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Properties of Bessel Function Solution to Kepler's Equation with Application to Opposition and Conjunction of Earth–Mars

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Abstract: In this article, a simple approach is suggested to calculate the approximate dates of opposition and conjunction of Earth and Mars since their opposition on August 28, 2003 (at perihelion of Mars). The goal of this article has been achieved via using accurate analytical solution to Kepler's equation in terms of Bessel function. The periodicity property of this solution and its particular values at specified times are discussed through some lemmas. The mathematical conditions of opposition and conjunction of the two planets are formulated. Moreover, the intervals of opposition and conjunction have been determined using the graphs of some defined functions. The calculations reveal that there are nine possible oppositions and conjunctions for Earth and Mars during 20 years started on August 28, 2003. The dates of such oppositions and conjunctions were approximately determined and listed in Tables. It is found that our calculations differ few days from the published real dates of Earth–Mars oppositions due to the neglected effects of the gravitational attraction of other planets in the Solar system on the motion of two planets. The period of 20 years can be extended for any number of years by following the suggested analysis. Furthermore, the current approach may be extended to study the opposition and conjunction of the Earth and any outer planet.

Keywords: Bessel Function Solution; Conjunction; Kepler's Equation; Opposition.

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1 Introduction

Opposition is a term used to indicate when one celestial body is on the opposite side of the sky when viewed from a particular place (usually the Earth). An opposition occurs when the planet is opposite from the Sun, relative to the Earth, see Figure 1. At opposition, the planet will rise as the Sun sets and will set as the Sun rises providing an entire night of observation (just like with a full moon). Also at opposition, the planet comes physically closest to the Earth in its orbit so it appears as large as possible. For planets outside the Earth's orbit (Mars, Jupiter, Saturn, Uranus, Neptune, and Pluto), the months around opposition are the best time to view these planets. Conversely, during a conjunction, a planet is in line with the sun and impossible to see at all.

Therefore, the study of opposition and conjunction of the Earth and any outer planet is significance in astrophysics. A Mars year (orbital period) is equal to 2.135 Earth years. Therefore, approximately every 2 earth years Mars is in opposition. When opposition occurs, it is ideal not only to observe the planets but also to send spacecraft out to the planet. The first space craft that visited Mars was the Mariner in 1964. 2010 was the first year that NASA did not send a new spacecraft during its opposition since 1996.

Looking up in the sky each night at the same time, usually you would observe that Mars is a little further east each night compared to the constellations. However, about every two years, there are a couple of months when Mars appears to move from east to west when observed at the same time (retrograde motion). This backward or retrograde motion was mysterious to the early observers and led to the use of the word "planet", from the Greek term which means "wanderer". It is in a short period including the time of opposition when Mars exhibits its retrograde motion to an observer on the Earth. As the Earth moves forward in its orbit, Mars will appear to slip backward compared to its more common eastward march across the sky.

Kepler's equation is one of the most investigated problems in over three centuries. Generally, it describes

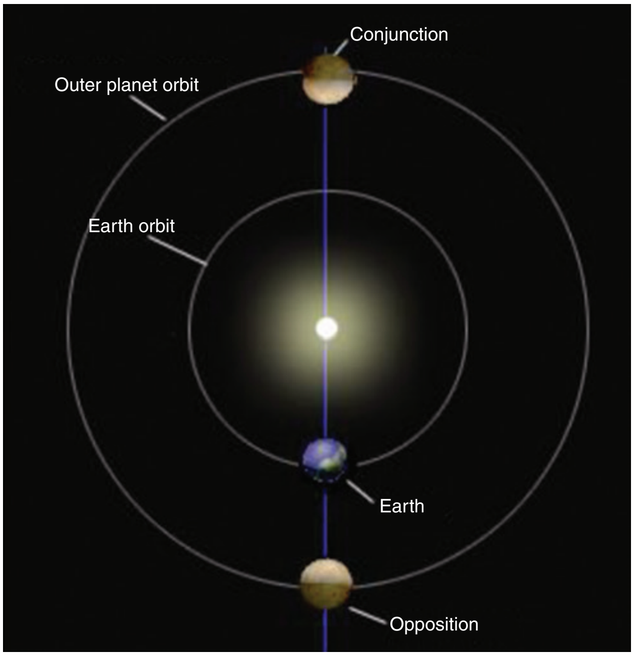

Figure 1: Opposition and conjunction of Earth and an outer planet.

the movement of celestial bodies such as planets, comets, and satellites under a central gravitational force. Determining the position of a planet in its orbit around the Sun at a given time depends on the solution of Kepler's equation. Although many authors [1–10] have devised various numerical and analytical solutions for Kepler's equation, it continues to be of scientific interest due to its various applications in astronomical physics. In [6], Colwell introduced the mathematical derivation for the solution of the elliptical Kepler equation in terms of the Bessel function. Although this analytical solution is simple, it converges for all planets of our Solar system, where the eccentricity $e<1$ [9]. Moreover, this solution enjoyed the continuity property on any interval of time, unlike the iterative numerical solutions [3–5, 9, 10]. Thus, it is the best for studying the current astronomical phenomenon.

The objective of this article is to introduce a simple approach to determine the possible oppositions and conjunctions of Earth–Mars within a specified period of time. Such period is considered in this article to be 20 years of full revolutions of Earth only for simplification and can be extended for any number of years, assuming that the Earth and Mars were initially at their opposition on August 28, 2003 (at perihelion of Mars). To achieve this goal, a popular solution to Kepler's equation in terms of Bessel function can be used. Some properties on the periodicity and the particular values of this solution have been discussed and proved through simple lemmas. Furthermore, the convergence of Bessel function solution iterations has been analysed, and hence, such a solution has been used with highly trust to study the possible dates of oppositions and conjunctions of Earth–Mars.

Moreover, the gravitational effects of other planets in the Solar system on the motion of Earth of Mars are neglected in this work for simplicity. This is the reason that our calculations differ few days from the real dates of Earth–Mars oppositions, as is clarified later in this article.

2 Analysis

Kepler's equation is one of the most investigated problems in recent decades. Generally, it describes the movement of celestial bodies such as planets, commits, and satellites under a central gravitational force. In the elliptical case, Kepler's first law states that the orbital motion of a planet *a* round the Sun is in the form of an ellipse and the radial distance of the Earth from the Sun is given by

$$r_E = \frac{a_E(1-e_E^2)}{1+e_E\cos\theta_E}, \tag{1}$$

and for Mars,

$$r_M = \frac{a_M(1-e_M^2)}{1+e_M\cos\theta_M}, \tag{2}$$

where a_E, e_E, a_M, and e_M are the semi-major axis of the Earth's orbit, the eccentricity of the Earth's orbit, the semi-major axis of the Mars's orbit, and the eccentricity of the Mars's orbit, respectively.

Besides, θ_E and θ_M are, respectively, the true anomalies which determines the locations of the planets Earth and Mars on their orbits. These two angles are related to other two angles ψ_E and ψ_M which determines the corresponding location on an auxiliary circle through the relationships in the following equations (see Fig. 2),

$$\tan\left(\frac{\theta_E}{2}\right) = \sqrt{\frac{1+e_E}{1-e_E}}\tan\left(\frac{\psi_E}{2}\right), \tag{3}$$

$$\tan\left(\frac{\theta_M}{2}\right) = \sqrt{\frac{1+e_M}{1-e_M}}\tan\left(\frac{\psi_M}{2}\right), \tag{4}$$

where ψ_E and ψ_M are defined as the eccentric anomalies of Earth and Mars, respectively, governed by Kepler's equation

$$\psi_{E,M}(t) - e_{E,M}\sin[\psi_{E,M}(t)] = \omega_{E,M}t, \quad \omega_{E,M} = \frac{2\pi}{T_{E,M}}, \tag{5}$$

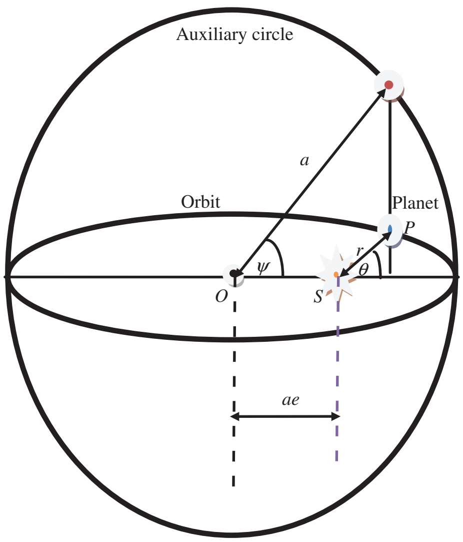

Figure 2: The geometric relationship between the eccentric anomaly ψ and the true anomaly θ.

where $\omega_{E,M}$ and $T_{E,M}$ refer to the angular velocities and the periods of Earth and Mars, respectively.

So, in order to find the true anomalies θ_E and θ_M, i.e. locations of Earth and Mars around the Sun at a specified time t, Kepler's equation (5) should be first solved for ψ_E and ψ_M at that time t and then (3) and (4) are used. Several authors [1–10] have reported on the numerical and the analytical solutions of Kepler's equation. Colwell [6] stated the mathematical derivation for the solution of (5) in terms of Bessel function as in (6) below:

$$\psi_{E,M}(t)=\omega_{E,M}t+\sum_{n=1}^{\infty}\frac{2}{n}J_n(n\,e_{E,M})\sin(n\,\omega_{E,M}t), \tag{6}$$

where J_n is known as the Bessel function of order n.

3 Properties of Bessel Function Solution

Some lemmas are discussed in this section for the periodicity, some particular values, and the remainder error for the successive iterations of the Bessel function solution. To do that, the infinity in the solution given by (6) is replaced by a finite number m and accordingly (6) is rewritten as

$$\psi_m(t)=\omega t+\sum_{n=1}^{m}\frac{2}{n}J_n(n\,e)\sin(n\,\omega\,t), \tag{7}$$

where

$$\psi(t)=\lim_{m\to\infty}\psi_m(t). \tag{8}$$

Let us begin with this lemma which shows some particular values of the Bessel function solution.

3.1 Lemma 1

The iterations of Bessel function solution $\psi_m(t)$, $\forall m\geq 1$, satisfy

– (I) $\psi_m\left((2k-1)\frac{T}{2}\right)=\pi$,
– (II) $\psi_m(kT)=2\pi$,

$\forall\, k=1, 2, 3, \ldots$, where T is the time of a full revolution of a planet around the Sun.

Proof: When $t=(2k-1)\frac{T}{2}$, we have $\omega t=(2k-1)\pi$. Therefore

$$\begin{aligned}\psi_m\left((2k-1)\frac{T}{2}\right)&=(2k-1)\pi+\sum_{n=1}^{m}\frac{2}{n}J_n(n\,e)\sin((2k-1)n\pi)\\&=\pi+\sum_{n=1}^{m}\frac{2}{n}J_n(n\,e)\sin(n\pi)\\&=\pi.\end{aligned} \tag{9}$$

This completes the proof of the first part. Using the same analysis when $t=kT$ yields

$$\begin{aligned}\psi_m(kT)&=2k\pi+\sum_{n=1}^{m}\frac{2}{n}J_n(n\,e)\sin(2kn\pi)\\&=2\pi+\sum_{n=1}^{m}\frac{2}{n}J_n(n\,e)\sin(2n\pi)\\&=2\pi,\end{aligned} \tag{10}$$

which completes the proof of the second part of this lemma.

3.2 Lemma 2

Each m-term of Bessel function solution $\psi_m(t)$ is periodic with period equals to T, i.e. $\psi_m(t+T)=\psi_m(t)$, $\forall\, m\geq 1$.

Proof: On using $\psi_m(t)$ in (7), it then follows

$$\begin{aligned}\psi_m(t+T)&=\omega T+\omega t+\sum_{n=1}^{m}\frac{2}{n}J_n(n\,e)\sin(n\,\omega\,t+n\,\omega\,T)\\&=2\pi+\omega t+\sum_{n=1}^{m}\frac{2}{n}J_n(n\,e)\sin(n\,\omega\,t+2n\pi)\\&=2\pi+\omega t+\sum_{n=1}^{m}\frac{2}{n}J_n(n\,e)\sin(n\,\omega\,t)\\&=2\pi+\psi_m(t)\\&=\psi_m(t).\end{aligned} \tag{11}$$

3.3 Lemma 3

Each m-term of the remainder error $RE_m(t)$ is periodic with period equals to T, i.e. $RE_m(t+T)=RE_m(t)$, $\forall\ m\geq 1$.

Proof: The remainder error $RE_m(t)$ using m-term of Bessel function solution $\psi_m(t)$ is given as

$$RE_m(t)=\psi_m(t)+e\sin(\psi_m(t))-\omega t, \tag{12}$$

On using $\psi_m(t)$ in (7) to the above remainder error, it then follows

$$RE_m(t)=\sum_{n=1}^{m}\frac{2}{n}J_n(n\,e)\sin(n\,\omega\,t) -\sin\left(\omega t+\sum_{n=1}^{m}\frac{2}{n}J_n(n\,e)\sin(n\,\omega\,t)\right), \tag{13}$$

and then

$$\begin{aligned}RE_m(t+T)=&\sum_{n=1}^{m}\frac{2}{n}J_n(n\,e)\sin(n\,\omega\,t+n\,\omega\,T)-\\ &\sin\left(\omega t+\sum_{n=1}^{m}\frac{2}{n}J_n(n\,e)\sin(n\,\omega\,t+n\,\omega\,T)\right)\\ =&\sum_{n=1}^{m}\frac{2}{n}J_n(n\,e)\sin(n\,\omega\,t+2n\pi)-\\ &\sin\left(\omega t+\sum_{n=1}^{m}\frac{2}{n}J_n(n\,e)\sin(n\,\omega\,t+2n\pi)\right)\\ =&\sum_{n=1}^{m}\frac{2}{n}J_n(n\,e)\sin(n\,\omega\,t)-\\ &\sin\left(\omega t+\sum_{n=1}^{m}\frac{2}{n}J_n(n\,e)\sin(n\,\omega\,t)\right)\\ =&RE_m(t).\end{aligned} \tag{14}$$

Although lemmas appear trivial and the proofs are too simple, they should be addressed to assert the periodicity and validity of the current solution over any long interval of time. This is because the nature of this study needs to a proof for the periodicity of the solution, even it is simple, to perform the calculations of Earth–Mars opposition and conjunction over any specified long intervals.

4 Convergence of Bessel Function Solution

Firstly, it was stated in [9] that the Bessel function solution converges for all values $0\leq e<1$. However, the number of terms m in the expansion given by (7) should be assigned so that the numerical results converge. This section is devoted to investigate the iterations for the solution (7) at different values of m. To achieve this task, the data for the two planets are fixed here. The eccentricities and the periods of Earth and Mars are given by

$$e_E=0.0167,\quad T_E=365\text{ days},\quad e_M=0.0934,\quad T_M=687\text{ days}. \tag{15}$$

In Figures 3 and 4, the iterations ψ_2, ψ_3, and ψ_4 are displayed using the data for Mars and Earth, respectively. It is shown from these figures that the iterations using few terms accelerate to certain solutions for full revolutions of the two planets. In order to stand on the accuracy of such approximations, the absolute remainder errors are depicted in Figures 5–7 for Mars and Figures 8–10 for Earth. Figures 5 and 6 indicate that small absolute errors have been achieved over the domain of one revolution of Mars. These results may refer to that the 4-term iteration

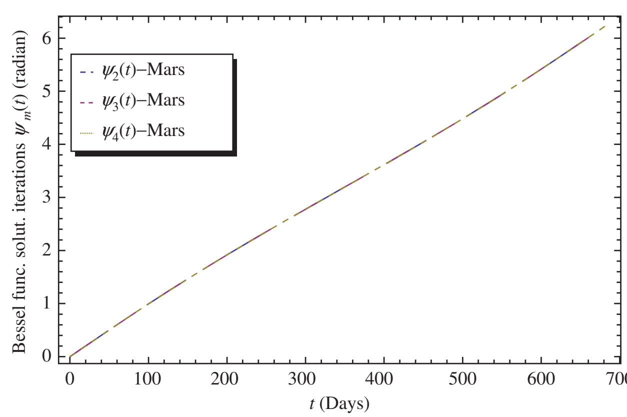

Figure 3: Solutions to Kepler' s equation for Mars using $\psi_2(t)$, $\psi_3(t)$, and $\psi_4(t)$.

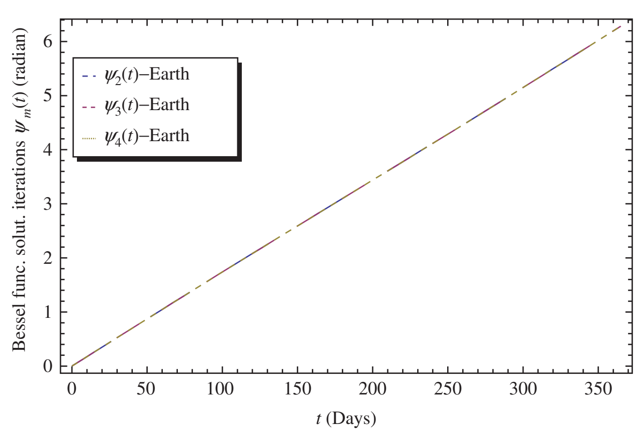

Figure 4: Solutions to Kepler' s equation for Earth using $\psi_2(t)$, $\psi_3(t)$, and $\psi_4(t)$.

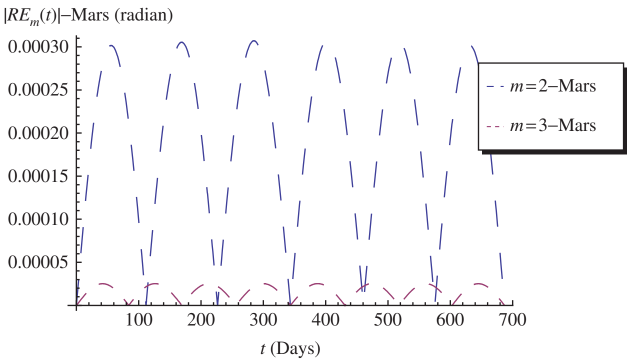

Figure 5: Absolute remainder errors in the domain of one revolution of Mars using 2 and 3 iterations.

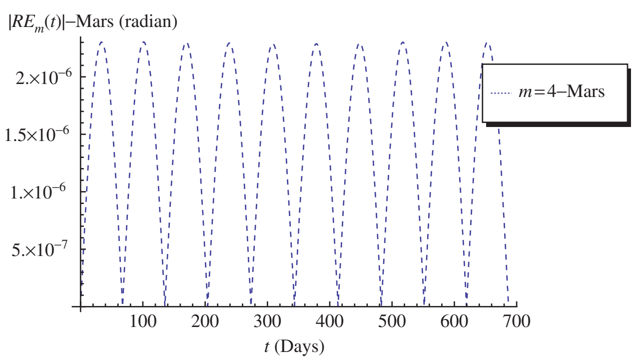

Figure 6: Absolute remainder errors in the domain of one revolution of Mars using 4 iterations.

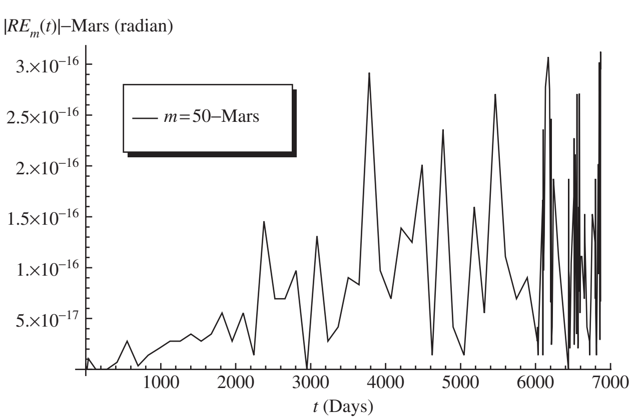

Figure 7: Absolute remainder errors in the domain of 10 revolutions of Mars using 50 iterations.

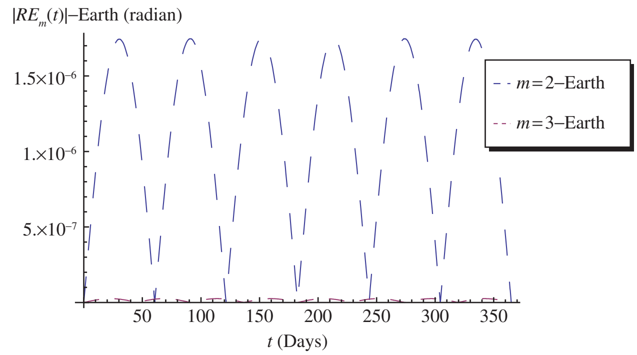

Figure 8: Absolute remainder errors in the domain of one revolution of Earth using 2 and 3 iterations.

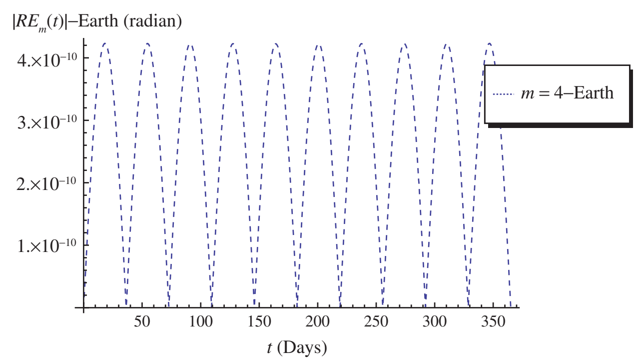

Figure 9: Absolute remainder errors in the domain of one revolution of Earth using 4 iterations.

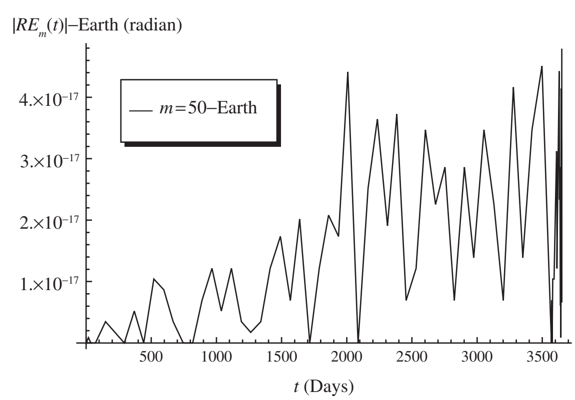

Figure 10: Absolute remainder errors in the domain of 10 revolutions of Earth using 50 iterations.

for the Bessel function solution is sufficient to conduct accurate numerical solution for Kepler's equation in the case of Mars. However, the number of terms can be freely increased to reach the desired accuracy. Regarding, smaller absolute errors have also been obtained and shown in Figure 7 by increasing the number of iterations, i.e. $m=50$, over the domain of 10 complete revolutions of Mars. The same conclusion can also be derived from Figures 8–10 for Earth. Therefore, the solution given in (7) can be used with highly trust when $m=50$ to studying the present physical phenomenon for opposition and conjunction of Earth–Mars.

5 Opposition and Conjunction of Earth–Mars

First of all, it assumed that the two planets started to move around the Sun from their locations on August 28, 2003, at perihelion of Mars. With progress of time, the curate positions of Earth and Mars, at a certain instant t, are determined by solving Kepler's equation for each planet at that instant t with substituting the resulted values of ψ_E and ψ_M into (3) and (4) for the final locations of the two planets on their ellipses. Naturally, the opposition holds when the two planets make the same angle on their orbits. Hence, the locations of Earth and Mars are in opposition when the following condition is satisfied:

$$\tan\left(\frac{\psi_E}{2}\right)-\sqrt{\frac{(1+e_M)(1-e_E)}{(1+e_E)(1-e_M)}}\tan\left(\frac{\psi_M}{2}\right)=0, \qquad (16)$$

or

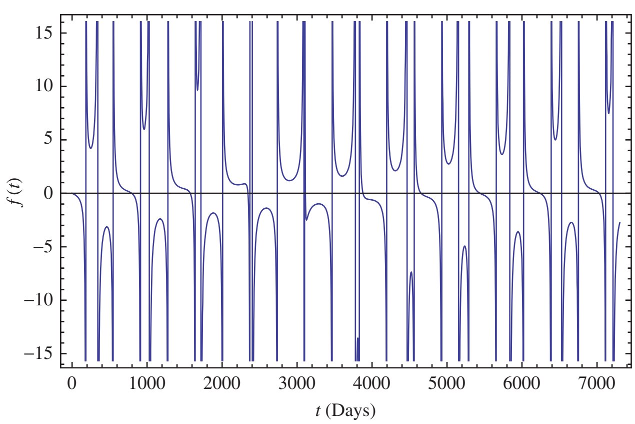

Figure 11: Intervals of oppositions for Earth–Mars.

$$\tan\left(\frac{\omega_E t}{2}+\sum_{n=1}^{\infty}\frac{1}{n}J_n(n\,e_E)\sin(n\,\omega_E\,t)\right)-\sqrt{\frac{(1+e_M)(1-e_E)}{(1+e_E)(1-e_M)}}\times\tan\left(\frac{\omega_M t}{2}+\sum_{n=1}^{\infty}\frac{2}{n}J_n(n\,e_M)\sin(n\,\omega_M\,t)\right)=0. \qquad (17)$$

Equation (17) is of course a very complex transcendental equation, and the intervals of the roots can be easily determined using the graph of the function:

$$f(t)=\tan\left(\frac{\omega_E t}{2}+\sum_{n=1}^{\infty}\frac{1}{n}J_n(n\,e_E)\sin(n\,\omega_E\,t)\right)-\sqrt{\frac{(1+e_M)(1-e_E)}{(1+e_E)(1-e_M)}}\times\tan\left(\frac{\omega_M}{2}+\sum_{n=1}^{\infty}\frac{2}{n}J_n(n\,e_M)\sin(n\,\omega_E\,t)\right). \qquad (18)$$

In Figure 11, $f(t)$ is plotted for 20 years, where the data in (19) for the two planets have been used,

$$a_E=150\times10^6\ \text{km},\quad a_M=227\times10^6\ \text{km}. \qquad (19)$$

Figure 11 shows that there are nine roots for (18) in the domain of 20 years, and this means that there are nine possible oppositional locations of Earth and Mars during that period of time. In view of Figure 11, the intervals of such roots are [600, 800], [1400, 1600], [2200, 2400], [3110, 3200], [3850, 4000], [4600, 4800], [5400, 5600], [6200, 6400], and [7000, 7150].

These nine roots are calculated using the command *Find root* in Mathematica 7, where effective proper starting values have been chosen. Such nine roots are tabulated in Table 1 with the corresponding true anomalies of Earth θ_E and Mars θ_M. It can be seen from this table that the equality of θ_E and θ_M occurs at each value of the calculated roots. Moreover, these numerical results can be easily verified by substituting each root into the (3) and (4) for checking the equality of θ_E and θ_M. In the language

Table 1: Times and angles for the oppositions of Earth–Mars within 20 years starting from their opposition on August 28, 2003 (at perihelion of Mars).

Sequence of oppositions	Interval of opposition (in days)	Time of opposition (in days)	Approximate date of opposition	Angle of opposition ($\theta_E=\theta_M$) (in degrees)
1	[600, 800]	795.613	November 1, 2005	66.4602
2	[1400, 1600]	1575.3	December 20, 2007	115.461
3	[2200, 2400]	2345.29	January 28, 2010	154.01
4	[3110, 3200]	3111.86	March 5, 2012	188.933
5	[3850, 4000]	3879.57	April 11, 2014	225.06
6	[4600, 4800]	4653.58	May 25, 2016	267.923
7	[5400, 5600]	5441.38	July 21, 2018	325.777
8	[6200, 6400]	6240.68	September 28, 2020	36.3181
9	[7000, 7150]	7027.95	November 24, 2022	93.5902

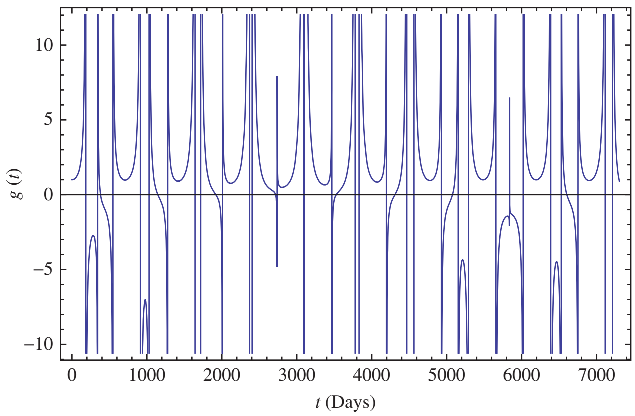

Figure 12: Intervals of conjunctions for Earth–Mars.

of years and days, the dates of such nine oppositions were approximately determined and listed in Table 1. It is found that our calculations differ few days from the published real dates of Earth–Mars oppositions (see http://cseligman.com/text/planets/marsoppositions.htm). This is due to the neglected effects of the gravitational attraction of the other planets in the Solar system on the motion of Earth and Mars.

Here, Earth and Mars are in a conjunction location when $\theta_E = \theta_M \pm \pi$, and this leads to the condition below:

$$\tan\left(\frac{\theta_E}{2}\right)\tan\left(\frac{\theta_M}{2}\right) = -1, \tag{20}$$

Table 2: Times and angles for the conjunctions of Earth–Mars within 20 years starting from their opposition on August 28, 2003 (at perihelion of Mars).

Sequence of conjunctions	Interval of conjunction (in days)	Time of conjunction (in days)	Approximate date of conjunctions	Angles of conjunctions (in degrees)	
				θ_M	θ_E
1	[270, 400]	381.18	September 12, 2004	196.495	16.495
2	[1100, 1200]	1145.94	October 17, 2006	231.737	51.737
3	[1800, 1950]	1919.94	November 29, 2008	275.544	95.544
4	[2600, 2800]	2717.02	February 4, 2011	340.453	160.453
5	[3420, 3600]	3526.53	April 14, 2013	56.611	236.611
6	[4220, 4400]	4309.53	June 16, 2015	108.693	288.693
7	[4960, 5180]	5077.6	July 23, 2017	147.016	327.016
8	[5800, 5900]	5840.5	August 25, 2019	180.511	0.511
9	[6600, 6750]	6603.47	September 24, 2021	214.075	34.0745

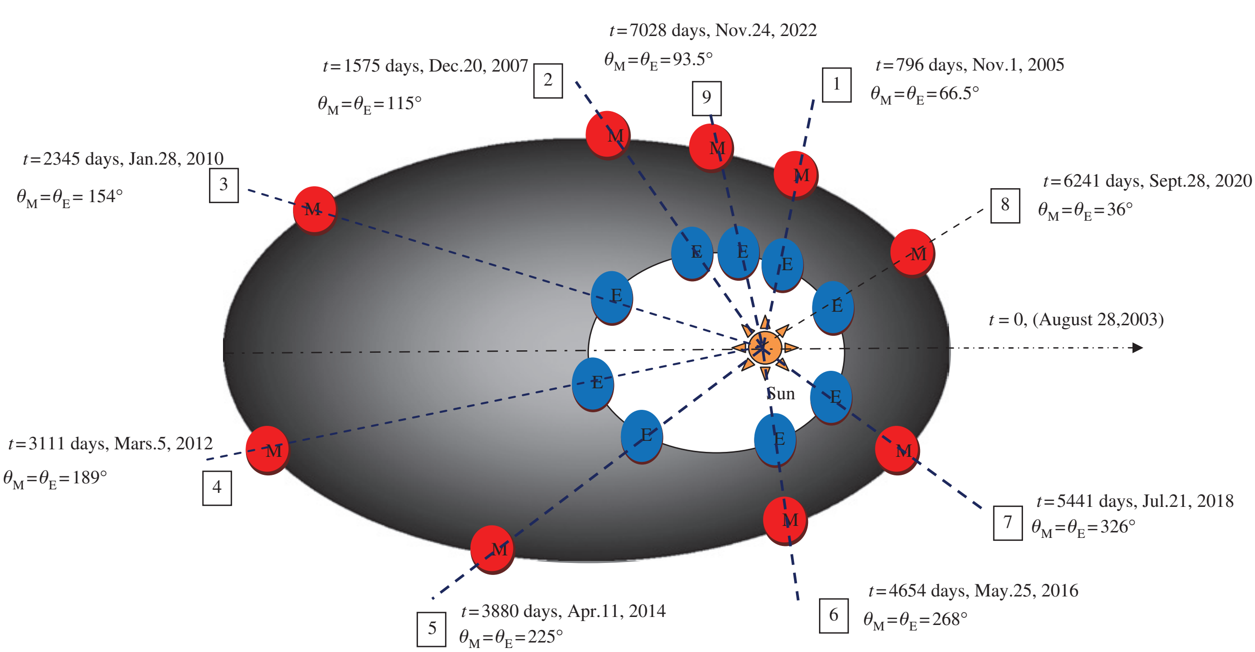

Figure 13: Sketch of opposition for Earth–Mars (2003–2022).

which implies that

$$1+\sqrt{\frac{(1+e_E)(1+e_M)}{(1-e_E)(1-e_M)}}\tan\left(\frac{\psi_E}{2}\right)\tan\left(\frac{\psi_M}{2}\right)=0. \quad (21)$$

Again, (21) is a complex transcendental equation and the intervals of its roots can be easily determined using the graph of the function:

$$g(t)=1+\sqrt{\frac{(1+e_E)(1+e_M)}{(1-e_E)(1-e_M)}}\tan\left(\frac{\psi_E}{2}\right)\tan\left(\frac{\psi_M}{2}\right). \quad (22)$$

Figure 12 indicates the possible roots of $g(t)$ for 20 full revolutions of Earth. The number of these roots is also nine which consequently leads to nine possible conjunctions. In view of Figure 12, the intervals of such roots are [270, 400], [1100, 1200], [1800, 1950], [2600, 2800], [3420, 3600], [4220, 4400], [4960, 5180], [5800, 5900], and [6600, 6750].

For accurate determination of such nine roots, the same above-mentioned analysis has been used and the results are tabulated in Table 2 along with the corresponding approximate dates of conjunctions. Figure 13 shows the final sketch of the calculated oppositions of the two planets. The analysis discussed in this article may be extended for further calculations of the opposition and conjunction for other planets. A remarkable observation here is that a conjunction of Earth and Mars occurred earlier than their opposition. Slightly after one full revolution of Earth, about 381 days (started on August 28, 2003), the first conjunction occurred and also slightly after two full revolutions of Earth, around 796 days, the two planets were in opposition.

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