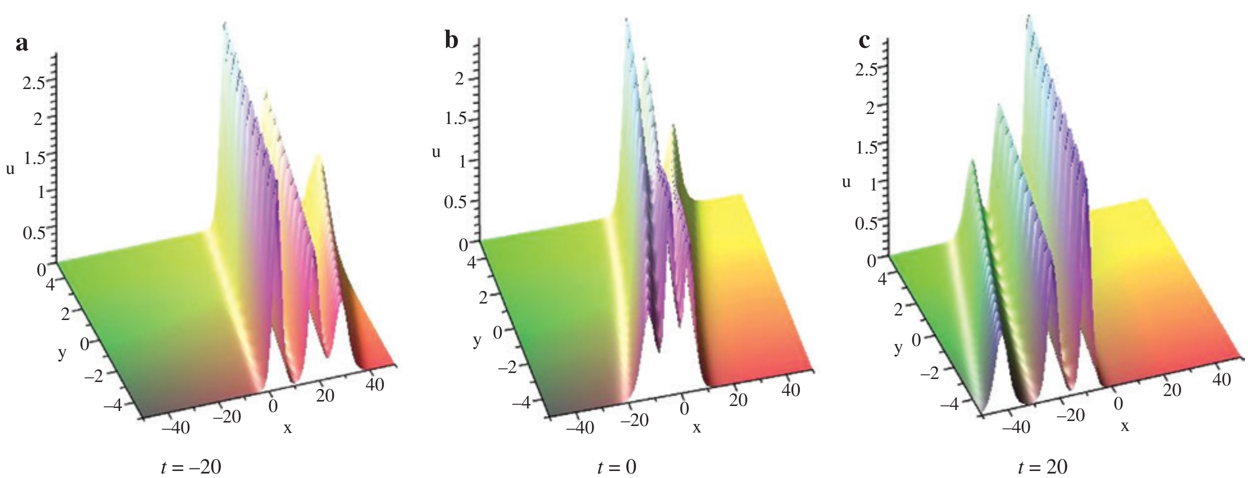

Figure 6: The overtaking collision of three solitary waves given by (28), where $N = 3$, $\lambda_1 = 2.2$, $\lambda_3 = 7.6$, $\lambda_5 = 2$, $k_1 = 0.6$, $k_2 = 0.8$, $k_3 = 1.0$, $l_1 = -1.5$, $l_2 = -1.2$, $l_2 = -1.35$, $\xi_{10} = \xi_{20} = \xi_{30} = 0$.

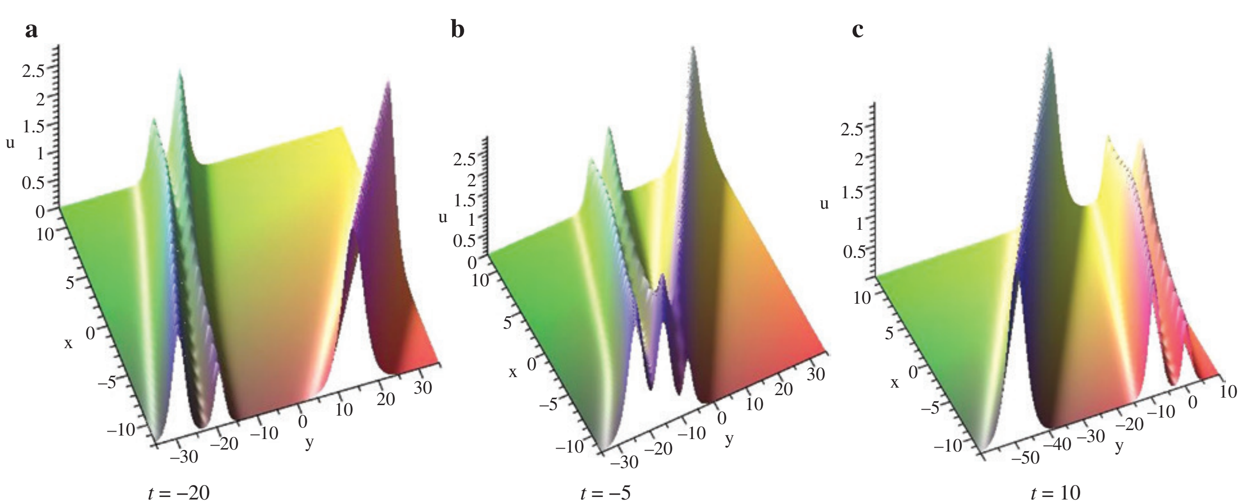

Figure 7: The head-on collision of three solitary waves given by (28), where $N = 3$, $\lambda_1 = 2.2$, $\lambda_3 = 7.6$, $\lambda_5 = 2$, $k_1 = 0.8$, $k_2 = 1.0$, $k_3 = 0.6$, $l_1 = -0.1$, $l_2 = -0.5$, $l_3 = -1.5$, $\xi_{10} = \xi_{20} = \xi_{30} = 0$.

Figures 2 and 3. Figure 2 show that two solitary waves are left-going along the x-axis and the smaller-amplitude soliton moves faster and then overtakes the larger. After the collision, the wave shapes and velocities remain unchanged. Figure 3 shows that two solitary waves are right-going along the y-axis and the larger-amplitude soliton overtakes the smaller one. The head-on collision of two solitary waves along opposite directions is depicted in Figure 4. When $k_1=k_2$, $l_1\neq l_2$, V-type collisions occur, as shown in Figure 5.

More interesting collisions of three solitary wave can be exhibited by graphs. For the sake of simplicity, here, only the overtaking collisions and head-on collisions are shown in Figures 6 and 7, respectively.

5 Conclusions

Using the WTC–Kruskal method and symbolic computation, we performed the Painlevé test for an extended 2+1-dimensional fifth-order KdV equation. Three distinct cases that pass the Painlevé test have been found, and two of those cases correspond to the generalizations of the integrable equations (2)–(7), whereas the first one turns out to be new.

For the new integrable model (20), we employed the Bell polynomials method to further prove its integrability. As a result, we derived the bilinear forms, bilinear BT, Lax pair, and infinite conservation laws systematically. Moreover, we also obtained the N-soliton solutions of the new obtained model. The collisions of multiple solitons have been discussed, which include overtaking, head-on as well as V-type collisions. Further studies on other integrable properties and exact solutions of this new unnamed model should be done in the future.

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