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Classical Equation of State for Dilute Relativistic Plasma

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Abstract: The aim of this paper is to calculate the analytical form of the equation of state for dilute relativistic plasma. We obtained the excess free energy and pressure in the form of a convergent series expansion in terms of the thermal parameter μ where $\mu=\frac{mc^2}{KT}$, m is the mass of charge, c is the speed of light, K is the Boltzmann's constant, and T is the absolute temperature. The results are discussed and compared with previous work of other authors.

Keywords: Dilute Relativistic Plasma; Equation of State; Excess Free Energy.

1 Introduction

The study of plasma thermodynamics is one of the attractive problems in the theoretical physics. Relativistic plasmas are objects encountered in many astrophysical situations. For instance, they occur in the magnetosphere of pulsars where they are strongly magnetised or in quasar jets [1]. Such plasmas may be created by heating a gas to very high temperatures. In the relativistic plasma, the relativistic corrections to a particle's mass and velocity are important. Such corrections typically become important when a significant number of electrons reach speeds >0.86c.

Hussein [2] used the virial expansion to calculate the osmotic pressure for plasma in the classical and quantum form. Hussein and Eisa [3] calculated the excess free energy until the third virial coefficient for the two-component plasma (TCP) model. Hussein et al. [4] calculated the equation of state of plasma under the influence of a weak magnetic field; the calculation was based on the magnetic binary Slater sum in the case of low density. Moreover, the analytical form of the equation of state until the third virial coefficient of a classical system interacting via an effective potential of fully ionised plasmas was calculated by Eisa [5].

Lapiedra and Santos [6] have undertaken an application of predictive methods to relativistic statistical mechanics for the case of a hot dilute plasma. Lapiedra and Santos [7] calculated the two-particle distribution function from the standard decoupling of the exactly relativistic BBGKY hierarchy. Then Barcons and Lapiedra [8] used it to calculate the thermodynamic functions of relativistic plasma in equilibrium. Barcons and Lapiedra [9] give explicit expressions for the thermodynamic functions of a high-temperature electron-positron plasma and give expression for distribution functions for a classical dilute arbitrarily hot plasma. Trubnikov and Kosachev [10] find relativistic corrections to the thermodynamic functions of a completely ionised plasma up to terms of order v^2/c^2. The standard two-body distribution function for dilute slightly relativistic plasma has been calculated previously by Kosachev and Trubnikov [11] starting from the Darwin Lagrangian.

Coulomb forces between point charges are purely repulsive and charges approach very close to each other only rarely whatever the plasma conditions. Coulomb systems such as plasma or electrolytes are made of charged particles interacting through Coulomb's law. The simplest model of a Coulomb system is the one-component plasma (OCP), also called jellium: an assembly of identical point charges, embedded in a neutralising uniform background of the opposite sign. Here we consider the classical (i.e. non-quantum) equilibrium statistical mechanics of the OCP. It is rather straightforward to calculate higher-order correlation functions from the measured configurations. Moreover, we study the model of TCP, i.e. neutral system of point like particles of positive and negative charges such as electrons and ions. For the numerical calculation, we restrict ourselves to the case of TCP which anti-symmetric with respect to the charges $e_e=-e_i=-e$ and therefore symmetrical with respect to the densities $\rho_e=\rho_i=\rho$.

The aim of this paper is to calculate the analytical form of the equation of state for dilute relativistic for OCP and TCP. Homogeneous plasma is characterised by two

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parameters: the density of particles n and the temperature T, if there are several kinds of particles it is also necessary to state their concentrations [7]. There are different energies associated with the plasma, namely:
(i) Energy of the rest mass per particle mc^2,
(ii) Kinetic energy per particle of order KT,
(iii) Coulomb energy of order $e^2n^{1/3}$ per particle.

The ratios of these energies give us the two main dimensionless parameters of the plasma: the thermal parameter of plasma $\mu=\frac{mc^2}{KT}$ and the dilution parameter $\epsilon_d=\frac{e^2n^{1/3}}{KT}$, the value of μ is very important. There are four different regimes characterised by it; plasma with $\mu\le 1$ is relativistic, $\mu\ge 1$ is weakly relativistic plasma, $\mu\gg 1$ is low plasma temperature, and $\mu\ll 1$ is ultra-relativistic high temperatures plasma. In our study, we note that the system is safely classical when $\lambda=\frac{\hbar}{P}\ll n^{1/3}$, where $\hbar=\frac{h}{2\pi}$, h is the Planck constant, P being the typical linear momentum of particles.

Our study is valid for dilute plasmas ($\epsilon_d<1$). The plasma temperatures and velocities also play a significant role for the stability of our system. Another important point is that of the walls. We shall study plasma which is homogeneous and isotropic but the plasma must be confined or it will expand. We may assume that the plasma is confined by some kind of walls which prevent the escape of particles, but when the container is so large, the effects of the walls are negligible.

A white dwarf, also called a degenerate dwarf, is a stellar remnant composed mostly of electron-degenerate matter. A white dwarf is very dense; its mass is comparable to that of the Sun, and its volume is comparable to that of Earth. A white dwarf's faint luminosity comes from the emission of stored thermal energy [12]. The nearest known white dwarf is Sirius B, at 8.6 light years, the smaller component of the Sirius binary star. There are currently thought to be eight white dwarfs among the hundred star systems nearest the Sun [13]. The unusual faintness of white dwarfs was first recognised in 1910 [14]. The name white dwarf was coined by Willem Luyten in 1922 [15].

Now it is well known from the nuclear structure studies that the kind of isotopes of He, C, O, Mg, Ne, and so on present in white dwarfs are all bosons. In case of such a compact object, the degenerate electron number density is so high (in white dwarfs it can be the order of 10^{30} cm^{-3}, even more see [16] and [17]) that the electron Fermi energy is comparable to the electron mass energy and the electron speed is comparable to the speed of light in a vacuum. We used plasma localised near the surface of white dwarf ($T\sim 10^7$ K) as a practical example of the application of the results we have obtained. The primary difference between dilute relativistic plasma and quantum plasma is a quantum plasma can be exist in the low-temperature and called in this time cold quantum plasma but dilute relativistic plasma must have high temperature. Also dilute relativistic plasma can be studied by classical statistical mechanics [18] but quantum plasma must be stuied by quantum statistical mechanics [19].

2 Internal Energy Excess Free Energy for Dilute Relativistic Plasma

We now consider a relativistic macroscopic system of N classical particles which are interacting among themselves, in the framework of the PRM. Let us define its generalised distribution function $F(t_a, \vec{x}_a, \vec{u}_a)$ ($\vec{u}_a$ is the three-vector consisting of the space components of $\vec{u}_a^{\alpha}$) as the probability density of finding particle 1 at $\vec{x}_1$ with velocity $\vec{u}_1$ at time t_1, particle 2 at $\vec{x}_2$ with velocity $\vec{u}_2$ at time t_2, and so on. Then we can define the reduced S-body distribution function as [9]

$$F^{(s)}(t_1, x_1, u_1, \ldots t_s, x_s, u_s)=F^{(1)}(1)\ldots F^{(1)}(s)\left[1+\sum_{\substack{i,j=1\\ i<j}}^{s}G^{(2)}(i,j)+\sum_{\substack{i,j,k=1\\ i<j<k}}^{s}G^{(3)}(i,j,k)+\ldots\right]. \quad (1)$$

With $G^{(R)}$ is the R-particle correlation function. Let us consider the case of homogeneous plasma in equilibrium. Then $F^{(1)}$ is the one-particle distribution function of an ideal gas. That is, in a frame relative to which the system is macroscopically at rest, we must set for $F^{(1)}$ the relativistic Maxwellian distribution [20]

$$F^{(1)}(1)=\frac{\mu_1}{4\pi(mc^3)K_2(\mu_1)}\exp\left(\frac{-\mu_1}{\sqrt{1-(v_1/c)^2}}\right), \quad (2)$$

where $K_2(\mu)$ denotes the modified Bessel function.

Let us consider the internal energy E which is given by Kraeft et al. [21] and Barcons and Lapiedra [9] in the following form:

$$E=\int F^N(t, x_1, u_{1,\ldots}, x_N, u_N)H(x_1, u_{1,\ldots}, x_N, u_N)\prod_{R=1}^{N}d^3x_R d^3u_R \quad (3)$$

Where

$$H=\sum_i H_0(i)+\sum_{i,j} H_1(i,j)+\sum_{i,j,k} H_2(i,j,k)+\sum_{i,j,k,l} H_3(i,j,k,l)+\ldots \tag{4}$$

We can define the microscopic energy of system of one charge $H_0(a)$ as

$$H_0(a)=\sum_a m_a\gamma_a \tag{5}$$

with $\gamma_a=1/\sqrt{1-(v_a/c)^2}$ is the Lorentz factor.

On the other hand, for the microscopic energy of two charges, we have

$$H(a,b)=H_0(a)+H_0(b)+H^{(1)}(a,b), \tag{6}$$

where

$$H_1(a,b)=\sum_{i=a,b} m_i\gamma_i^3\left(\frac{1}{i\vec{K}\cdot(\vec{v}_1-\vec{v}_2)}-\pi\delta(\vec{K}\cdot\vec{v}_1-\vec{K}\cdot\vec{v}_2)\right) * v_i\cdot\hat{a}_i^{(1)}(\vec{K},\vec{v}_1,\vec{v}_2). \tag{7}$$

Now we can write

$$E=E_1+E_2+E_3+E_4+\ldots, \tag{8}$$

where

$$E_1=\sum_a\int H^{(0)}(a)F^{(1)}(a)d^3x_a d^3u_a, \tag{9}$$

$$E_2=\sum_{a,b}\int H^{(0)}(a,b)F^{(1)}(a)F^{(1)}(b)G(a,b)d^3x_a d^3u_a d^3x_b d^3u_b, \tag{10}$$

$$E_3=\sum_{a,b}\int[H^{(0)}(a)+H^{(0)}(b)]F^{(1)}(a)F^{(1)}(b)G(a,b)d^3x_a d^3u_a d^3x_b d^3u_b, \tag{11}$$

$$E_4=\sum_{a,b}\int H^{(1)}(a,b)F^{(1)}(a)F^{(1)}(b)d^3x_a d^3u_a d^3x_b d^3u_b. \tag{12}$$

Now let us proceed with the calculation of these terms. The integral (9) is the energy of an ideal gas then the corresponding energy per particle is in evident notation (see Appendix)

$$\frac{E_1}{N}=\sum_{i=1,2}\frac{\mu_i V}{8}\left[\frac{K_4(\mu_i)-K_0(\mu_i)}{K_2(\mu_i)}\right], \tag{13}$$

where $\mu_1=\frac{m_1c^2}{KT}$, $\mu_2=\frac{m_2c^2}{KT}$.

With m_1 is the electron mass and m_2 is the positron mass. Also we can write it as

$$E_1=\sum_a\left(\frac{3}{\mu_a}+\frac{\mu_a}{2}+\frac{1}{4}(\Gamma-\ln 2+\ln\mu_a)\mu_a^3+\frac{1}{16}(-1+2\Gamma-\ln 2+2\ln\mu_a)\mu_a^5+o(\mu_a^6)\right), \tag{14}$$

where we have used the expansion of the modified Bessel functions [22] for $0<\mu<1$ and Γ is Euler's constant, with numerical value $\Gamma\simeq 0.577216$.

Then we can define the correlation function $G(a,b)$ as

$$G(a,b)=G_{\mathrm{DH}}(a,b)+G_R(a,b), \tag{15}$$

where $G_{\mathrm{DH}}=-\frac{e_1e_2}{KTr_{12}}e^{-\kappa r_{12}}$ is the Debye-Hückel correlation function and G_R is a relativistic correction given by Barcons and Lapiedra [9] as

$$G_R(a,b)=-\frac{e_1e_2}{KTr_{12}}e^{-\kappa r_{12}}\left(\frac{\vec{v}_1.\vec{v}_2}{c^2}+\frac{(\vec{x}_{12}.\vec{v}_1)(\vec{x}_{12}.\vec{v}_2)}{2c^2r_{12}^2}\right). \tag{16}$$

And to calculate E_2 we must substituting by (15) into (10) then we can get

$$\frac{E_{2\mathrm{DH}}}{N}=-e^3\sqrt{\pi\rho}\sum_{i=1,2}\sqrt{\frac{\mu_i}{m_i}} \tag{17}$$

and

$$\frac{E_{2R}}{N}=-\frac{\pi e^4}{\alpha}\sum_{i=1,2}\frac{\mu_i}{m_i}\left[\left(\frac{1}{\mu_i}\frac{K_1(\mu_i)}{K_2(\mu_i)}+\frac{1}{\mu^2}\frac{K_0(\mu_i)}{K_2(\mu_i)}\right)^2+\left(\frac{K_0(\mu_i)}{\mu^2K_2(\mu_i)}\right)^2\right]. \tag{18}$$

Hence, the sum of (13), (17), and (18) gives the energy per particle in a classical dilute relativistic plasma for all temperatures. Then by integrate the sum of (13), (17), and (18), which gives the internal energy per particle in a classical dilute relativistic plasma for all temperatures, we can get the excess free energy for the TCP model as

$$\frac{F^{ex}}{NKT}=\sum_i n_i\left(-e^3\sqrt{\frac{n_i\pi}{m_1}}\mu_1^{1/2}-e^3\sqrt{\frac{n_i\pi}{m_2}}\mu_2^{1/2}\right)+\frac{1}{2!}\sum_i\sum_j n_in_j\left(\frac{-1}{\mu_1}+\frac{-1}{\mu_2}+\frac{\mu_1}{K}+\frac{\mu_2}{K}+\left(-\frac{1}{2}+\frac{e^4\pi}{\alpha m_2}(-1.2486+2.829\ln\mu_2-1.25\ln^2\mu_2)\right)\mu_2+\left(0.2898-\frac{\ln\mu_2}{4}+\frac{e^4\pi}{\alpha m_2}\left(-0.5758+1.307\ln\mu_2-\frac{1}{4}\ln^2\mu_2\right)\right)\mu_2^3\right.$$

$$+\left(-\frac{1}{4}+\frac{e^4\pi}{\alpha m_1}\left(-1.003+0.0575\ln\mu_1-\frac{1}{4}\ln^2\mu_1\right)\right)\mu_1$$
$$+\left(0.0144-\frac{\ln\mu_1}{8}+\frac{e^4\pi}{\alpha m_1}\left(0.098-\frac{1.386}{\mu_1^2}\right.\right.$$
$$\left.\left.-\frac{1}{2}\Gamma\mu_1+\frac{2\ln\mu_1}{\mu_1^2}-0.817\ln\mu_1-\frac{1}{4}\ln^2\mu_1\right)\right)\mu_1^3$$
$$+O[\mu_1]^4+O[\mu_2]^4. \tag{19}$$

We notice in (19) our results are in agreement with the results of Barcons and Lapiedra [8] up to order $\frac{1}{\mu}$.

3 Equation of State for Dilute Relativistic Plasma

The equation of state is the fundamental relation between the macroscopically quantities describing a physical system in equilibrium. Our aim in this section is to derive it using the distribution function for OCP and TCP. We can start with the simplest model of a Coulomb system which is known as the OCP model. Now we want to find the equation of state of this model to this goal, let us begin with the $F^{\text{ex}}_{(\text{OCP})}$ which can eassliy caculate by integrate (14) as

$$\frac{F^{\text{ex}}_{(\text{OCP})}}{\text{NKT}}=\sum_a\left(\frac{\mu_a^2}{4}+3\ln\mu_a+\frac{1}{64}\mu_a^4(-1+4\Gamma+4\ln 2+4\ln\mu_a)\right.$$
$$\left.+\frac{1}{288}\mu_a^6(-4+6\Gamma-3\ln 4+6\ln\mu_a)\right)+o(\mu_a^7). \tag{20}$$

Then, we shall compute the pressure using the relation

$$\frac{P}{nKT}=1-\left(\frac{\partial F^{\text{ex}}}{\partial V}\right)_{T,N}. \tag{21}$$

By substituting from (20) into (21), we get for OCP model the equation of state in the following form:

$$\frac{P}{nKT}=1-\sum_a\left(\frac{1}{16\mu_a}[48+8\mu_a^2+4\mu_a^4(\Gamma-\ln 2)\right.$$
$$\left.+\mu_a^6(2\Gamma-1-\ln 4)+2\mu_a^4\ln\mu_a(2+\mu_a^2)]\right). \tag{22}$$

Now by substituting from (19) into (21), we can get the equation of state for TCP model as

$$\frac{P}{nKT}=1-\sum_i\frac{-n_i}{4}\left(e^3\sqrt{\frac{n_i\pi}{m_1}}\mu_1^{-1/2}+e^3\sqrt{\frac{n_i\pi}{m_2}}\mu_2^{-1/2}\right)$$
$$-\frac{1}{V}\sum_i\sum_j n_i n_j\left(\frac{1}{\mu_1^2}+\frac{1}{\mu_2^2}+\left(-0.5+\frac{e^4\pi}{\alpha m_2}(-1.2486\right.\right.$$
$$\left.+2.829\ln\mu_2-1.25\ln^2\mu_2)\right)\mu_2+\left(0.2898-\frac{\ln\mu_2}{4}\right.$$
$$\left.+\frac{e^4\pi}{\alpha m_2}\left(-0.5758+1.307\ln\mu_2-\frac{1}{4}\ln^2\mu_2\right)\right)\mu_2^3$$
$$+\left(-\frac{1}{4}+\frac{e^4\pi}{\alpha m_1}\left(-1.003+0.0575\ln\mu_1-\frac{1}{4}\ln^2\mu_1\right)\right)\mu_1$$
$$+\left(0.0144-\frac{\ln\mu_1}{8}+\frac{e^4\pi}{\alpha m_1}\left(0.098-\frac{1.386}{\mu_1^2}-\frac{1}{2}\Gamma\mu_1+\frac{2\ln\mu_1}{\mu_1^2}\right.\right.$$
$$\left.\left.\left.-0.817\ln\mu_1-\frac{1}{4}\ln^2\mu_1\right)\right)\mu_1^3\right)+O[\mu_1]^4+O[\mu_2]^4. \tag{23}$$

We notice in (23), our results are in agreement with the results of Barcons and Lapiedra [8] up to order $\frac{1}{\mu}$.

We restricted our calculations to the following values:
$K=1.3806568\times10^{-23}$ J/K;
$m=1.6749286\times10^{-27}$ kg;
$c=2.99792458\times10^{8}$ m/s;
$T=10^{13}$ to 10^{15} K.

4 Conclusions

In this article, we calculated the classical equation of state and the excess free energy up to order μ^6 for OCP model and up to order μ^3 for TCP model. For the numerical calculation, we restrict ourselves to the case of TCP which anti-symmetric with respect to the charges $e_e=-e_i=-e$ and therefore symmetrical with respect to the densities $n_1=n_2=n$. To simplify the numeric investigations, we simulated so far only mass symmetrical electron-ion plasma with $m_1=m_2=m$ and also $\mu_1=\mu_2=\mu$ and plotted the curves for it.

We consider only the thermal equilibrium plasma. Among the presently known formulas, to our knowledge (19) represents the of the excess free energy of TCP in the classical relativistic form for effective potential; our results include terms of the higher order of the plasma thermal parameter. The excess free energy of Barcons and Lapiedra [8] is in agreement with the corresponding terms in our results for the Coulomb potential. The

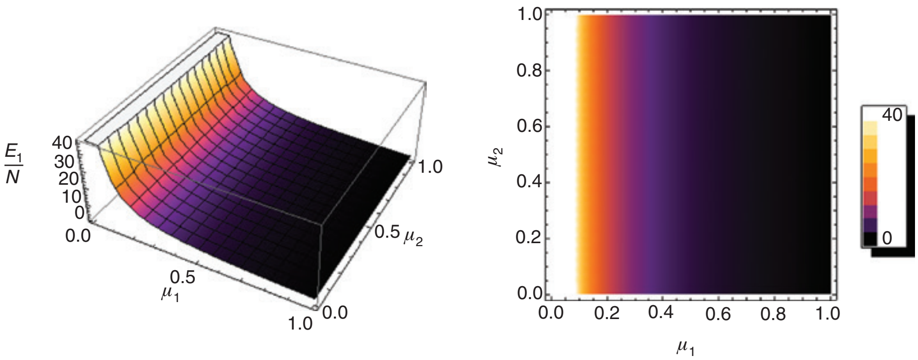

Figure 1: Relation between E_1/N and the thermal parameters μ_1 and μ_2 for two-component plasma from (13).

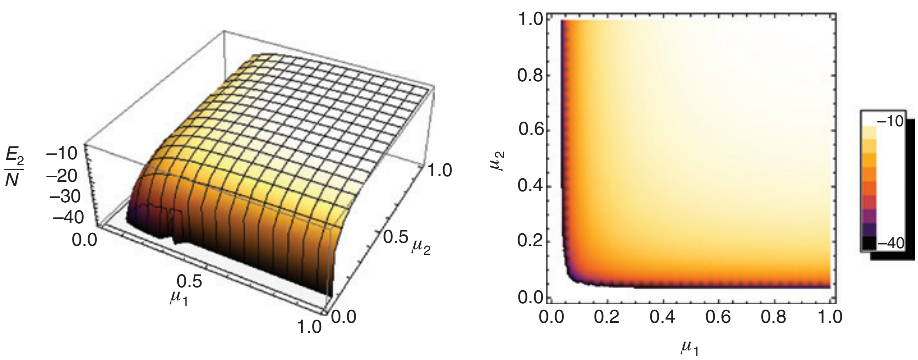

Figure 2: Relation between E_2/N and the thermal parameters μ_1 and μ_2 for two-component plasma from the sum of (17) and (18).

results are used to derive the equation of state; among the presently known formulas, to our knowledge the classical equation of state (23) represents the first formulas of the classical equation of state of a two-component, classical, dilute, and slightly relativistic plasma up to order μ^3. During a thermodynamic process, internal energy of a system can be increase or decrease; Figure 1 show the relation between internal energy from equation (13) and the thermal parameters for two-component plasma. Also Figure 2 show the relation between internal energy from the sum of equations (17) and (18) and the thermal parameters for two-component plasma.

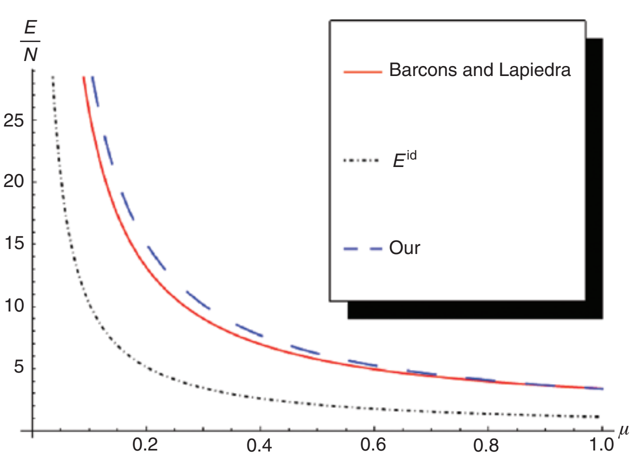

Figure 3: Comparison between the internal energy for OCP model from Barcons and Lapiedra [8] (solid line), the internal energy for ideal gas [9] (dotted dashed line), and our results up to μ^5 (dashed line).

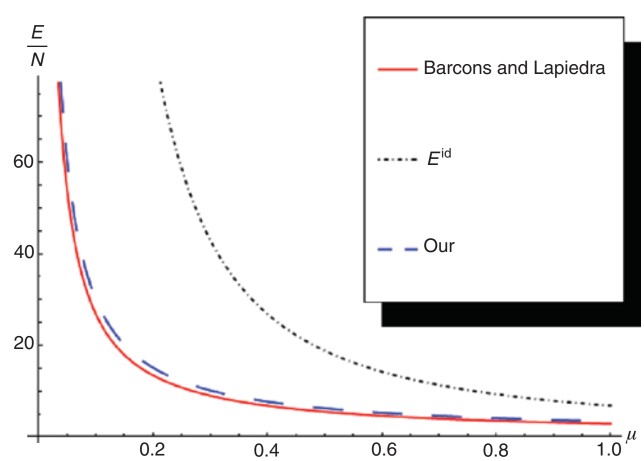

Figure 4: Comparison between the internal energy for OCP model from Barcons and Lapiedra [8] (solid line), the internal energy for ideal gas ref [9] (dotted dashed line), and our results up to μ^5 (dashed line). (Neutron mass $m_n = 939.56563$ MeV.)

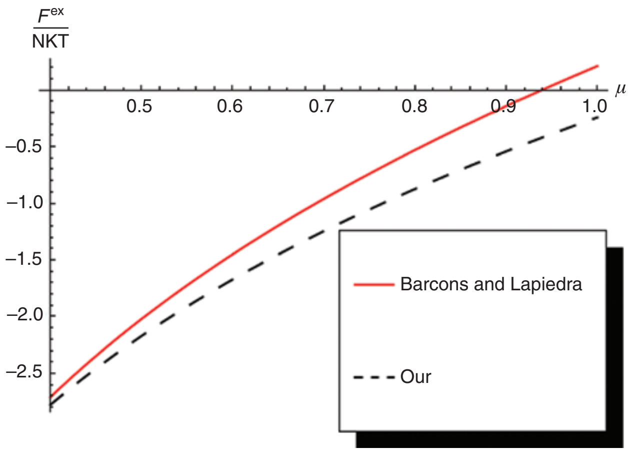

Figure 5: Comparison between the excess free energy for OCP model from Barcons and Lapiedra [8] (solid line) and our results up to μ^5 (dashed line). (Neutron mass $m_n = 939.56563$ MeV).

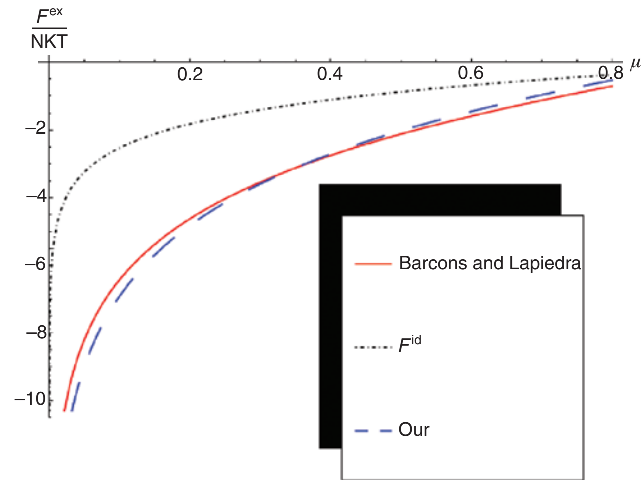

Figure 7: Comparison between the excess free energy for OCP model from Barcons and Lapiedra [8] (solid line), the internal energy for ideal gas [9] (dotted dashed line), and our results up to μ^5.

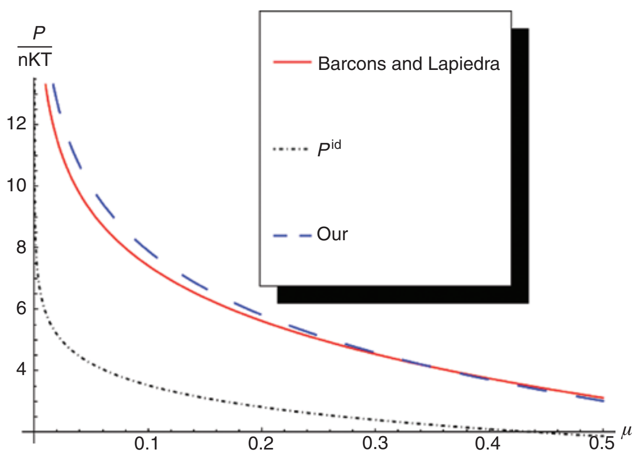

Figure 6: Comparison between the pressure for OCP model from Barcons and Lapiedra [8] (solid line), the internal energy for ideal gas [9] (dotted dashed line), and our results up to μ^5 (dashed line).

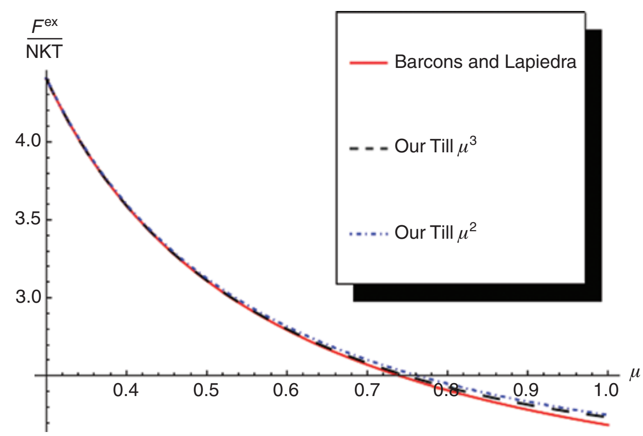

Figure 8: Comparison between the excess free energy for TCP model from Barcons and Lapiedra [8] (solid line), our results of the classical excess free energy up to μ^3 (dashed line), and our results up to μ^2 (dotted dashed line).

Figures 3, 4, 5 and 6 show the comparison between our results for OCP model with the results of Barcons and Lapiedra [8] and ideal gas model [9]. We see from this figures in limit $T \to \infty$ or $\mu \to 0$, we get a good agreement with ideal gas model. Also when plasma thermal parameter increases our results near to Barcons and Lapiedra [8]. We note from Figure 7 the excess free energy for OCP model from Barcons and Lapiedra [8] and our results up to μ^5 is the same in the range of μ=0.3:0.4 .

From Figures 8 and 9, we note a good agreement in our results for TCP model with the results of Barcons and Lapiedra [8] at small values of μ but when plasma thermal parameter increases they are far from each other.

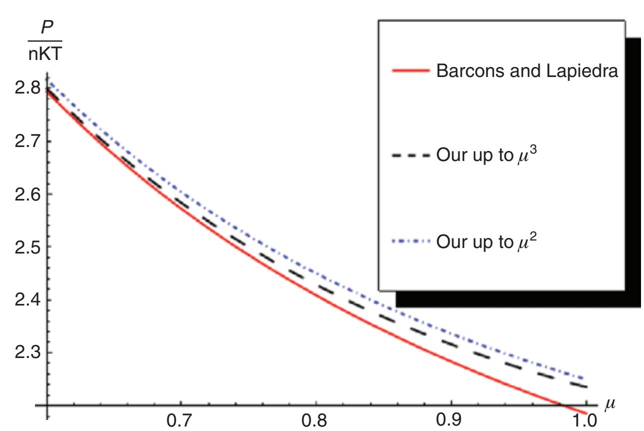

Figure 9: Comparison between the classical equation of state for TCP model from Barcons and Lapiedra [8] (solid line), our results of the classical equation of state up to μ^3 (dashed line), and our results up to μ^2 (dotted dashed line).

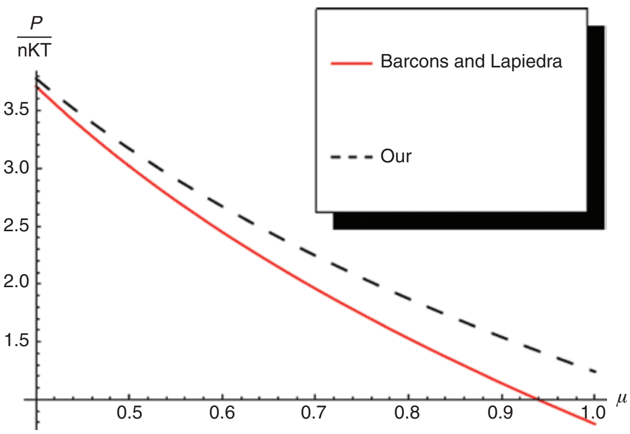

Figure 10: Comparison between pressure for OCP model from Barcons and Lapiedra [8] (solid line) and our results up to μ^5 (dashed line). (Neutron mass $m_n = 939.56563$ MeV.)

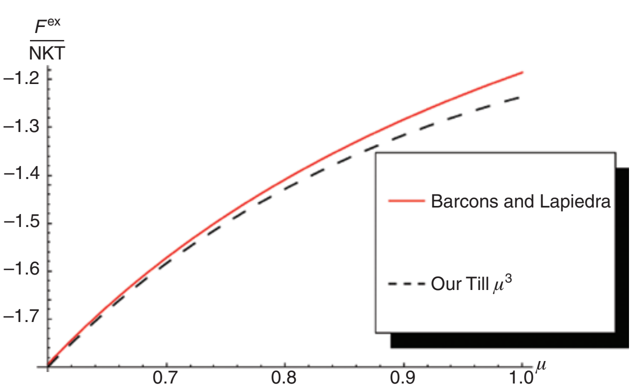

Figure 11: Comparison between the excess free energy for plasma localised near the surface of white dwarf ($T \sim 10^7$ K) from Barcons and Lapiedra [8] (solid line) and our results up to μ^3 (dashed line).

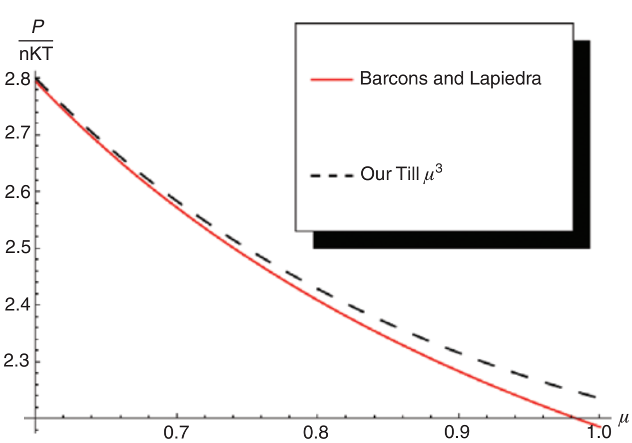

Figure 12: Comparison between pressure for plasma localised near the surface of white dwarf ($T \sim 10^7$ K) Barcons and Lapiedra [8] (solid line) and our results up to μ^3 (dashed line).

The existence of heavy elements is found to form in a prestellar stage of the evolution of the universe, when all matter was compressed to extremely high densities and possessed correspondingly high temperatures. Thus, the ionised condition within the massive compact stars occur due to this high density and temperature. In case of such a compact object (i.e. white dwarfs), the degenerate electron number density is so high (in white dwarfs it can be of the order of 10^{30} cm^{-3} or even more [16] and [23]). Also Figure 10 show comparison between pressure for OCP model from Barcons and Lapiedra [9] and our results in the case of Neutron mass. We used plasma localised near the surface of white dwarf ($T \to 10^7$ K) as a practical example of the results we have obtained; see Figures 11 and 12. We see from this figures in limit $\mu \to 0$, we get a good agreement with ideal gas model Barcons and Lapiedra [8].

Appendix

In this Appendix, we derive (13) in the framework of PRM

$$\begin{aligned} E_{(\mathrm{OCP})} &= \sum_i \iint H_0(i) F^{(1)}(i) d^3x_i d^3u_i \\ &= \sum_i \iint m_i \gamma_i \frac{\mu_i}{4\pi(m_i c^3) K_2(\mu_i)} \exp(-\mu_i \gamma_i) d^3x_i d^3u_i \\ &= \sum_i \frac{\mu_i}{4\pi c^3 K_2(\mu_i)} \int_V d^3x_i \int \gamma_i \exp(-\mu_i \gamma_i) d^3u_i \\ &= \sum_i \frac{\mu_i V}{4\pi c^3 K_2(\mu_i)} \int \sqrt{1+\frac{u_i^2}{c^2}} \exp\left(-\mu_i \sqrt{1+\frac{u_i^2}{c^2}}\right) d^3u_i \\ &= \sum_i \frac{\mu_i V}{8 K_2(\mu_i)} \int_0^\infty [\cosh 4y - 1] \exp(-\mu_i \cosh y) dy. \end{aligned} \tag{A.1}$$

Now using the integral representation of the modified Bessel function of the second kind which is given as

$$K_n(z) = \int_0^\infty \cosh(ny) \exp(-z \cosh y) dy. \tag{A.2}$$

Then we can get

$$E_{(\mathrm{OCP})} = \sum_i \frac{\mu_i V}{8} \left[\frac{K_4(\mu_i) - K_0(\mu_i)}{K_2(\mu_i)} \right]. \tag{A.3}$$

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