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# Exact Solutions for Stokes' Flow of a Non-Newtonian Nanofluid Model: A Lie Similarity Approach

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**Abstract:** The fully developed time-dependent flow of an incompressible, thermodynamically compatible non-Newtonian third-grade nanofluid is investigated. The classical Stokes model is considered in which the flow is generated due to the motion of the plate in its own plane with an impulsive velocity. The Lie symmetry approach is utilised to convert the governing nonlinear partial differential equation into different linear and nonlinear ordinary differential equations. The reduced ordinary differential equations are then solved by using the compatibility and generalised group method. Exact solutions for the model equation are deduced in the form of closed-form exponential functions which are not available in the literature before. In addition, we also derived the conservation laws associated with the governing model. Finally, the physical features of the pertinent parameters are discussed in detail through several graphs.

**Keywords:** Conservation Laws; Group Invariant Solutions; Lie Symmetry Analysis; Nanofluid Flow; Third-Grade Fluid.

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## 1 Introduction

The study of classical Stokes' model for the flat plate problem has been the subject of fundamental theoretical interest in the literature of fluid dynamics. Stokes' problem occurs in many applied models such as acoustic streaming around an oscillating body and unsteady boundary layer with fluctuation [1, 2]. Many studies related to Stokes' flow for Navier–Stokes fluid and other different classes of non-Newtonian fluids are available in the literature [3–9].

Nanofluids is a term coined by Choi [10] by introducing the nanoparticles in the base fluids and theoretically demonstrating the feasibility of the concept of nanofluids. The materials which are commonly used as nanoparticles include chemically stable metals, metal oxides, oxide ceramics, metal carbides, metal nitrides, and carbon in various forms. Examples are gold, copper, alumina, silica, zirconia, titania, $Al_2O_3$, CuO, SiC, AlN, SiN, diamond, graphite, and carbon nanotubes. The common base fluids are water, oil, and ethylene glycol. The size of nanoparticles (usually <100 nm) in liquids mixture gives them the ability to interact with liquids at the molecular level and so conduct heat better than today's heat transfer fluids. The characteristic feature of nanofluids is thermal conductivity enhancement, a phenomenon observed by Choi et al. [11] and Masuda et al. [12]. Eastman et al. [13] in their work observed an unusual thermal conductivity enhancement in copper (Cu) nanofluids at small nanoparticle volume fraction. Experimental studies conducted by [14–16] show that the effective thermal conductivity increases under macroscopically stationary conditions. Since then, authors have demonstrated that nanofluids can have significantly better heat transfer characteristics than the conventional fluids depending on the nanoparticles used, size of nanoparticles, and concentration of colloidal suspension. A comprehensive survey of convective transport in nanofluids was done by Buongiorno [17], who considered seven slip mechanisms that can produce a relative velocity between the nanoparticles and the base fluid.

In real situations, nanofluids do not satisfy the properties of Newtonian fluids; hence, it is more justified to consider them as non-Newtonian fluids. Non-Newtonian nanofluids are widely encountered in many industrial and technology applications, such as industrial cooling applications, melts of polymers, biological solutions, micro-electromechanical systems, paints, drug delivery, cryopreservation, instrumentation, automobiles, asphalts, and glues, but a careful review of the literature reveals that non-Newtonian nanofluids have so far received very little attention. The numerical study of the magnetohydrodynamic (MHD) boundary layer flow of a Maxwell nanofluid past a stretching sheet was accomplished by Nadeem et al. [18]. Ramzan and Bilal [19] studied the unsteady MHD second-grade incompressible nanofluid towards a stretching sheet. Santra et al. [20] simulated the forced convection of Cu–water nanofluid in a channel with both Newtonian and non-Newtonian models. For the non-Newtonian case, the power-law rheology is applied in which the fluid consistency coefficient and the flow behaviour index are interpolated and extrapolated from the experimental results with Cu–water nanofluid [21]. Ellahi et al. [22] recently presented the analytical series solutions of third-grade non-Newtonian nanofluids with Reynolds' and Vogel's model. In addition to the above, the concept of nanofluid is well described in detail in [23–32].

The connection between the integrability properties of differential equations (DEs) goes back to the concept of invariance introduced by Sophus Lie [33, 34]. This theory, now called the Lie group method, is central to the modern technique for studying nonlinear DEs. It uses the notion of symmetry to generate solutions in a systematic manner. A symmetry of a DE is a special class of transformation which maps any solution of a DE to another solution of the same DE. It is also possible to use the symmetry groups to introduce new dependent and independent variables, called similarity variables. These new variables can be utilised to reduce the number of independent variables. Today, the Lie symmetry approach to DEs is widely applied in various fields of mathematics, mechanics, and theoretical physics, and many results published in these areas demonstrate that Lie's theory is an efficient tool for solving nonlinear problems formulated in terms of DEs. The Lie symmetry approach has been widely applied by several authors to solve difficult nonlinear problems dealing with the flows of non-Newtonian fluids [35–43].

In the study of DEs, conservation laws play a vital role. It is well known that conservation laws play an important role in the solution process of DEs. In fact, conservation laws describe physical conserved quantities, such as mass, energy, momentum, and angular momentum, as well as charge and other constants of motion. They have been used in investigating the existence, uniqueness, and stability of solutions of nonlinear partial differential equations (PDEs). Recently, conservation laws were used to obtain exact solutions of some PDEs, see for example [44–47] and the references therein. Therefore, it is important to study the conservation laws of PDEs.

In the aforementioned studies, the nanofluid flow problems described by nonlinear equations are either presented experimental results or the numerical solutions. To the best of our knowledge, no such study has so far been investigated which focuses on the exact closed-form solutions of a non-Newtonian nanofluid flow problem. The objective of this paper is therefore to formulate the exact solutions for the time-dependent flow of a non-Newtonian nanofluid flow. The flow is caused due to the arbitrary motion of the plate in its own plane with an impulsive velocity. Various classes of group invariant solutions are derived for the flow model equation by employing the group theoretical approach.

## 2 Mathematical Model

The constitutive relation for an incompressible and thermodynamically compatible third-grade nanofluid has the form

$$\mathbf{T}=-p\mathbf{I}+\mu_{\mathrm{nf}}\mathbf{A}_1+\alpha_1\mathbf{A}_2+\alpha_2\mathbf{A}_1^2+\beta_3(tr\mathbf{A}_1^2)\mathbf{A}_1, \tag{1}$$

where $\mathbf{T}$ is the Cauchy stress tensor, $p$ the pressure, $\mathbf{I}$ the identity tensor, $\rho_{\mathrm{nf}}$ the density of nanofluid, $\mu_{\mathrm{nf}}$ the dynamic viscosity of nanofluid, $\alpha_1$, $\alpha_2$, and $\beta_3$ are the material constants, and $\mathbf{A}_i$ $(i=1-3)$ are the Rivlin–Ericksen tensors which are defined through the following equations:

$$\mathbf{A}_1=(\text{grad }\mathbf{V})+(\text{grad }\mathbf{V})^T, \tag{2}$$

$$\mathbf{A}_n=\frac{\mathrm{d}\mathbf{A}_{n-1}}{\mathrm{d}t}+\mathbf{A}_{n-1}(\text{grad }\mathbf{V})+(\text{grad }\mathbf{V})^T\mathbf{A}_{n-1}\quad (n>1), \tag{3}$$

where $\mathbf{V}=[u(y,t),0,0]$ denotes the velocity field and $\mathrm{d}/\mathrm{d}t$ is the material time derivative defined by

$$\frac{\mathrm{d}}{\mathrm{d}t}=\frac{\partial}{\partial t}+(\mathbf{V}\cdot\nabla), \tag{4}$$

in which $\nabla$ is the gradient operator. The density and viscosity of nanofluid are defined as

$$\rho_{\mathrm{nf}}=(1-\varphi)\rho_f+\varphi\rho_s, \tag{5}$$

$$\mu_{\mathrm{nf}}=\frac{\mu_f}{(1-\varphi)^{2.5}}, \tag{6}$$

with $\varphi$ being the nanoparticle volume concentration, $\rho_f$ the density of the base fluid, and $\rho_s$ the density of the nanoparticles.

Here we consider the Stokes flow of an incompressible third-grade nanofluid bounded by an infinite rigid plate. The flow is caused by the impulsive motion of the rigid plate. The fluid occupies the porous half space $y>0$. The plate is infinite in the $XZ$-plane and therefore all the physical quantities except the pressure depend on $y$ only. The time-dependent motion through a porous medium is governed by

$$\rho_{\mathrm{nf}}\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t}=\mathrm{div}\,\mathbf{T}+\mathbf{R}, \tag{7}$$

where $\mathbf{R}$ is Darcy's resistance in the porous medium.

The constitutive relationship between the pressure drop and the velocity for the unidirectional flow of a third-grade nanofluid is

$$\frac{\partial p}{\partial x}=-\frac{\phi}{\kappa}\left[\frac{\mu_f}{(1-\varphi)^{2.5}}+\alpha_1\frac{\partial}{\partial t}+2\beta_3\left(\frac{\partial u}{\partial y}\right)^2\right]u, \tag{8}$$

where $\kappa$ is the permeability and $\phi$ the porosity of the porous medium.

The pressure gradient in (8) is regarded as a measure of the flow resistance in the bulk of the porous medium. If $R_x$ is a measure of the flow resistance due to the porous medium in the $x$-direction, then $R_x$ through (8) is given by

$$R_x=-\frac{\phi}{\kappa}\left[\frac{\mu_f}{(1-\varphi)^{2.5}}+\alpha_1\frac{\partial}{\partial t}+2\beta_3\left(\frac{\partial u}{\partial y}\right)^2\right]u. \tag{9}$$

Making use of (9) into momentum equation (7) by keeping in mind (1)–(6), one obtains the following governing equation in the absence of the modified pressure gradient

$$\rho_{\mathrm{nf}}\frac{\partial u}{\partial t}=\mu_{\mathrm{nf}}\frac{\partial^2 u}{\partial y^2}+\alpha_1\frac{\partial^3 u}{\partial y^2\partial t}+6\beta_3\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2}$$
$$-\frac{\phi}{\kappa}\left[\mu_{\mathrm{nf}}+\alpha_1\frac{\partial}{\partial t}+2\beta_3\left(\frac{\partial u}{\partial y}\right)^2\right]u, \tag{10}$$

along with the boundary conditions

$$u(y,\,0)=g(y),\quad y>0, \tag{11}$$

$$u(0,\,t)=u_0V(t),\quad t>0, \tag{12}$$

$$u(\infty,\,t)=0,\quad t>0, \tag{13}$$

where $u_0$ is the reference velocity, and $g(y)$ and $V(t)$ are as yet undetermined. These are specified through the Lie group approach.

Let us introduce the following non-dimensional parameters:

$$\bar{u}=\frac{u}{u_0},\quad \bar{y}=\frac{u_0y}{\nu_f},\quad \bar{t}=\frac{u_0^2t}{\nu_f},\quad \bar{\alpha}=\frac{\alpha_1u_0^2}{\rho_f\nu_f^2},\quad \bar{\beta}=\frac{2\beta_3u_0^4}{\rho_f\nu_f^3},$$
$$\bar{\kappa}=\frac{\phi\nu_f^2}{\kappa u_0^2}, \tag{14}$$

where $\nu_f=\mu_f/\rho_f$. Making use of the non-dimensional quantities given in (14), the dimensionless forms of the governing equations (10), after dropping bars for simplicity, lead to the following non-dimensional PDE:

$$\left(1-\varphi+\varphi\frac{\rho_s}{\rho_f}\right)\frac{\partial u}{\partial t}=\frac{1}{(1-\varphi)^{2.5}}\frac{\partial^2 u}{\partial y^2}+\alpha\frac{\partial^3 u}{\partial y^2\partial t}+3\beta\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2}$$
$$-\kappa\left[\frac{u}{(1-\varphi)^{2.5}}+\alpha\frac{\partial u}{\partial t}+\beta u\left(\frac{\partial u}{\partial y}\right)^2\right], \tag{15}$$

subject to the boundary conditions

$$u(y,\,0)=g(y),\quad y>0, \tag{16}$$

$$u(0,\,t)=V(t),\quad t>0, \tag{17}$$

$$u(\infty,\,t)=0,\quad t>0. \tag{18}$$

By defining

$$\varphi_*=\left(1-\varphi+\varphi\frac{\rho_s}{\rho_f}+\alpha\kappa\right), \tag{19}$$

we can rewrite (15) as

$$\frac{\partial u}{\partial t}=\frac{1}{\varphi_*(1-\varphi)^{2.5}}\frac{\partial^2 u}{\partial y^2}+\frac{\alpha}{\varphi_*}\frac{\partial^3 u}{\partial y^2\partial t}+\frac{3\beta}{\varphi_*}\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2}$$
$$-\frac{\kappa\beta}{\varphi_*}\left(\frac{\partial u}{\partial y}\right)^2u-\frac{\kappa}{\varphi_*(1-\varphi)^{2.5}}u. \tag{20}$$

The PDE (20) is solved subject to conditions (16)–(18).

## 3 Classical Lie Symmetry Analysis

In this section, we briefly discuss how to determine Lie point symmetry generators admitted by (20). We use these generators to solve (20) analytically subject to conditions (16)–(18).

We look for transformations of the independent variables $t$, $y$ and the dependent variable $u$ of the form

$$\bar{t}=\bar{t}(t,y,u,\epsilon),\quad \bar{y}=\bar{y}(t,y,u,\epsilon),\quad \bar{u}=\bar{u}(t,y,u,\epsilon),\tag{21}$$

which constitute a group where $\epsilon$ is the group parameter such that (20) is left invariant. From Lie's theory, the transformations in (21) are obtained in terms of the infinitesimal transformations

$$\begin{aligned}\bar{t}&\simeq t+\epsilon\xi^1(t,y,u),\quad \bar{y}\simeq y+\epsilon\xi^2(t,y,u),\\ \bar{u}&\simeq u+\epsilon\xi^3(t,y,u),\end{aligned}\tag{22}$$

or the operator

$$\chi=\xi^1(t,y,u)\partial_t+\xi^2(t,y,u)\partial_y+\xi^3(t,y,u)\partial_u,\tag{23}$$

which is a generator of the Lie point symmetry of (23) if the following condition holds:

$$\chi^{[3]}\left[u_t-\frac{1}{\varphi_*(1-\varphi)^{2.5}}u_{yy}-\frac{\alpha}{\varphi_*}u_{yyt}-\frac{3\beta}{\varphi_*}(u_y)^2u_{yy}+\frac{\kappa\beta}{\varphi_*}u(u_y)^2+\frac{\kappa}{\varphi_*(1-\varphi)^{2.5}}u\right]\Bigg|_{20=0}=0.\tag{24}$$

Here $\chi^{[3]}$ denotes the third prolongation of the operator (23) that includes all the derivatives of the dependent variable up to the third order; it is defined by

$$\chi^{[3]}=\chi+\zeta^t\partial u_t+\zeta^y\partial u_y+\zeta^{yy}\partial u_{yy}+\zeta^{yyy}\partial u_{yyy}+\zeta^{tyy}\partial u_{tyy},\tag{25}$$

with

$$\begin{aligned}\zeta^t&=D_t\xi^3-u_tD_t\xi^1-u_yD_t\xi^2,\\ \zeta^y&=D_y\xi^3-u_tD_y\xi^1-u_yD_y\xi^2,\\ \zeta^{yy}&=D_y\zeta^y-u_{ty}D_y\xi^1-u_{yy}D_y\xi^2,\\ \zeta^{yyy}&=D_y\zeta^{yy}-u_{tyy}D_y\xi^1-u_{yyy}D_y\xi^2,\\ \zeta^{tyy}&=D_t\zeta^{yy}-u_{tyy}D_t\xi^1-u_{yyy}D_t\xi^2,\end{aligned}\tag{26}$$

and the total derivative operators

$$\begin{aligned}D_t&=\partial_t+u_t\partial_u+u_{tt}\partial u_t+u_{ty}\partial u_y+\cdots,\\ D_y&=\partial_y+u_y\partial_u+u_{yy}\partial u_y+u_{ty}\partial u_t+\cdots.\end{aligned}\tag{27}$$

Substituting the expansions of (29) into the symmetry condition (27) and separating by powers of the derivatives of $u$, as $\xi^1$, $\xi^2$, and $\xi^3$ are independent of the derivatives of $u$, lead to the overdetermined system of linear homogeneous PDEs

$$\begin{aligned}&\xi^1_u=0,\\ &\xi^1_y=0,\\ &\xi^2_u=0,\\ &\xi^2_y=0,\\ &\xi^2_t=0,\\ &\xi^3_{uu}=0,\\ &\xi^3_y=0,\\ &\frac{1}{\varphi_*(1-\varphi)^{2.5}}\xi^1_t+\frac{\alpha}{\varphi_*}\xi^3_{tu}=0,\\ &\frac{\kappa}{\varphi_*(1-\varphi)^{2.5}}\xi^3+\frac{u\kappa}{\varphi_*(1-\varphi)^{2.5}}\xi^1_t+\xi^3_t-\frac{u\kappa}{\varphi_*(1-\varphi)^{2.5}}\xi^3_u=0,\\ &\xi^3+u\xi^1_t+u\xi^3_u=0,\\ &\xi^1_t+2\xi^3_u=0.\end{aligned}\tag{28}$$

By solving the system (28), we obtain a three-dimensional Lie algebra generated by

$$\chi_1=\frac{\partial}{\partial t},\quad \chi_2=\frac{\partial}{\partial y},\tag{29}$$

$$\chi_3=-\left(\frac{\varphi_*(1-\varphi)^{2.5}}{\kappa}\right)e^{\left(\frac{2\kappa}{\varphi_*(1-\varphi)^{2.5}}\right)t}\frac{\partial}{\partial t}+ue^{\left(\frac{2\kappa}{\varphi_*(1-\varphi)^{2.5}}\right)t}\frac{\partial}{\partial u}.\tag{30}$$

## 4 Compatibility Criterion and Generalised Groups

Here we briefly discuss the compatibility criterion developed by Aziz et al. [48]. In [48], the general compatibility criterion/compatibility test is established for a fifth-order ordinary differential equation (ODE) to be compatible with a first-order ODE. Various research examples taken from the literature have been presented in [48] to which the compatibility approach actually worked out. In this particular work, we only have to discuss the third-order ODEs. Thus, we confined ourselves to discussing only the compatibility criterion for solving a third-order ODE subject to a first-order ODE.

Let us consider a third-order ODE in one independent variable $x$ and one dependent variable $y$,

$$f(x,y,y^{(1)},y^{(2)},y^{(3)})=0,\tag{31}$$

and a first-order ODE

$$e(x,y,y^{(1)})=0,\tag{32}$$

such that

$$J=\frac{\partial(e, f)}{\partial(y^{(1)}, y^{(2)}, y^{(3)})}\neq 0. \tag{33}$$

Then, we one can solve for the highest derivatives as

$$y^{(3)}=F(x, y, y^{(1)}, y^{(2)}), \tag{34}$$

and

$$y^{(1)}=E(x, y), \tag{35}$$

where $F$ and $E$ are smooth and continuously differentiable functions of $x$, $y$ and, in the case of $F$, their derivatives. Now (34) depends on $y^{(1)}$, $y^{(2)}$, and $y^{(3)}$ which are obtained by differentiating (35). This gives

$$y^{(2)}=E_x+EE_y, \tag{36}$$

$$y^{(3)}=E_{xx}+2EE_{xy}+E^2E_{yy}+E_xE_y+EE_y^2. \tag{37}$$

By equating the right-hand side of (34) with (27), we obtain

$$F[x, y, E, E_x+EE_y]=E_{xx}+2EE_{xy}+E^2E_{yy}+E_xE_y+EE_y^2, \tag{38}$$

which gives the compatibility criterion or compatibility test for a third-order ODE to be compatible with a first-order ODE.

The connection has also been made in [48] among the compatibility of higher order ODEs subject to the lower order ODEs through conditional symmetries or generalised groups.

We give here a precise definition of conditional symmetries [49].

**Definition 1** *An nth-order scalar ODE, $n=2, 3$, is called conditionally classifiable by a symmetry algebra with respect to a first-order ODE called the root ODE if and only if the nth-order ODE jointly with the first-order ODE forms an over-determined compatible system and the first-order ODE has symmetry algebra which is the conditional symmetry algebra of the nth-order ODE.*

Now the algorithm of computing the conditional symmetries of an $n$th-order scalar ODE (see [49]) is given below.

Let $\chi$ be the vector field of dependent and independent variables given by

$$\chi=\xi^1(x, y)\frac{\partial}{\partial x}+\xi^2(x, y)\frac{\partial}{\partial y}, \tag{39}$$

where $\xi^1$ and $\xi^2$ are the coefficient functions of the vector field $\chi$. Suppose that the vector field $\chi$ is a conditional symmetry generator of an $n$th-order scalar ODE subject to a first-order ODE. Then the conditional symmetry condition

$$\chi^{[n]}[y^{(n)}-P(x, y, y^{(1)}, \ldots, y^{(n-1)})]|_{y^{(n)}-P=0,\, y^{(m)}-Q=0}=0, \tag{40}$$

holds, where $n=2, 3$, $m$ is taken as 1 herein, and $\chi^{[n]}$ denotes the $n$th prolongation of the generator $\chi$ defined as

$$\chi^{[n]}=\chi+\sum_{j=1}^{n}\varsigma_j\frac{\partial}{\partial y^{(j)}}, \tag{41}$$

where the additional coefficient functions are defined as

$$\varsigma_j=D_x(\varsigma_{j-1})-y^{(j)}D_x(\xi), \qquad j=1, \ldots, n,\ \varsigma_0=\eta, \tag{42}$$

and $D_x$ is the total differentiation operator.

We now state the following propositions of the work [49].

**Proposition 1** [49] *If a scalar nth-order, $n\geq 2$, ODE of the form*

$$E_\alpha(x, y, y', y'',\ldots, y^{(n)})=0, \quad \alpha=1, \ldots, p, \tag{43}$$

*is completely integrable by quadratures, then it admits a conditional symmetry subject to the first-order ODE related to the invariant curve condition which arises from the known solution curves.*

**Proposition 2** [49] *If a scalar nth-order, $n\geq 2$, ODE of the form (43) has exact solutions $\phi(x, y)=0$ or $\phi(x, y, C_1, \ldots, C_r)=0$, where $r$ ranges from 1 to $r<n$, then it admits a conditional symmetry subject to the first-order ODE related to the invariant curve condition which arises from the known solution curves.*

The proofs of these propositions are given in [49]. Now we state the following result which is the consequence of the propositions defined above.

We have that the conditional symmetry of our $n$th-order scalar ODE is given by

$$X=\frac{\partial}{\partial x}+e(x, y)\frac{\partial}{\partial y}, \tag{44}$$

where the first-order ODE is given by

$$y'=e(x, y). \tag{45}$$

## 5 Travelling Wave Solutions

Travelling wave solutions are special kind of group invariant solutions which are invariant under a linear combination of the time-translation and the space-translation generators.

We search for an invariant solution under the operator

$$\chi=\frac{\partial}{\partial t}+m\frac{\partial}{\partial y}, \tag{46}$$

which denotes wave-front-type travelling wave solutions with constant wave speed $m$. The characteristic system of (46) is

$$\frac{\mathrm{d}y}{m}=\frac{\mathrm{d}t}{1}=\frac{\mathrm{d}u}{0}. \tag{47}$$

Solving (47), the invariant is given as

$$u(y,\, t)=f(\eta), \quad \text{where} \quad \eta=y-mt. \tag{48}$$

Using (48) into (20) results in a third-order ordinary differential for $f(\eta)$, namely,

$$-m\frac{\mathrm{d}f}{\mathrm{d}\eta}=\frac{1}{\varphi_*(1-\varphi)^{2.5}}\frac{\mathrm{d}^2f}{\mathrm{d}\eta^2}-\frac{\alpha m}{\varphi_*}\frac{\mathrm{d}^3f}{\mathrm{d}\eta^3}+\frac{3\beta}{\varphi_*}\left(\frac{\mathrm{d}f}{\mathrm{d}\eta}\right)^2\frac{\mathrm{d}^2f}{\mathrm{d}\eta^2}$$
$$-\frac{\kappa\beta}{\varphi_*}f\left(\frac{\mathrm{d}f}{\mathrm{d}\eta}\right)^2-\frac{\kappa}{\varphi_*(1-\varphi)^{2.5}}f, \tag{49}$$

with the transformed boundary conditions given by

$$f(0)=l_1, \quad f(\eta)\to 0 \text{ as } \eta\to\infty, \tag{50}$$

where $l_1$ can take a sufficiently large value.

## 5.1 Solution for $f(\eta)$ via Compatibility Approach

Now we obtain the exact solution of the reduced ODE (49) subject to boundary conditions (50) by using a compatibility criterion. We check that the third-order ODE (49) is compatible with the first-order ODE

$$\frac{\mathrm{d}f}{\mathrm{d}\eta}+\delta f=0 \quad \text{with} \quad \delta\neq 0, \tag{51}$$

where $\delta$ is constant. The general solution of (51) is

$$f(\eta)=A\exp(-\delta\eta). \tag{52}$$

The parameters to be determined are $A$ and $\delta$. Using the compatibility test (38) for a third-order ODE to be compatible with a first-order ODE, we obtain

$$-m\delta f+\frac{1}{\varphi_*(1-\varphi)^{2.5}}\delta^2 f+\frac{\alpha_* m}{\varphi_*}\delta^3 f+\frac{3\beta}{\varphi_*}\delta^4 f^3$$
$$-\frac{\kappa\beta}{\varphi_*}\delta^2 f^3-\frac{\kappa}{\varphi_*(1-\varphi)^{2.5}}f=0. \tag{53}$$

Equating the above equation in powers of a dependent variable, we obtain

$$f:-m\delta+\frac{1}{\varphi_*(1-\varphi)^{2.5}}\delta^2+\frac{\alpha_* m}{\varphi_*}\delta^3-\frac{\kappa}{\varphi_*(1-\varphi)^{2.5}}=0, \tag{54}$$

$$f^3:\frac{3\beta}{\varphi_*}\delta^4-\frac{\kappa\beta}{\varphi_*}\delta^2=0. \tag{55}$$

From (55), we obtain

$$\delta=\pm\sqrt{\frac{\kappa}{3}}. \tag{56}$$

We choose

$$\delta=\sqrt{\frac{\kappa}{3}}, \tag{57}$$

so that our solution satisfy the second boundary condition at infinity. Using the value of $\delta$ in (54), we get

$$\sqrt{\frac{\kappa}{3}}\left[\frac{\alpha m\kappa}{3\varphi_*}-m\right]-\frac{2\kappa}{3\varphi_*(1-\varphi)^{2.5}}=0. \tag{58}$$

which is the compatibility condition for a third-order ODE (49) to be compatible with a first-order ODE (51). Thus, the solution of a third-order ODE subject to a first-order ODE (provided that compatibility condition (58) holds) is written as

$$f(\eta)=\exp\left(-\sqrt{\frac{\kappa}{3}}\eta\right). \tag{59}$$

Finally, the exact solution $u(y,\, t)$, which satisfies the compatibility condition (58), is

$$u(y,\, t)=\exp\left(-\sqrt{\frac{\kappa}{3}}(y-mt)\right). \tag{60}$$

We observe that the compatibility condition (58) gives the speed $m$ of the travelling wave

$$m=\frac{2\kappa}{(1-\varphi)^{2.5}\sqrt{\frac{\kappa}{3}}(\alpha\kappa-3\varphi_*)}. \tag{61}$$

Making use of the value of $m$ from (61) into (60), the solution $u(y,\, t)$ takes the form

$$u(y,\, t)=\exp\left[-\sqrt{\frac{\kappa}{3}}y+\frac{2\kappa}{(1-\varphi)^{2.5}(\alpha\kappa-3\varphi_*)}t\right]. \tag{62}$$

Finally, substituting the value of $\varphi_*$ from(19) into (62), the solution is written as

$$u(y,\,t)=\exp\left[-\sqrt{\frac{\kappa}{3}}y+\frac{2\kappa}{(1-\varphi)^{2.5}\left\{\alpha\kappa-3\left(1-\varphi+\varphi\frac{\rho_s}{\rho_f}+\alpha\kappa\right)\right\}}t\right]. \tag{63}$$

We note that this solution satisfies the boundary condition (16)–(18) with

$$V(t)=\exp\left[\frac{2\kappa}{(1-\varphi)^{2.5}\left(\alpha\kappa-3\left(1-\varphi+\varphi\frac{\rho_s}{\rho_f}+\alpha\kappa\right)\right)}t\right]$$

$$\text{and}\quad g(y)=\exp\left[-\sqrt{\frac{\kappa}{3}}y\right]. \tag{64}$$

We remark here that $V(t)$ and $g(y)$ depend on the physical parameters of the flow.

**Remark 1** *We note that the symmetry of the first-order ODE*

$$\frac{\mathrm{d}f}{\mathrm{d}\eta}+\delta f=0,\quad \textit{with}\quad \delta\neq 0, \tag{65}$$

*is found to be*

$$X=\frac{\partial}{\partial\eta}-\sqrt{\frac{\kappa}{3}}f\frac{\partial}{\partial f}. \tag{66}$$

*The operator given in* (66) *is the conditional symmetry of the third-order ODE* (49) *subject to* (65). *Thus the physical solution of these compatible equations can also be found by using the conditional symmetry structure of these equations.*

## 6 Group Invariant Solutions Corresponding to $\chi_3$

The operator $\chi_3$ is given as

$$\chi_3=-\left(\frac{\varphi_*(1-\varphi)^{2.5}}{\kappa}\right)e^{\left(\frac{2\kappa}{\varphi_*(1-\varphi)^{2.5}}\right)t}\frac{\partial}{\partial t}+ue^{\left(\frac{2\kappa}{\varphi_*(1-\varphi)^{2.5}}\right)t}\frac{\partial}{\partial u}. \tag{67}$$

By solving the corresponding characteristics system of (67), the invariant solution is given by

$$u(y,\,t)=F(y)\exp\left[-\left(\frac{\kappa}{\varphi_*(1-\varphi)^{2.5}}\right)t\right], \tag{68}$$

where $F(y)$ as yet is an undetermined function of $y$. Substituting (68) into (20) yields the linear second-order ODE

$$\frac{\mathrm{d}^2F}{\mathrm{d}y^2}-\frac{\kappa}{3}F=0. \tag{69}$$

Using conditions (17) and (18), one can write the boundary conditions for (69) as

$$F(0)=1,\quad F(l)=0,\quad l\to\infty, \tag{70}$$

where

$$V(t)=\exp\left[-\left(\frac{\kappa}{\varphi_*(1-\varphi)^{2.5}}\right)t\right]. \tag{71}$$

We solve (69) subject to the boundary conditions given in (70) for positive $\kappa$; we obtain

$$F(y)=\exp\left(-\sqrt{\frac{\kappa}{3}}y\right). \tag{72}$$

Substituting this $F(y)$ in (68), we deduce the solution for $u(y,\,t)$ in the form

$$u(y,\,t)=\exp\left[-\left\{\left(\frac{\kappa}{\varphi_*(1-\varphi)^{2.5}}\right)t+\sqrt{\frac{\kappa}{3}}y\right\}\right]. \tag{73}$$

## 7 Group Invariant Solutions Corresponding to $\chi_1$

The time translation generator $\chi_1$ is given by

$$\chi_1=\frac{\partial}{\partial t}. \tag{74}$$

The invariant solution admitted by $\chi_1$ is the steady-state solution

$$u(y,\,t)=G(y). \tag{75}$$

Introducing (75) into (20) yields the third-order ODE for $G(y)$, namely,

$$\frac{1}{\varphi_*(1-\varphi)^{2.5}}\frac{\mathrm{d}^2G}{\mathrm{d}y^2}+\frac{3\beta}{\varphi_*}\left(\frac{\mathrm{d}G}{\mathrm{d}y}\right)^2\frac{\mathrm{d}^2G}{\mathrm{d}y^2}-\frac{\kappa\beta}{\varphi_*}G\left(\frac{\mathrm{d}G}{\mathrm{d}y}\right)^2-\frac{\kappa}{\varphi_*(1-\varphi)^{2.5}}G, \tag{76}$$

with the boundary conditions

$$G(0)=v_0,\quad G(y)\to 0,\quad y\to\infty, \tag{77}$$

where $v_0$ can take a sufficiently large value with $V=v_0$ a constant. Again by using the compatibility and generalised group method [48], as discussed in the previous section, the above equation (76) admits the exact solution of the form (which we also require to be zero at infinity due to the second boundary condition)

$$G(y)=v_0\exp\left(-\sqrt{\frac{\kappa}{3}}y\right), \tag{78}$$

provided that the compatibility condition

$$\frac{\kappa}{3\varphi_*(1-\varphi)^{2.5}}-\frac{\kappa}{\varphi_*(1-\varphi)^{2.5}}=0, \tag{79}$$

holds.

# 8 Conservation Laws

In this section, we derive the conservation laws for the governing PDE (20). The conservation laws for this equation are constructed for the first time by using the new conservation theorem of Ibragimov [50]. For details of related definitions and theorems, we refer the reader to [50].

The PDE (20) and its adjoint equation are

$$E\equiv\beta\kappa u_y^2u+c\kappa u+\varphi_*u_t-cu_{yy}-\alpha u_{yyt}-3\beta u_{yy}u_y^2=0, \tag{80}$$

$$\begin{aligned}E^*\equiv{}&\kappa v(-2\beta u_{yy}u-\beta u_y^2+c)-2\beta\kappa u_yv_yu-\varphi_*v_t-cv_{yy}+\alpha v_{yyt}\\&-3\beta v_{yy}u_y^2-6\beta u_{yy}u_yv_y=0,\end{aligned} \tag{81}$$

where $c=1/(1-\varphi)^{2.5}$. The second-order Lagrangian for the system (80)–(81) is given by

$$£=v(\beta\kappa u_y^2u+c\kappa u+\varphi_*u_t-cu_{yy}-\alpha u_{yyt}-3\beta u_{yy}u_y^2). \tag{82}$$

We now have the following two cases.

**Case 1:** Firstly, we consider the time-translation symmetry $X_1=\partial/\partial t$ of PDE (20). The Lie characteristic functions associated to $X_1$ are

$$W^1=-u_t \text{ and } W^2=-v_t. \tag{83}$$

Consequently, by using the Ibragimov theorem [50], the conservation law associated with $X_1$, which gives the conservation law of energy, has components that are given by

$$\begin{aligned}C_1^t={}&\frac{1}{3}[v\{3\kappa u(\beta u_y^2+c)-3u_{yy}(3\beta u_y^2+c)-2\alpha u_{yyt}\}\\&+\alpha(u_tv_{yy}-v_yu_{ty})],\\C_1^y={}&\frac{1}{3}[-u_t\{6\beta\kappa u_yuv+3v_y(3\beta u_y^2+c)-2\alpha v_{ty}\}\\&+u_{ty}\{3v(3\beta u_y^2+c)-\alpha v_t\}+2\alpha vu_{tty}-\alpha u_{tt}v_y].\end{aligned} \tag{84}$$

**Case 2:** Likewise, the space-translation symmetry $X_2=\partial/\partial y$ has the Lie characteristic functions

$$W^1=-u_y \text{ and } W^2=-v_y. \tag{85}$$

The associated conservation law, which gives conservation of linear momentum, has components given by

$$\begin{aligned}C_2^t={}&\frac{1}{3}[u_y(\alpha v_{yy}-3\varphi_*v)+\alpha u_{yyy}v-\alpha u_{yy}v_y],\\C_2^y={}&\frac{1}{3}[v(3\varphi_*u_t-\alpha u_{tyy})+3\kappa uv(c-\beta u_y^2)\\&+u_y(2\alpha v_{ty}-3cv_y)-\alpha(v_tu_{yy}+v_yu_{ty})-9\beta u_y^3v_y].\end{aligned} \tag{86}$$

It is important to remark here that it is very rare in the literature that the conservation laws are found for a non-Newtonian fluid model equation. One can use the notion of conservation laws and associated Lie point symmetries to formulate exact solutions of such type of complicated equations arising in the study of both experimental and theoretical non-Newtonian fluid mechanics. Such study will be the subject of our future investigations.

# 9 Graphical Results and Discussion

The acquired velocity profiles from the applicable sections are contained herewith by means of graphical plots versus $y$ and these are demonstrated in Figures 1–4. The objective of such an enterprise is to study the behaviour of a number of meaningful parameters relative to third-grade nanofluid flow on the structure of the velocity field. In doing so, we would like to make some inferences and observations with regard to their physical significance for the third-grade nanofluid flow model.

The influence of the nanofluid particles on the structure of the flow model is described in Figure 1. The graphs are plotted for copper–water nanofluid and it is clear from the figure that an increase in $\varphi$ caused an increase in fluid velocity. These profiles are in agreement with physical-based nanofluids. Moreover, the velocity attains its maximum value near the surface but gradually decreases to zero at the free stream far away from the plate satisfying

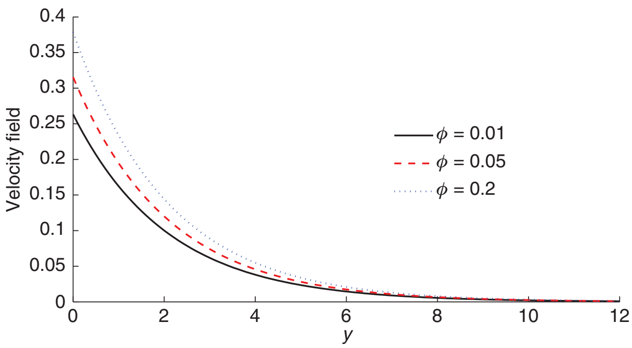


**Figure 1:** Influence of the nanofluid volume concentration $\varphi$ on the velocity field (41) with $\rho_s = 8933$, $\rho_f = 997.1$, $\alpha = 0.1$, $\kappa = 0.7$, and $t = \pi$ fixed.

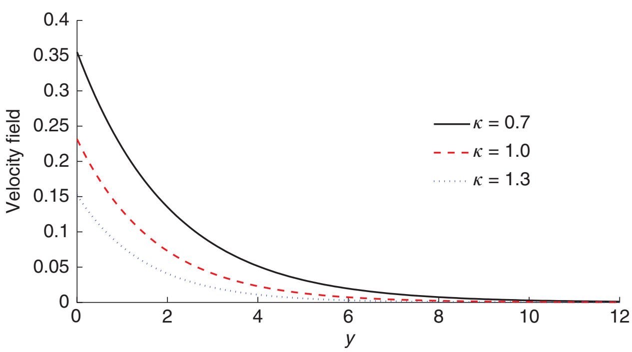


**Figure 2:** Influence of the porosity parameter $\kappa$ on the velocity field (41) with $\rho_s = 8933$, $\rho_f = 997.1$, $\alpha = 0.1$, $\varphi = 0.1$, and $t = \pi$ fixed.

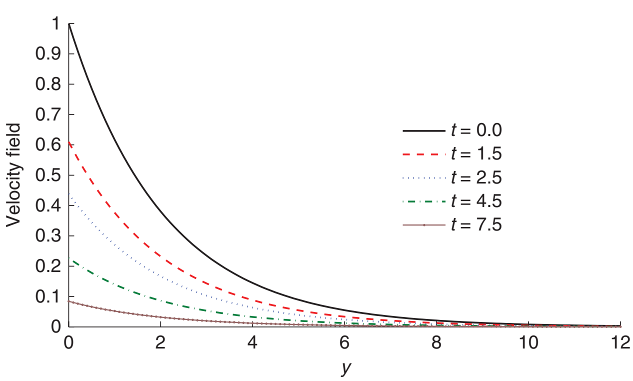


**Figure 3:** Influence of the time $t$ on the velocity field (41) with $\rho_s = 8933$, $\rho_f = 997.1$, $\alpha = 0.1$, $\varphi = 0.1$, $\kappa = 1$, and $t = \pi$ fixed.

the boundary conditions, thus supporting the validity of the obtained results.

Figure 2 has been plotted to show the influence of the porosity of the porous medium $\kappa$ on the velocity field (41). As anticipated, with an increase in the porosity of the porous medium causes an increase in the drag force and hence the velocity decreases.

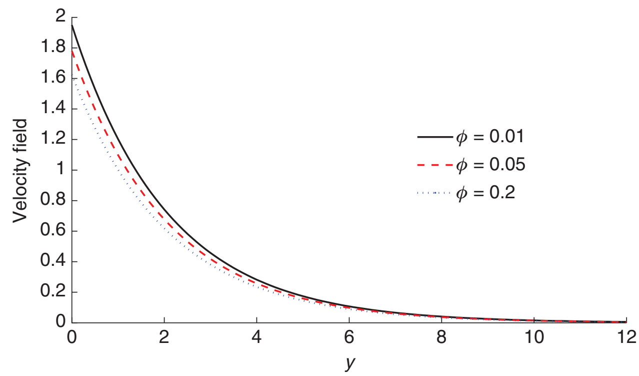


**Figure 4:** Influence of the nanofluid volume concentration $\varphi$ on the velocity field (49) with $\rho_s = 8933$, $\rho_f = 997.1$, $\alpha = 0.2$, $\kappa = 0.7$, and $t = \pi$ fixed.

The graphical behaviour of the travelling wave solution (41) for varying values of time $t$ is shown in Figure 3. This figure depicts that the velocity decreases as the time increases. Clearly, the variation of velocity is observed for $0 \le t \le 7.5$. For $t > 7.5$, the velocity profile remains the same. In other words, we can say that the steady-state behaviour for the velocity is achieved for $t > 7.5$.

In Figure 4, the group-invariant solution (49) is plotted for the varying values of the nanofluid volume concentration parameter $\varphi$. With the increase in $\varphi$ the velocity profile decreases. This in turn decreases the thickness of the momentum boundary layer.

## 10 Conclusions

In this article, we have focused on highlighting the study of a non-Newtonian nanofluid flow model. The porous medium is also taken into consideration. The governing PDE of the non-Newtonian model along with nanoparticles is solved by employing the group theoretical and compatibility approach. The classically invariant solutions are formulated in the form of exponential functions. Furthermore, the conservation laws for the governing nonlinear PDE are also derived by employing the Ibragimov method. The importance of constructing the conservation laws has been discussed in the introduction.

The physical mechanism in the problem is diffusion. The fluid velocity is generated by the no-slip boundary condition when the plate is impulsively set in motion with a time-dependent velocity and diffuses in the direction towards the axis of the flow. This causes the velocity profiles to flatten out and the shear stress across the medium to steadily decrease and vanish as $t \to \infty$. The model has some features in common with the classical Stokes' model for flow induced in a half-space of viscous fluid when a plate is impulsively set in motion. However, the present study can be described as a generalised Stokes' flow for which the plate is impulsively set in motion with some time-dependent velocity which cannot be prescribed arbitrarily but depends on the physical parameters of the flow model. The results presented in this paper will now be available for experimental verification of the same type of nonlinear boundary value problems. It should be remarked that the flow model for this particular study has not been solved earlier by any traditional numerical approach.

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