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Conservation laws and Exact Solutions of Phi-Four (Phi-4) Equation via the $(G'/G, 1/G)$-Expansion Method

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Abstract: In this article, we constructed formal Lagrangian of Phi-4 equation, and then via this formal Lagrangian, we found adjoint equation. We investigated if the Lie point symmetries of the equation satisfy invariance condition or not. Then we used conservation theorem to find conservation laws of Phi-4 equation. Finally, the exact solutions of the equation were obtained through the $(G'/G, 1/G)$-expansion method.

Keywords: Conservation Law; Exact Solution; $(G'/G, 1/G)$-Expansion Method; Phi-4 Equation.

AMS 2010: 33F10; 70S10; 83C15.

1 Introduction

The concept of conservation laws has a long and profound history in physics. Whatever the physical laws considered: classical mechanics; fluid mechanics; solid-state physics; and quantum mechanics, quantum field theory, or general relativity and whatever the constituents of the theory and the intricate dynamic processes involved, quantities left dynamically invariant have always been essential ingredients to describe nature. At the mathematical level, conservation laws are deeply connected with the existence of a variational principle that admits symmetry transformations. This crucial fact was fully acknowledged by Emmy Noether in 1918 [1, 2].

Construction of conservation laws for a given system is generally a nontrivial task. However, for a system arising from a Lagrangian formulation, there exists a fundamental theorem by Emmy Noether. He proved that for every infinitesimal transformation that is admitted by the action integral of a Lagrangian system, one can constructively find a conservation law [3, 4].

There are many different approaches to the construction of conservation laws such as characteristic method, variational approach [5], symmetry and conservation law relation [6, 7], direct construction method for conservation laws [8], partial Noether's approach [9], Noether's approach [10], and conservation theorem [11–15]. In this work, we will use conservation theorem approach for finding Phi-4 equation's conservation laws.

Searching exact solutions of nonlinear partial differential equations (PDEs) plays an important and significant role in the study on the dynamics of those phenomena. Many mathematical techniques have been employed to find exact solutions to these equations. These methods include the inverse scattering method [16], Hirota's bilinear method [17], the tanh method [18], transformed rational function method [19], the sine–cosine method [20, 21], the homogeneous balance method [22], the exp-function method [23], the extended trial equation method [24], the modified simple equation (MSE) method [25, 26], the first integral method [27], the auxiliary equation method [28], the (G'/G)-expansion method [29], the $(1/G')$-expansion method [30, 31], Kudryashov's method [32], and so on.

The fundamental principle of the (G'/G)-expansion method is that the exact solutions of nonlinear PDEs can be expressed by a polynomial in (G'/G) in which $G=G(\xi)$ satisfies the differential equation $G''(\xi)+\lambda G'(\xi)+\mu G(\xi)=0$. Here, λ and μ are constants [29].

The $(G'/G, 1/G)$-expansion method can be considered as a generalisation of the (G'/G)-expansion method. The fundamental principle of the $(G'/G, 1/G)$-expansion method is that the exact travelling wave solutions of nonlinear PDEs can be expressed by a polynomial in the two variables (G'/G) and $(1/G)$, in which $G=G(\xi)$ satisfies a second-order linear ordinary differential equation (ODE), namely $G''(\xi)+\lambda G(\xi)=\mu$, where λ and μ are constants. The degree of this polynomial can be determined via the

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homogeneous balance method. Also, the coefficients of this polynomial can be determined by solving a set of algebraic equations resulted from the process of using the method [33–36].

The rest of this article has been arranged as follows. In Section 2, basic definitions about the conservation laws and the basic ideas of the $(G'/G, 1/G)$-expansion method are given. In Section 3, the conservation laws of the Phi-4 equation are found. In Section 4, the method is employed for obtaining the exact solutions of Phi-4 equation. Finally in Section 5, some conclusions are presented.

2 Definitions

Firstly, it would be helpful to give some definitions and formulations of the conservation laws theory in this part of the article.

2.1 Conservation Laws

Firstly, we consider the k-th-order PDEs in the following form:

$$F(x, u, u_1, u_2, \dots, u_k)=0, \tag{1}$$

where $x=(x_1, x_2, \dots, x_n)$, n independent variables; $u=(u^1, u^2, \dots, u^m)$, m dependent variables; and u_j is j-th-order derivation with respect to x, $(j=1, 2, \dots, k$ and $i_j=1, 2, \dots, n)$. The infinitesimal generator for the governing equation (1) can be written as follows:

$$X=\xi^i \frac{\partial}{\partial x_i}+\eta^\alpha \frac{\partial}{\partial u^\alpha}, \tag{2}$$

where ξ^i and η^α are the infinitesimal functions, $i=1, 2, \dots, n$ and $\alpha=1, 2, \dots, m$. These functions are only the functions of dependent and independent variables. Namely, we investigate only the Lie point symmetries in this articel. The k-th prolongation of the infinitesimal generator is

$$X^{(k)}=X+\eta_i^{(1)\alpha} \frac{\partial}{\partial u_i^\alpha}+\dots+\eta_{i_1 i_2 \dots i_k}^{(k)\alpha} \frac{\partial}{\partial u_{i_1 i_2 \dots i_k}^\alpha}, \; k\geq 1, \tag{3}$$

where

$$\eta_i^{(1)\alpha}=D_i\eta^\alpha-(D_i\xi^j)u_j^\alpha,$$
$$\eta_{i_1 i_2 \dots i_k}^{(k)\alpha}=D_{i_k}\eta_{i_1 i_2 \dots i_{k-1}}^{(k-1)\alpha}-(D_{i_k}\xi^j)u_{i_1 i_2 \dots i_{k-1} j}^\alpha,$$

where $i, j=1, 2, \dots, n$ and $\alpha=1, 2, \dots, m$ and $i_l=1, 2, \dots, m$ for $l=1, 2, \dots, k$ and D_i is the total derivative operator. In this case, Lagrangian in the lower form for (1) is

$$L=L(x, u, u_1, \dots, u_k). \tag{4}$$

Then the Euler–Lagrange equation can be founded by

$$\frac{\delta L}{\delta u^\alpha}=0, \; (\alpha=1, \dots, m), \tag{5}$$

where $\delta/\delta u^\alpha$ is the variational derivative with

$$\frac{\delta}{\delta u^\alpha}=\frac{\partial}{\partial u^\alpha}+\sum_{s=0}^{\infty}(-1)^s D_{i_1}\dots D_{i_s}\frac{\partial}{\partial u_{i_1\dots i_s}^\alpha}, (\alpha=1, \dots, m).$$

Formal Lagrangian is the multiplication of a new adjoint variable $w=(w^1, w^2, \dots, w^\beta)$, with a given equation. Namely,

$$L=w^\beta F. \tag{6}$$

Adjoint equation can be obtained as follows:

$$F_\alpha^*(x, u, w, \dots, u_{(k)}, w_{(k)})\equiv\frac{\delta(w^\beta F)}{\delta u^\alpha}=0, \; \alpha=1, 2, \dots, m \tag{7}$$

Adjoint equation F_α^* inherits symmetries of the original equation. By substituting $w=u$ in (7) if we obtain (1), then (1) is said self-adjoint equation.

When formal Lagrangian and infinitesimal functions are substituted in the following formula,

$$T^i=\xi^i L+W^\alpha\frac{\delta L}{\delta u_i^\alpha}+\sum_{s\geq 1}D_{i_1}\dots D_{i_s}(W^\alpha)\frac{\partial}{\partial u_{i_1\dots i_s}^\alpha}, \; (i=1, \dots, n) \tag{8}$$

where $W^\alpha=\eta^\alpha-\xi^j u_j^\alpha$ and $\alpha=1, \dots, m$, T^i conserved quantities are obtained. The conserved quantities obtained from (8) involves the arbitrary solutions w of the adjoint equation, and hence one obtains an infinite number of conservation laws for (1) by specifying w.

If

$$D_i(T^i)=0, \tag{9}$$

then the conserved quantities are conservation laws of (1). Here, the total differentiation operator is that

$$D_i=\frac{\partial}{\partial x^i}+u_i^\alpha\frac{\partial}{\partial u^\alpha}+u_{ij}^\alpha\frac{\partial}{\partial u_j^\alpha}+\cdots.$$

2.2 $(G'/G, 1/G)$-Expansion Method

In the current section, the main steps of the $(G'/G, 1/G)$-expansion method for finding exact solutions of nonlinear PDEs are described [37–41]. As preparations, let us consider the following second-order linear ODE:

$$G''(\xi)+\lambda G(\xi)=\mu, \tag{10}$$

where λ and μ are constants. We choose here

$$\phi = G'/G, \quad \psi = 1/G \tag{11}$$

for simplicity. Using (10) and (11) yields

$$\phi' = -\phi^2 + \mu\psi - \lambda, \quad \psi' = -\phi\psi. \tag{12}$$

For general solutions of (10), we get the following three cases.

Case 1 ($\lambda > 0$)*:* The general solution of (10) is

$$G(\xi) = A_1 \sin(\sqrt{\lambda}\xi) + A_2 \cos(\sqrt{\lambda}\xi) + \frac{\mu}{\lambda},$$

and we have

$$\psi^2 = \frac{\lambda}{\lambda^2\sigma - \mu^2}(\phi^2 - 2\mu\psi + \lambda), \tag{13}$$

where A_1 and A_2 are two arbitrary constants, and $\sigma = A_1^2 + A_2^2$.

Case 2 ($\lambda > 0$)*:* The general solution of (10) is

$$G(\xi) = A_1 \sinh(\sqrt{-\lambda}\xi) + A_2 \cosh(\sqrt{-\lambda}\xi) + \frac{\mu}{\lambda},$$

and we have

$$\psi^2 = \frac{-\lambda}{\lambda^2\sigma + \mu^2}(\phi^2 - 2\mu\psi + \lambda), \tag{14}$$

where A_1 and A_2 are two arbitrary constants, and $\sigma = A_1^2 - A_2^2$.

Case 3 ($\lambda = 0$)*:* The general solution of (10) is

$$G(\xi) = \frac{\mu}{2}\xi^2 + A_1\xi + A_2,$$

and we have

$$\psi^2 = \frac{1}{A_1^2 - 2\mu A_2}(\phi^2 - 2\mu\psi), \tag{15}$$

where A_1 and A_2 are two arbitrary constants.

After these preparations, we consider a general nonlinear PDE in the following form:

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \ldots) = 0, \tag{16}$$

where x and t are independent variables, and (16) is a polynomial of u and its partial derivatives. The main steps of the $(G'/G, 1/G)$-expansion method are as follows.

Step 1 Using the following transformation $\xi = x - vt$, $u(x, t) = u(\xi)$, (16) reduces to following ODE:

$$P(u, -vu', u', v^2u'', -vu', u'', \ldots) = 0, \tag{17}$$

where v is the wave velocity.

Step 2 We suppose that the formal solution of (17) can be expressed as follows:

$$u(\xi) = \sum_{i=0}^{N} a_i\phi^i + \sum_{i=1}^{N} b_i\phi^{i-1}\psi, \tag{18}$$

where $a_i (i = 0, \ldots, N)$ and $b_i (i = 1, \ldots, N)$ are constants to be determined later, and the positive integer N can be determined by using the homogeneous balance method between the highest-order derivatives and the nonlinear terms in (17).

Step 3 Substituting (18) into (17), using (12) and (13) [here Case 1 is taken as example, so (13) is used], the left-hand side of (17) can be converted into a polynomial in ϕ and ψ. Note that the degree of ψ is not larger than one. Equating each coefficient of this polynomial to zero, we can get an algebraic system of equations in terms of $a_i (i = 0, \ldots, N)$ $b_i (i = 1, \ldots, N)$, v, $\lambda (\lambda < 0)$, μ, A_1, and A_2.

Step 4 We solve the algebraic equations verified in Step 3 with the aid of Maple. Substituting the values of $a_i (i = 0, \ldots, N)$ $b_i (i = 1, \ldots, N)$, v, λ, μ, A_1 and A_2 obtained into (18), the travelling wave solutions of (17) expressed by the hyperbolic functions can be obtained.

Step 5 Similar to Steps 3 and 4, substituting (18) into (17), using (12) and (14) [or (12) and (15)], we can get the travelling wave solutions of (17) expressed by trigonometric functions (or expressed by rational functions).

3 Conservation Laws of the Phi-4 Equation

The Phi-4 equation plays an important role in nuclear and particle physics over the decades:

$$u_{tt} - u_{xx} + m^2u + cu^3 = 0, \tag{19}$$

where m and c are real constants. In this section, we will find conservation laws for the Phi-4 equation.

Let us construct formal Lagrangian for (19) in the following form:

$$L = (u_{tt} - u_{xx} + m^2u + cu^3)w(x, t), \tag{20}$$

where $w = w(x, t)$ is the adjoint variable. If we substitute (20) into (7), then the adjoint equation can be verified as follows:

$$F^{*}=\frac{\partial L}{\partial u}+D_{xx}\left(\frac{\partial L}{\partial u_{xx}}\right)+D_{tt}\left(\frac{\partial L}{\partial u_{tt}}\right)$$
$$=m^2w+3cu^2w-w_{xx}+w_{tt}. \tag{21}$$

Now, we can see that (19) is not obtained by substituting $w=u$ in (21). Consequently, we can say that (19) is not self-adjoint.

The Phi-4 equation admits following three Lie point symmetry generators:

$$X_1=\frac{\partial}{\partial x},$$
$$X_2=\frac{\partial}{\partial t}, \tag{22}$$
$$X_3=t\frac{\partial}{\partial x}+x\frac{\partial}{\partial t}.$$

In addition, above Lie point symmetry generators' commutator table is as follows:

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	0	X_2
X_2	0	0	X_1
X_3	$-X_2$	$-X_1$	0

Now, we will find conservation laws of (19).

Case 1: We consider

$$X_1=\frac{\partial}{\partial x}, \tag{23}$$

the Lie point symmetry generator. Presently, we will show that X_1 satisfies by invariance test:

$$X(L)+L(D_i(\xi^i))=0. \tag{24}$$

Second prolonged vector of X_1 is

$$X_1^{(2)}=\xi^x\frac{\partial}{\partial x}+\xi^t\frac{\partial}{\partial t}+\eta\frac{\partial}{\partial u}+\eta^x\frac{\partial}{\partial u_x}+\eta^t\frac{\partial}{\partial u_t}+\eta^{xx}\frac{\partial}{\partial u_{xx}}+\eta^{tt}\frac{\partial}{\partial u_{tt}}. \tag{25}$$

Here,

$$\eta_i^{(1)}=D_i\eta-(D_i\xi_j)u_j \tag{26}$$

and

$$\eta^{(k)}_{i_1i_2\ldots i_k}=D_{i_k}\eta^{(k-1)}_{i_1i_2\ldots i_{k-1}}-(D_{i_k}\xi_j)u_{i_1i_2\ldots i_{(k-1)j}},\ k=2,\ l=1,\ 2,\ i_l=1,\ 2. \tag{27}$$

Here, $i=1, 2, \ldots$. We get from the above formulations:

$$\eta^x=\eta^{xx}=\eta^t=\eta^{tt}=0. \tag{28}$$

By substituting (28) into (25), we get $X_1^{(2)}=\partial/\partial x$. If we substitute these values into (24), invariance condition is clearly satisfied:

$$X^{(2)}(L)+L(D_x(\xi^x)+D_t(\xi^t))=0.$$

For X_1,

$$\xi^x=1,$$
$$\xi^t=0, \tag{29}$$
$$\eta=0.$$

If we use (8), conserved vectors' formulations are

$$T^t=\xi^tL+W\left[-D_t\left(\frac{\partial L}{\partial u_{tt}}\right)\right]+D_t(W)\frac{\partial L}{\partial u_{tt}},$$
$$T^x=\xi^xL+W\left[-D_x\left(\frac{\partial L}{\partial u_{xx}}\right)\right]+D_x(W)\frac{\partial L}{\partial u_{xx}}. \tag{30}$$

We can find W with (29) in the following form:

$$W=\eta-\xi^xu_x-\xi^tu_t$$
$$=-u_x. \tag{31}$$

As a result, conserved vectors are

$$T_1^t=u_xw_t-u_{xt}w,$$
$$T_1^x=wu_{tt}+wm^2u+wcu^3-u_xw_x, \tag{32}$$

for (19) with $X_1=\partial/\partial x$ Lie point symmetry generator.

If we substitute (32) into (9),

$$D_t(T_1^t)+D_x(T_1^x)=u_{xt}w_t+u_xw_{tt}-u_{xtt}w-u_{xt}w_t+u_{tt}w_x+wu_{xtt}$$
$$+w_xm^2u+wm^2u_x+w_xcu^3+3wcu^2u_x-u_{xx}w_x-u_xw_{xx}$$
$$=u_x\underbrace{(m^2w+3cu^2w-w_{xx}+w_{tt})}_{F^*}$$
$$+w_x\underbrace{(u_{tt}-u_{xx}+m^2u+cu^3)}_{F}$$
$$=0 \tag{33}$$

is satisfied.

Case 2: Secondly, we consider

$$X_2=\frac{\partial}{\partial t}. \tag{34}$$

Equation (19) satisfies invariance test:

$$X^{(2)}(L)+L(D_x(\xi^x)+D_t(\xi^t))=0,$$

for $X_2=\partial/\partial t$ Lie-point symmetry.

Let us construct conservation laws corresponding to symmetry X_2. From X_2,

$$\begin{aligned}\xi^x&=0,\\ \xi^t&=1,\\ \eta&=0\end{aligned}\tag{35}$$

are obtained. If we use $W=\eta-\xi^x u_x-\xi^t u_t$ and (35), we obtain

$$W=-u_t.\tag{36}$$

By substituting (36) into (30), we obtain the following conserved vectors:

$$\begin{aligned}T_2^t&=-wu_{xx}+wm^2u+wcu^3+u_tw_t,\\ T_2^x&=-u_tw_x+u_{xt}w.\end{aligned}\tag{37}$$

For (37), (9)

$$\begin{aligned}D_t(T_2^t)+D_x(T_2^x)=&u_t\underbrace{(m^2w+3cu^2w-w_{xx}+w_{tt})}_{F^*}\\ &+w_t\underbrace{(u_{tt}-u_{xx}+m^2u+cu^3)}_{F}\\ &=0\end{aligned}\tag{38}$$

is satisfied. Therefore, T_2^t and T_2^x are conservation laws for (19).

Case 3: Thirdly, we will use the following Lie point symmetry:

$$X_3=t\frac{\partial}{\partial x}+x\frac{\partial}{\partial t}.\tag{39}$$

Equation (19) satisfies invariance test for X_3. For the operator X_3,

$$\begin{aligned}\xi^x&=t,\\ \xi^t&=x,\\ \eta&=0.\end{aligned}\tag{40}$$

We have

$$W=-u_xt-u_tx.\tag{41}$$

If we substitute (41) into (30), conserved vectors are

$$\begin{aligned}T_3^t&=wx(-u_{xx}+m^2u+cu^3)+w_t(u_xt+u_tx)-w(u_{xt}t+u_x),\\ T_3^x&=wt(u_{tt}+m^2u+cu^3)-w_x(u_xt+u_tx)+w(u_{xt}x+u_t).\end{aligned}\tag{42}$$

Using (9), for (42) it is shown that

$$\begin{aligned}D_t(T_3^t)+D_x(T_3^x)=&tw_x\underbrace{(u_{tt}-u_{xx}+m^2u+cu^3)}_{F}\\ &+tu_x\underbrace{(m^2w+3cu^2w-w_{xx}+w_{tt})}_{F^*}\\ &+xw_t\underbrace{(u_{tt}-u_{xx}+m^2u+cu^3)}_{F}+xu_t\underbrace{(m^2w+3cu^2w-w_{xx}+w_{tt})}_{F^*}\\ &=0.\end{aligned}\tag{43}$$

Therefore, invariance condition is satisfied.

4 Exact Solutions of Phi-4 Equation

In this section, the presented method is applied to the Phi-4 equation (7). For our purpose, we introduce the following transformation as $u(x,t)=u(\xi)$, $\xi=x-vt$, where v is a constant. With this transformation, (7) reduced to following ODE:

$$(v^2-1)u''+m^2u+cu^3=0,\tag{44}$$

where prime denotes the derivative with respect to ξ.

According to the homogeneous balance method, we get the balancing number as $N=1$. Therefore, the solution (18) takes the following form:

$$u(\xi)=a_0+a_1\phi+b_1\psi,\tag{45}$$

where a_0, a_1, and b_1 are constants to be determined. Let us examine the three cases mentioned above.

Case 1: ($\lambda>0$ Trigonometric function solutions)

Substituting (45) into (44), using (12) and (13), the left-hand side of (44) becomes a polynomial in ϕ and ψ. Equating each coefficient of this polynomial to zero yields a set of the following algebraic equations in a_0, a_1, b_1, m, v, σ, μ, and λ:

$$\begin{aligned}\phi^3\quad:\quad&-2a_1\lambda^4\sigma^2+ca_1^3\mu^4-2ca_1^3\lambda^2\sigma\mu^2+2v^2a_1\lambda^4\sigma^2+ca_1^3\lambda^4\sigma^2\\ &+2v^2a_1\mu^4+4a_1\lambda^2\sigma\mu^2+3ca_1b_1^2\lambda^3\sigma-3ca_1b_1^2\lambda\mu^2\\ &-4v^2a_1\lambda^2\sigma\mu^2-2a_1\mu^4=0,\\ \phi^2\psi^1\quad:\quad&-4v^2b_1\lambda^2\mu^2+2v^2b_1\mu^4-cb_1^3\lambda\mu^2+2v^2b_1\lambda^4\sigma^2\\ &+3ca_1^2b_1\lambda^4\sigma^2+3ca_1^2b_1\mu^4-6ca_1^2b_1\lambda^2\sigma\mu^2+4b_1\lambda^2\sigma\mu^2\\ &-2b_1\mu^4+cb_1^3\lambda^3\sigma-2b_1\lambda^4\sigma^2=0,\\ \phi^2\psi^0\quad:\quad&3ca_0b_1^2\lambda^3\sigma-v^2b_1\mu\lambda^3\sigma+3ca_0a_1^2\lambda^4\sigma^2-2cb_1^3\lambda^2\mu\\ &-3ca_0b_1^2\lambda\mu^2-b_1\mu^3\lambda+b_1\mu\lambda^3\sigma+v^2b_1\mu^3\lambda+3ca_0a_1^2\mu^4\\ &-6ca_0a_1^2\lambda^2\sigma\mu^2=0,\end{aligned}$$

$$\begin{aligned}
\phi^1\psi^1 \quad : \quad & -4v^2b_1\lambda^2\sigma\mu^2+2v^2b_1\mu^4+2v^2b_1\lambda^4\sigma^2+3ca_1^2b_1\lambda^4\sigma^2\\
& -6ca_1^2b_1\lambda^2\sigma\mu^2-cb_1^3\lambda\mu^2+4b_1\lambda^2\sigma\mu^2-2b_1\mu^4+cb_1^3\lambda^3\sigma\\
& +3ca_1^2b_1\mu^4-2b_1\lambda^4\sigma^2=0,\\
\phi^1\psi^0 \quad : \quad & 3ca_0b_1^2\lambda^3\sigma-v^2b_1\mu\lambda^3\sigma+3ca_0a_1^2\lambda^4\sigma^2-2cb_1^3\lambda^2\mu\\
& -3ca_0b_1^2\lambda\mu^2-b_1\mu^3\lambda+b1\mu\lambda^3\sigma+v^2b_1\mu^3\lambda+3ca_0a_1^2\mu^4\\
& -6ca_0a_1^2\lambda^2\sigma\mu^2=0,\\
\phi^0\psi^1 \quad : \quad & 3a_1\mu^5-3v^2a_1\mu\lambda^4\sigma^2+6ca_1b_1^2\lambda\mu^3-12ca_0a_1b_1\lambda^2\tau\mu^2\\
& -3v^2a_1\mu^5+6ca_0a_1b_1\lambda^4\tau^2-6ca_1b_1^2\lambda^3\mu\tau+6ca_0a_1b_1\mu^4\\
& +3a_1\mu\lambda^4\tau^2-6a_1\mu^3\lambda^2\tau+6v^2a_1\mu^3\lambda^2\tau=0,\\
\phi^0\psi^0 \quad : \quad & 3ca_0^2a_1\mu^4+2v^2a_1\lambda^5\sigma^2+2v^2a_1\lambda\mu^4+4a_1\lambda^3\mu^2\\
& +3ca_1b_1^2\lambda^4\sigma-a_0^3\lambda^4\sigma^2-3ca_1b_1^2\lambda^2\mu^2+3ca_0^2a_1\lambda^4\sigma^2\\
& -4v^2a_1\lambda^3\sigma\mu^2-6ca_0^2a_1\lambda^2\sigma\mu^2-2m^2a_1\lambda^2\sigma\mu^2\\
& +m^2a_1\lambda^4\sigma^2+m^2a_1\mu^4-2a_1\lambda^5\sigma^2-2a_1\lambda\mu^4=0.
\end{aligned}$$

Solving this system with the aid of Maple, we get

$$a_0=0,\quad a_1=\sqrt{-\frac{(v^2-1)}{2c}},\quad b_1=\sqrt{-\frac{(v^2-1)(\lambda^2\tau-\mu^2)}{2c\lambda}},\quad m=\sqrt{-\frac{1}{2}\lambda(v^2-1)},$$

where λ, $c\neq 0$. By substituting these values into (45), using (11) and (13), we obtain the exact solutions of Phi-4 as follows:

$$u(\xi)=\frac{\sqrt{-\frac{\lambda(v^2-1)}{2c}}(A_1\cos(\sqrt{\lambda}\xi)-A_2\sin(\sqrt{\lambda}\xi))}{A_1\sin(\sqrt{\lambda}\xi)+A_2\cos(\sqrt{\lambda}\xi)+\frac{\mu}{\lambda}}+\frac{\sqrt{\frac{((A_1^2+A_2^2)\lambda^2-\mu^2)(v^2-1)}{2c\lambda}}}{A_1\sin(\sqrt{\lambda}\xi)+A_2\cos(\sqrt{\lambda}\xi)+\frac{\mu}{\lambda}}, \tag{46}$$

where

$$\xi=x-vt \text{ and } \sigma=A_1^2+A_2^2.$$

The Figure 1 of this obtained solution can be given as follows by choosing special values for arbitrary constants:

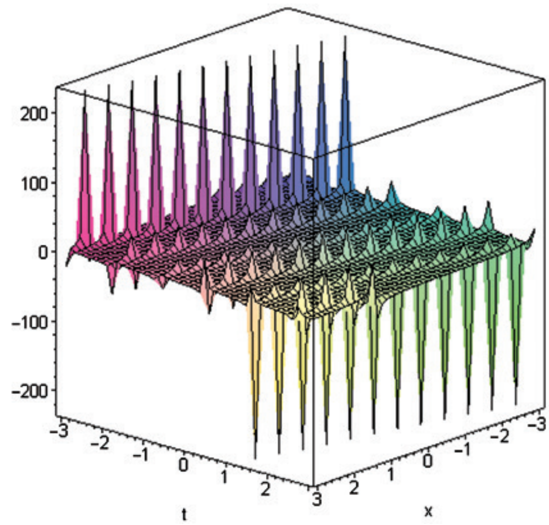

Figure 1: Graph of the $u(x, t)$ corresponding to the values: $v=-4$, $A_1=1$, $A_2=-1$, $\mu=2$, $\lambda=2$, $c=-5$.

Case 2: ($\lambda<0$ hyperbolic function solutions)

Substituting (45) into (44), using (12) and (14), the left-hand side of (44) becomes a polynomial in ϕ and ψ. By equating each coefficient of this system to zero, we get a set of algebraic equations. Solving this system with Maple, we obtain

$$a_0=0,\ a_1=\sqrt{-\frac{(v^2-1)}{2c}},\ b_1=\sqrt{\frac{(v^2-1)(\lambda^2\tau+\mu^2)}{2c\lambda}},\quad m=\sqrt{-\frac{1}{2}\lambda(v^2-1)}.$$

where λ, $c\neq 0$. Substituting these values into (45), using (11) and (14), we get exact solutions of Phi-4 as follows:

$$u(\xi)=\frac{\sqrt{\frac{\lambda(v^2-1)}{2c}}(A_1\cosh(\sqrt{-\lambda}\xi)+A_2\sinh(\sqrt{-\lambda}\xi))}{A_1\sinh(\sqrt{-\lambda}\xi)+A_2\cosh(\sqrt{-\lambda}\xi)+\frac{\mu}{\lambda}}+\frac{\sqrt{\frac{(v^2-1)[\lambda^2(A_1^2-A_2^2)+\mu^2]}{2c\lambda}}}{A_1\sinh(\sqrt{-\lambda}\xi)+A_2\cosh(\sqrt{-\lambda}\xi)+\frac{\mu}{\lambda}}, \tag{47}$$

where

$$\xi=x-vt \text{ and } \sigma=A_1^2-A_2^2.$$

The Figure 2 of this obtained solution can be represented by choosing special values for arbitrary constants.

Case 3: ($\lambda=0$ rational function solutions)

Substituting (45) into (44), using (12) and (15), the left-hand side of (44) becomes a polynomial in ϕ and ψ. By using the same procedure above, we have a set of algebraic equations. If we solve this system, we get

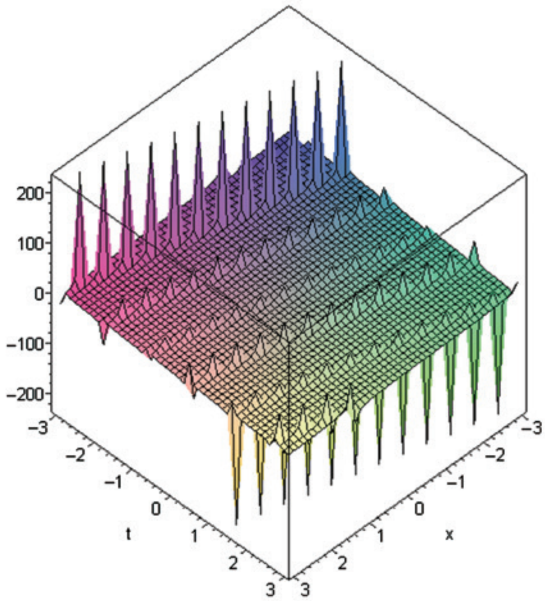

Figure 2: Graph of the $u(x, t)$ corresponding to the values: $v=-4$, $A_1=1$, $A_2=-1$, $\mu=2$, $\lambda=2$, $c=-5$.

$$a_0=0,\quad a_1=\sqrt{-\frac{(v^2-1)}{2c}},\quad b_1=\sqrt{-\frac{(v^2-1)(A_1^2-2A_2\mu)}{2c}},$$
$$m=0,$$

where $c\neq 0$. Substituting these values into (45), using (11) and (15), we get exact solutions of Phi-4 as follows:

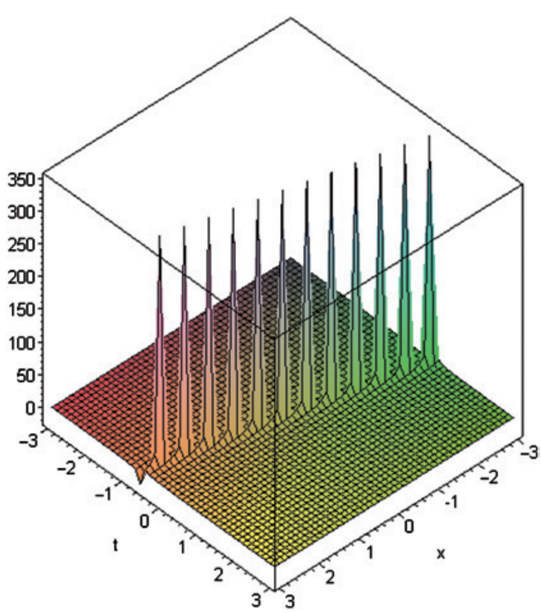

Figure 3: Graph of the $u(x, t)$ corresponding to the values: $v=-4$, $A_1=1$, $A_2=-1$, $\mu=2$, $c=-5$.

$$u(\xi)=\frac{\sqrt{-\frac{(v^2-1)}{2c}}(\mu\xi+A_1)}{\frac{\mu\xi^2}{2}+A_1\xi+A_2}+\frac{\sqrt{\frac{(v^2-1)(A_1^2-2A_2\mu)}{2c}}}{\frac{\mu\xi^2}{2}+A_1\xi+A_2},$$

where $\xi=x-vt$. We can graph the Figure 3 of the solution by choosing special values for arbitrary constants.

Note that our solutions are new and more extensive than the given ones in [42]. Comparing our solutions with [42], it can be seen that by choosing suitable values for the parameters, similar solutions can be verified.

5 Conclusion

In this study, we used conservation theorem for obtaining conservation laws of the Phi-4 equation. Also through the $(G'/G, 1/G)$-expansion method, we have investigated exact travelling wave solutions of the Phi-4 equation. The method proposed in this work is expected to be further employed to solve similar nonlinear problems in applied mathematics and mathematical physics. This work illustrates the validity and great potential of the $(G'/G, 1/G)$-expansion method for nonlinear PDEs. All the obtained solutions in this article have been checked by Maple software.

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