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Spinless Particle in a Magnetic Field Under Minimal Length Scenario

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Abstract: In this article, we studied the Klein–Gordon equation in a generalised uncertainty principle (GUP) framework which predicts a minimal uncertainty in position. We considered a spinless particle in this framework in the presence of a magnetic field, applied in the z-direction, which varies as $\frac{1}{x^2}$. We found the energy eigenvalues of this system and also obtained the correspounding eigenfunctions, using the numerical method. When GUP parameter tends to zero, our solutions were in agreement with those obtained in the absence of GUP.

Keywords: Generalised Uncertainty Principle (GUP); Klein–Gordon Equation; Magnetic Field; Minimal Length.

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1 Introduction

One of the fundamental principles of quantum mechanics is the Heisenberg uncertainty principle, expressing that position and momentum of a particle cannot be measured simultaneously. But at high energy limit where gravity effects gain so much importance and space-time discreteness may occur, this uncertainty principle should be modified as the so-called generalised uncertainty principle (GUP). This generalisation leads to a nonzero minimal uncertainty in position measurements. Various candidates of quantum gravity, such as the string theory, loop quantum gravity, and quantum geometry, all indicate the existence of such a minimal measurable length of the order of the Planck length $l_p=\sqrt{\frac{G\hbar}{c^2}}=1.6\times10^{-35}$ m, [1, 2].

In recent years, the study of relativistic wave equations, particularly the Klein–Gordon (K-G) equation, has been taken under much consideration. Quantum gravitational corrections to the real K-G field in the presence of a minimal length have been considered [2, 3]. The authors obtained the exact solution of the K-G equation in the presence of a minimal length; also, the exact solution of K-G equation was obtained for charged particles in a magnetic field with the shape invariant method.

In this article, we study spinless particles in the presence of a magnetic field within the GUP framework that predicts minimal uncertainty in position. In Section 2, we state the GUP. In Section 3, we find the generalised K-G equation which describes the spinless particles that have reshaped under Lorentz invariants the GUP deformed relativity equation that describes these particles. In Section 4, we solve the generalised K-G equation in the presence of a magnetic field with specific gauge $\frac{1}{x^2}$. In part two of Section 4, we gain a generalised 4-degree K-G equation where we explain particles with zero spin that have reshaped under Lorentz invariants. We solve these equations for special magnetic fields with zero scalar fields and specified vector fields that we reach from the shown answers in these sections. Moreover, we study the mentioned differential equation for a special state of vector potential in which the stagnation magnetic field is along the z axis and it charges as $\frac{1}{x^2}$. Finally, we present our conclusions in Section 5.

2 Generalised Uncertainty Principle

We consider the GUP proposed by Kempf et al. [4–6]:

$$\Delta X\Delta P\geq(\hbar/2)(1+\beta(\Delta P)^2+\gamma), \tag{1}$$

where β is the GUP parameter and γ is a positive constant. We also have $\beta=\beta_0/(M_{Pl}c)^2$, where M_{Pl} is the Planck mass and β_0 is of order one. The equation's inequality relation results in the existence of an absolute minimal length uncertainty as $(\Delta x)_{min}=\hbar\sqrt{\beta}$. In one dimension, the following deformed commutation relation is

$$[X, P]=i\hbar(1+\beta P^2), \tag{2}$$

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which leads to an inequality relation. As Kempf, Mangano, and Mann mentioned in their seminal article [5], X and P can be written as

$$P\phi(p)=p\phi(p),\quad X\phi(p)=i\hbar(1+\beta P^2)\partial_p\phi(p), \tag{3}$$

in momentum space representation. In (3), p can be interpreted as the momentum operator at low energies and P as the momentum operator at high energies. Using (3), the Hamiltonian

$$H=\frac{P^2}{2m}+V(x), \tag{4}$$

is transformed into

$$H=H_0+\beta H_1+O(\beta^2), \tag{5}$$

where $H_0=\frac{p^2}{2m}+V(x)$ and $H_1=\frac{p^4}{3m}$. In the quantum domain, the Hamiltonian equation results in the following generalised Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2}+\beta\frac{p^4}{3m}\psi(x)+V(x)\psi(x)+O(\beta^2)=E\psi(x), \tag{6}$$

which has an extra term in comparison to an ordinary Schrödinger equation due to the modified commutation relation. It is not an easy task to solve this equation, because it is a fourth-order differential equation. Therefore, the perturbation method is used in order to obtain the solutions.

3 Generalised K-G Equation in the Presence of a Magnetic Field

In this section, we study the spinless particles in the presence of a magnetic field. For this purpose, first we write the Lagrangian density of a complex field without considering the gravity effects, and then we obtain the generalised form of the K-G equation. The Lagrangian density of complex field in the presence of magnetic field is written as

$$\mathfrak{L}=-[(i\partial_\mu+qA_\mu)\phi^*][(i\partial^\mu-qA^\mu)\phi]-m^2\phi^*\phi, \tag{7}$$

where A_μ is the four axis magnetic potential, ϕ, ϕ^* are the complex fields, q is the charge of particle, and m is considered to be the mass of the particle. Using the Euler Lagrange equation, we obtained the K-G equation with gauge invariant as

$$[(i\partial_\mu+qA_\mu)(i\partial^\mu-qA^\mu)-m^2]\psi(x)=0. \tag{8}$$

Also, we could obtain this relation by using the K-G equation with a gauge invariant $p^\mu\to p^\mu-A^\mu$. Thus, we have

$$[\partial_\mu\partial^\mu+m^2+U(x)]\psi(x)=0. \tag{9}$$

In (9), $U(x)$ includes scalar and vector potential

$$U(x)=iq\partial_\mu A^\mu+iqA^\mu\partial_\mu-q^2A^\mu A_\mu, \tag{10}$$

$$=i\partial_\mu V^\mu+iV^\mu\partial_\mu+S, \tag{11}$$

where

$$S=q^2A^\mu A_\mu,\; V^\mu=qA^\mu. \tag{12}$$

We obtained the following equation by expanding (8):

$$[-(1+\beta\Box)^2\Box-iq(1+\beta\Box)\partial_\mu A^\mu-iqA_\mu(1+\beta\Box)\partial^\mu +q^2A_\mu A^\mu-m^2]\psi(x)=0, \tag{13}$$

where $c=\hbar=1$ is chosen as a unique unit. With a look at (13), it is clear that if $\beta\to 0$, we reach (9), that is, the K-G equation. By taking this algebra into account, the simplified form of (13) is as follows:

$$[(1+\beta\Box)^2\beta+m^2+U_G(x)]\psi(x)=0, \tag{14}$$

where $U_G(x)$ is written as

$$U_G(x)=i(1+\beta\Box)\partial_\mu V^\mu+iV^\mu(1+\beta\Box)\partial_\mu-S. \tag{15}$$

In (15), V_μ and S are the scalar and vector potentials, respectively. If we disregard the sentences with β^2 and higher, (14) is reduces to

$$[(1+2\beta\Box)\Box+iq(1+\beta\Box)\partial_\mu A^\mu+iqA_\mu(1+\beta\Box)\partial^\mu-q^2A_\mu A^\mu +m^2]\psi(x)=0. \tag{16}$$

Now, in order to find the plane wave solutions of (16), we choose $\psi=ce^{-ikx}$ and replace it in (16) to obtain some volumes of K which, according to them, ψ is an answer of (16).

$$[(-1+q\beta\vec{A}.\vec{K})K^2+2\beta(K^2)^2+q^2\vec{A}^2+m^2]=0. \tag{17}$$

We use coulomb gauge $\nabla\cdot\vec{A}=0$ and specific state $A_0=0$. Then, K^2 is obtained as follows:

$$K^2=\frac{-(-1+q\vec{A}\cdot\vec{K})\pm\sqrt{(-1+q\beta\vec{A}\cdot\vec{K})^2-8\beta(q^2\vec{A}^2+m^2)}}{4\beta}. \tag{18}$$

By considering the vector field as

$$\vec{A}=\vec{a}_0 e^{i(\vec{K}\cdot\vec{r}-\omega t)}, \tag{19}$$

and using the coulomb gauge, we obtain $\nabla\cdot\vec{A}=i\vec{K}\cdot\vec{A}=0$. So, limitation of β is obtained as

$$\beta<\frac{1}{8(q^2A^2+m^2)}.$$

According to what Moayedi et al. [7] have shown, in the same study in this condition we have

$$[M_+]^2=K_+^2=\left[\frac{(1+2\sqrt{2\beta}M)^{1/2}+(1-2\sqrt{2\beta}M)^{1/2}}{2\sqrt{2\beta}}\right]^2, \tag{20}$$

$$[M_-]^2=K_-^2=\left[\frac{(1+2\sqrt{2\beta}M)^{1/2}-(1-2\sqrt{2\beta}M)^{1/2}}{2\sqrt{2\beta}}\right]^2. \tag{21}$$

In (20) and (21) relations, $M^2=q^2A^2+m^2$. The important point is that if $M_+=M_-=\sqrt{2}M$, then it will be $\beta=\frac{1}{8M^2}$. By knowing that $K^2=K_\mu K^\mu=K_0^2-\vec{K}^2$, $K_0=\omega$ and by applying relations in (20) and (21), the reformed energy momentum relation will be achieved for this system as

$$\begin{aligned} E_p^{(+)2}&=M_+^2+|\vec{p}|^2,\\ E_p^{(-)2}&=M_-^2+|\vec{p}|^2,\ E_p^{\pm}=\omega_k^{(\pm)}. \end{aligned} \tag{22}$$

The effective masses M_+ and M_- could be rewritten in (20) and (21) up to $O(\beta)$ as

$$M_+=\frac{1}{\sqrt{2\beta}}-\frac{M^2}{2}\sqrt{2\beta} \tag{23}$$

$$M_-=M+\beta M^3, \tag{24}$$

for $\beta=0$, the effective mass M_- reduces to the usual mass, M. By putting (24) in (22), we can determine the shape of the energy momentum relation in the gravitational framework as follows:

$$E_p^{(-)2}=M^2+|\vec{p}|^2+2\beta M^4. \tag{25}$$

By comparing this equation with (27) in work by Moayedi et al. [7], it will be clarified that the general mathematical shape of energy momentum relation is the same for free particles in the presence of a magnetic field, and it is different only in the volume of M, which is dependent on vector potential and the ruling magnetic field.

4 Solving Generalised K-G Equation in the Presence of a Magnetic Field with Specific Gauge $\frac{1}{x^2}$

In this section, we consider a specified gauge introduced by Maharana [8] as

$$\vec{A}=\left(A_x=0,\ A_y=\frac{\alpha}{x},\ A_z=0\right),\ A_0=0. \tag{26}$$

Therefore, magnetic field will be $\vec{B}=\nabla\times\vec{A}=\left(B_x=0,\ B_y=0,\ B_z=-\frac{\alpha}{x^2}\right)$ where α is a constant. By considering vector potential and then separating time and position apart from the wave as $\psi(x, y, z, t)=\phi(x, y, z)e^{-iEt}$ and putting this relation into (16), we obtain the following relation:

$$\begin{aligned}&(E^2+2\beta E^4+m^2)\phi(x,y,z)=\\ &-\left[(1+4\beta E^2)\nabla^2+2\beta(\nabla^2)^2+iq\frac{\alpha}{x}\frac{\partial}{\partial y}(1+\beta(E^2+\nabla^2))+\frac{q^2\alpha^2}{x^2}\right]\\ &\phi(x,y,z)\end{aligned} \tag{27}$$

The third and fourth sentences from the right-hand side of (27) are clearly dependent on x, so it is possible to have $\phi(x,y,z)=u(x)e^{i(K_y y+K_z z)}$. By putting this relation in (27), we obtain the following fourth-order differential equation:

$$2\beta\frac{d^4u}{dx^4}+a\frac{d^2u}{dx^2}+\left[b+\frac{q^2\alpha^2}{x^2}-\frac{c}{x}\right]u=0, \tag{28}$$

which contains the expressions

$$\begin{aligned} &a=1+4\beta(E^2-K_y^2-K_z^2),\\ &\omega_\alpha=q\alpha,\\ &b=-(K_y^2+K_z^2)[1+4\beta E^2-2\beta(K_y^2+K_z^2)]+(E^2+2\beta E^4+m^2),\\ &c=\omega_\alpha K_y[1+\beta(E^2-K_y^2-K_z^2)]. \end{aligned} \tag{29}$$

In (28), if $\beta\to 0$, it reduces to (15), obtained by Setare and Hatami [9]. To solve the differential of (28), we have used two methods. Initially the perturbation method and secondly the variational iteration method (VIM) was suggested by studying various research of others [10–13]. Now, we return to the modified K-G (28), and as previously has been said, the solution of this differential equation is complicated. We tried to obtain the rest energy correction up to $O(\beta)$ in β via the usual perturbation method of quantum mechanics. Because the amount of energy was calculated by Setare and Hatami [9] without the gravitational effects and differential (28) gained this energy regardless of β, thus the perturbation Hamiltonian is $H^{pert}=2\beta\frac{d^4}{dx^4}$.

$$[H^0+H^{\text{pert}}]u(x)=0, \tag{30}$$

$$H^0=a\frac{d^2}{dx^2}+\left[b+\frac{q^2\alpha^2}{x^2}-\frac{c}{x}\right]. \tag{31}$$

From Setare and Hatami [9], we have

$$E_n^{\beta=0}=\pm\sqrt{m^2+K_y^2+K_z^2-\frac{\omega_\alpha^2K_y^2}{(n+1/2+\sqrt{1/4+\omega_\alpha^2})^2}}, \tag{32}$$

and

$$u_n(x)=x^{a_0}e^{-\frac{b_0}{a_0+n}x}L_n^{2a_0-1}\left(\frac{2b_0}{a_0+n}x\right), \tag{33}$$

which $a_0=1/2+\sqrt{1/4+\omega_\alpha^2}$ and $b_0=\omega_\alpha K_y$. In Figure 1, we have shown (33) with different values, and mode n = 3 will be as following. In Figure 1, the function $u(x)$ is presented. In this figure, the gravity effect is omitted. As can be seen, figures with different modes have the same general shape.

In the following, we use the perturbation theory in quantum mechanics to obtain the first correction in the energy levels. For this goal, the expectation value of the eigenfunction (33) was considered:

$$\Delta E_n^\beta=\frac{\langle u_n(x)|H^{\text{pert}}|u_n(x)\rangle}{\langle u_n(x)|u_n(x)\rangle};\quad E_n^\beta=E_n^{\beta=0}+\Delta E_n^\beta. \tag{34}$$

As we describe, the VIM provides efficient algorithm for analytic approximate solutions and numeric simulations for real world applications in sciences [13–17]. In order to solve the fourth ODEs, we have used the following relation as in Wazwaz [10]:

$$u_{n+1}(x)=u_n(x)+\frac{1}{3!}\int_0^x(t-x)^3Lu_n(t)dt, \tag{35}$$

where L is the linear operator that in this research is $2\beta\frac{d^4}{dt^4}+a\frac{d^2}{dt^2}+\left[b+\frac{q^2\alpha^2}{t^2}-\frac{c}{t}\right]$. The final answer is $u(x)=\lim_{n\to 0}u_n(x)$. In this method, with respect to the order of differential equation (fourth-order in this case), we need four initial conditions. Because initial conditions were not clear, we selected general form:

$$u(0)=\mathcal{A},\quad u'(0)=\mathcal{B},\quad u''(0)=\mathcal{C},\quad u'''(0)=\mathcal{D}, \tag{36}$$

$$u_0(x)=u(0)+\frac{u'(0)}{1!}+\frac{u''(0)}{2!}+\frac{u'''(0)}{3!}+\dots. \tag{37}$$

As it was indicated, we put the different n's as $n=0$, 1, 2, … into (30), and we are able to obtain $u_0, u_1, u_2, \dots, u_n$ as

$$u_0(x)=\mathcal{A}+\mathcal{B}x+\frac{\mathcal{C}}{2}x^2+\frac{\mathcal{D}}{6}x^3, \tag{38}$$

$$\begin{aligned}u_1(x)&=-\frac{1}{720}x^6b\mathcal{C}+\frac{1}{120}x^5\left(\frac{c}{2}\right)\mathcal{C}-\frac{1}{24}x^4\left(a+\frac{\omega_\alpha^2}{2}\right)\mathcal{C}+x^2\frac{\mathcal{C}}{2},\\ u_2(x)&=\cdots\end{aligned} \tag{39}$$

To calculate (39) using (35), answers do not necessarily converge to zero. Therefore, to avoid this problem, we considered coefficients containing these sentences as zero. Hence, $\mathcal{A}=0$, $\mathcal{B}=\frac{\omega_\beta K_y}{q^2\alpha}\mathcal{C}$. To obtain (39) according to a complex form, general response $\mathcal{D}$ has been omitted. Equation (28) is a differential equation with coefficient variables of the fourth-order. This form of coefficients and $u_1(x)$ informs us that we must go a long route ahead to obtain $u_2(x)$. Therefore, we let the general form remain as $u_1(x)$ and use numerical methods to obtain eigenfunction.

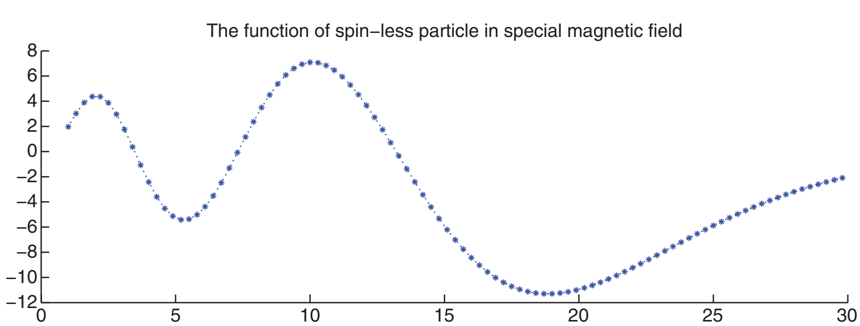

Figure 1: The diagram of the spin-less wave function in special magnetic field without gravity.

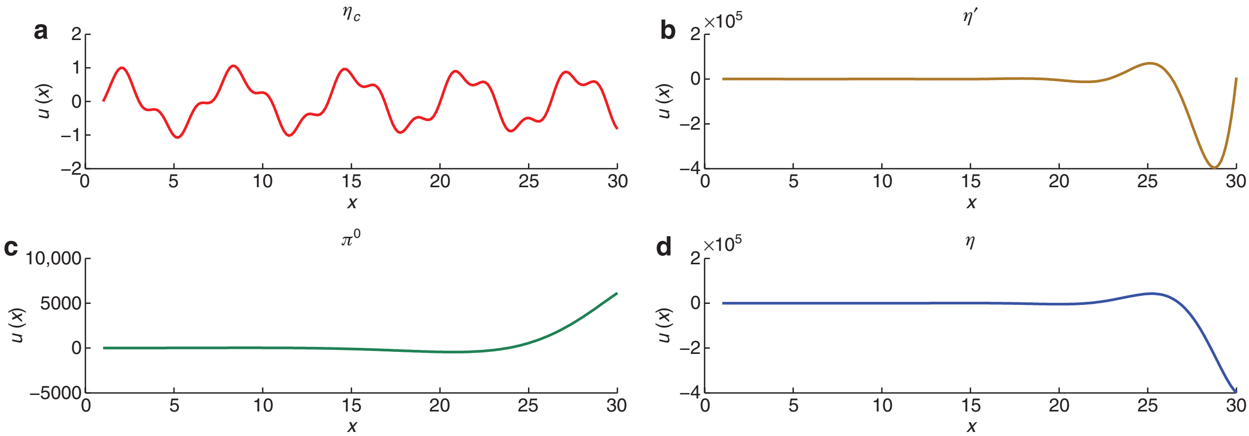

Figure 2: Numerical results for $u(x)$ with $\beta \neq 0$.

We found the graph of function of (28) by using Matlab software.

In the aforementioned figures, the function $u(x)$ is presented. These figures represent wave functions for four particles with gravity effects. Figure 2a is η_c with quark content $c\bar{c}$ and $\beta = 0.049$, Figure 2b shows the η' with quark content $(u\bar{u} + d\bar{d} + c\bar{c})/\sqrt{6}$ and $\beta = 0.477$, Figure 2c is π^0 particle with $(u\bar{u} - d\bar{d})/\sqrt{2}$ quark details with $\beta = 1.453$, and Figure 2d is η with $(u\bar{u} + d\bar{d} - 2s\bar{s})/\sqrt{6}$ with gravity coefficient $\beta = 24.027$ [7]. It appears that the effects of gravity, i.e. β, is small, the figures limited to Figure 1, and whenever β is longer, the general figures shape out of exponential form.

5 Conclusions

In this article, we considered a spinless particle in the presence of a magnetic field in the GUP framework that predicts a minimal uncertainty in position. We obtained the modifed K-G equation in this framework. Having investigated the plane wave, we found that there is no difference between the obtained equations and previous ones, and only the value of the magnetic field can lead to different mass and energy, which is consistent with the results of Moayedi et al. [7], but magnetic field can influence mass and energy of the particles. We tried to solve the generalised K-G equation for a special magnetic field by two methods. In the first method (pertubation theory) we provided a relation to the energy levels. In Section 4, we tried to solve the K-G equation for a special state of a magnetic field which, after mathematical process, we obtained as (28). Relation (34) and Figure 1, respectively, show the system energy and special state. When $\beta \to 0$, all states and limitations are established, and problems are created according to what is in (32) and (33). It is anticipated that the results and numerical answers in this study will stimulate further investigations.

References

[1] X. Calmet, Mod. Phys. Lett. **A22**, 2027 (2007).
[2] M. Merad and M. Falek, Phys. Scr. **79**, 015010 (2009).
[3] P. Pedram, M. Amirfakhrian, and H. Shababi, Int. Mod. Phys. A **24**, 1550016 (2015).
[4] C. Quesne and V. M. Tkachuk, J. Phys. A: Math. Gen. **38**, 1747 (2005).
[5] A. Kempf, G. Mangano, and R. B. Mann, Phys. Rev. D **52**, 1108 (1995).
[6] M. M. Stetsko and V. M. Tkachuk, Phys. Rev. A **74**, 012101 (2006).
[7] S. K. Moayedi, M. R. Setare, and H. Moayeri, arXiv:1004.0563v1.
[8] K. Maharana, arXiv:math-ph/0306069v1.
[9] M. R. Setare and O. Hatami, Theor. Phys. China **51**, 10001002 (2009).
[10] A. M. Wazwaz, Appl. Math. Comput. **212**, 120134 (2009).
[11] A. M. Wazwaz, Appl. Math. Comput. **105**, 1119 (1999).
[12] J. H. He, Appl. Math. Comput. **114**, 115123 (2000).
[13] J. H. He, J. Comput. Appl. Math. **207**, 317 (2007).
[14] H. T. Davis, Introduction to Nonlinear Dierential and Integral Equations, Dover, New York 1962.
[15] H. Carslaw and J. Jaeger, Conduction of Heat in Solids, Oxford University Press, Oxford, UK 1947.
[16] R. C. Roberts, Unsteady Flow of Gas through a Porous Medium, Proc. First U.S. Natl. Congr. Appl. Mech., Ann Arbor, MI, **773**, 776 (1952).
[17] R. E. Kidder, J. Appl. Mech. **27**, 329 (1067).