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Noether Symmetry Analysis of the Dynamic Euler-Bernoulli Beam Equation

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Abstract: We study the fourth-order dynamic Euler-Bernoulli beam equation from the Noether symmetry viewpoint. This was earlier considered for the Lie symmetry classification. We obtain the Noether symmetry classification of the equation with respect to the applied load, which is a function of the dependent variable of the underlying equation. We find that the principal Noether symmetry algebra is two-dimensional when the load function is arbitrary and extends for linear and power law cases. For all cases, for each of the Noether symmetries associated with the usual Lagrangian, we construct conservation laws for the equation via the Noether theorem. We also provide a basis of conservation laws by using the adjoint algebra. The Noether symmetries pick out the special value of the power law, which is –7. We consider the Noether symmetry reduction for this special case, which gives rise to a first integral that is used for our numerical code. For this, we then find numerical solutions using an in-built function in MATLAB called bvp4c, which is a boundary value solver for differential equations that are depicted in five figures. The physical solutions obtained are for the deflection of the beam with an increase in displacement. These are given in four figures and discussed.

Keywords: Conservation Laws; Dynamic Euler-Bernoulli Beam Equation; Lagrangian; Noether Point Symmetries.

1 Introduction

Daniel Bernoulli was the first to formulate the partial differential equation (PDE) governing the motion of a thin vibrating beam. Later on, Leonhard Euler used Bernoulli's theory to study the shape of elastic beams subject to various loading conditions. Thus, the Euler-Bernoulli beam theory is due to Daniel Bernoulli and Leonard Euler.

In the Euler-Bernoulli beam theory, the transversal motion of a loaded thin elastic beam is given by the uniform fourth-order dynamic Euler-Bernoulli beam PDE:

$$\frac{\partial^4 u}{\partial x^4}+\frac{\partial^2 u}{\partial t^2}-f(u)=0, \quad (x,\, t)\in D\subset\mathbb{R}^2, \tag{1}$$

where D is an open set in $\mathbb{R}^2$, u is smooth in its arguments, and $f(u)$ is the applied load, which is a smooth function of the transversal displacement or deflection u of the beam. The deflection $u(t, x)$ is a function of time t and position x from one end of the beam (see [1–5] and the references therein). The beam PDE (1) is the Euler-Lagrange equation for a function that minimizes the deflection u [1].

Han et al. [2] deduced the uniform beam PDE (1) from a Hamilton variational principle. They invoke the potential and kinetic energies of the uniform beam due to bending to obtain a Lagrangian. Then by the extended Hamilton principle, they arrive at (1).

In this paper, we study (1) from the Noether symmetry point of view. We perform the Noether point symmetry classification with respect to a standard Lagrangian of the PDE (1) and determine the possible functional forms of the unknown load function $f(u)$ by using the determining equation for the Noether point of the equation. We find that for the arbitrary values of the load function, the principle Noether symmetry algebra is spanned by the two Noether symmetries. Moreover, we have shown that this algebra extends in two cases, viz, the linear and the power law. For each of these extended cases, we also obtain the Noether symmetries. A comparison of the results obtained in this work is made with that one related to the Lie group classification of the PDE (1) derived by Bokhari et al. [5]. For all the cases, including $f(u)$ given by a power law of u, for every Noether symmetries admitted by the PDE (1), we construct conservation laws for the equation via the Noether theorem. We use the physically relevant conservation law to construct a first integral and also as a consequence of physical solutions numerically, which are depicted in four graphs and discussed. In a previous study, Bokhari et al. [5] obtained only an asymptotic solution.

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The outline of the paper is as follows. In Section 2, we provide the notation and some important results used in the derivation of our main results. In Section 3, we present the Noether symmetry classification of the PDE (1) with respect to a usual Lagrangian. In Section 4, for all cases, including that which corresponds to the power law of u, for each of the Noether symmetries, we construct conservation laws for the PDE (1) via the Noether theorem. Herein, we further deduce a basis of conservation laws for each case as well as perform a Noether symmetry reduction for an exceptional power law case, viz, $\sigma=-7$. The conserved components and the results of the first integral obtained form the basis of our numerical code. Numerical solutions are obtained by invoking an in-built function in MATLAB called bvp4c, which is a boundary value solver for differential equations for the nonlinear equations with boundary conditions in Section 5 taking into account the first integral. These are represented Figures 1–5. Concluding remarks are made in Section 6.

2 Preliminaries

In this section, we present the notation and some important results that are used in the following sequel. For details, the avid reader is referred to [6–12].

Consider a kth-order system of PDEs of n independent variables $x=(x^1, x^2, \ldots, x^n)$ and m dependent variables $u=(u^1, u^2, \ldots, u^m)$, namely,

$$E_\alpha(x, u, u_{(1)}, \ldots, u_{(k)})=0, \quad \alpha=1, \ldots, m, \tag{2}$$

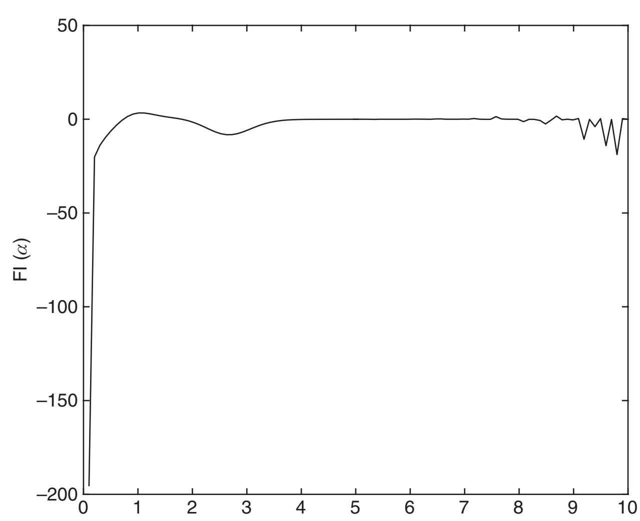

Figure 1: Values obtained for the first integral for $\delta_1=1$ with $F_0=1.5$ and clamped end boundary conditions.

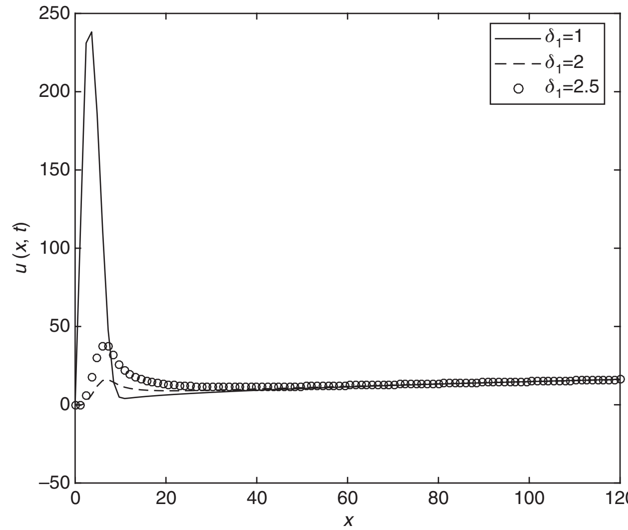

Figure 2: Numerical solution for the deflection $u(x, t)$ for different δ_1 with $F_0=1.5$ and time $T=8$ for clamped end boundary conditions where $x\in[0, 120]$.

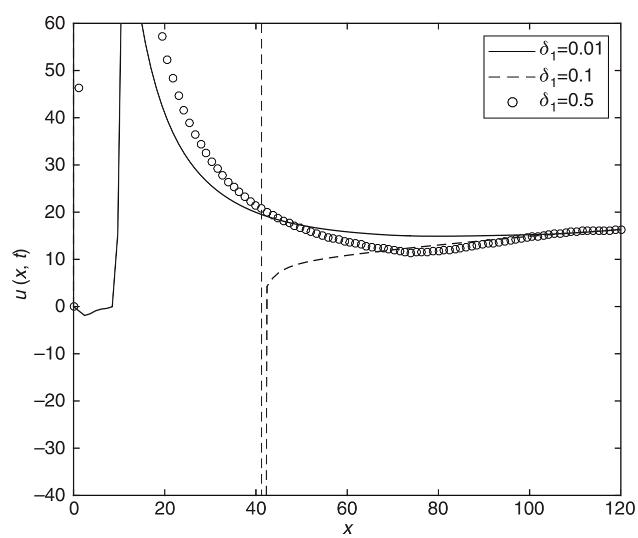

Figure 3: Numerical solution for the deflection $u(x, t)$ with various small values of δ_1 with $F_0=1$ and time $T=4$ for clamped end boundary conditions where $x\in[0, 120]$.

where $u_{(1)}$, $u_{(2)}$, …, $u_{(k)}$ denote the collections of all first up to kth-order partial derivatives, i.e. $u_i^\alpha=D_i(u^\alpha)$, $u_{ij}^\alpha=D_jD_i(u^\alpha)$, …, $u_{i_1\ldots i_k}=D_{i_k}\ldots D_{i_1}(u^\alpha)$, respectively, with the total derivative operator with respect to x^i given by

$$D_i=\frac{\partial}{\partial x^i}+u_i^\alpha\frac{\partial}{\partial u^\alpha}+u_{ij}^\alpha\frac{\partial}{\partial u_j^\alpha}+\ldots, \quad i=1, \ldots, n, \tag{3}$$

where the summation convention is used whenever appropriate. This operator truncates when it acts on a

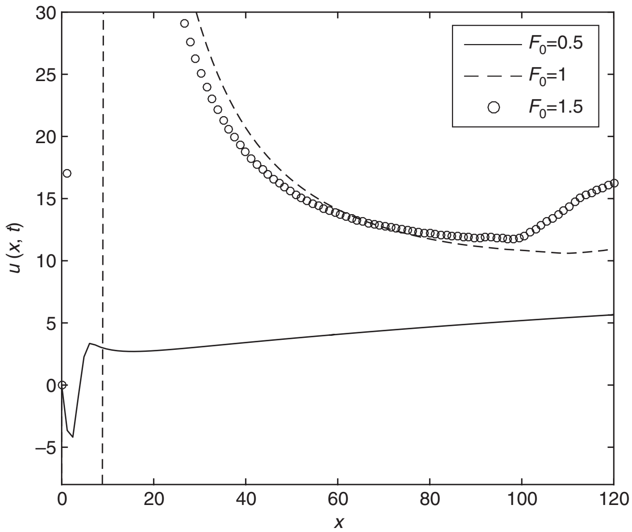

Figure 4: Numerical solution for the deflection $u(x, t)$ with various values of F_0 with $\delta_1=1$ and time $T=4$ for clamped end boundary conditions where $x\in[0, 120]$.

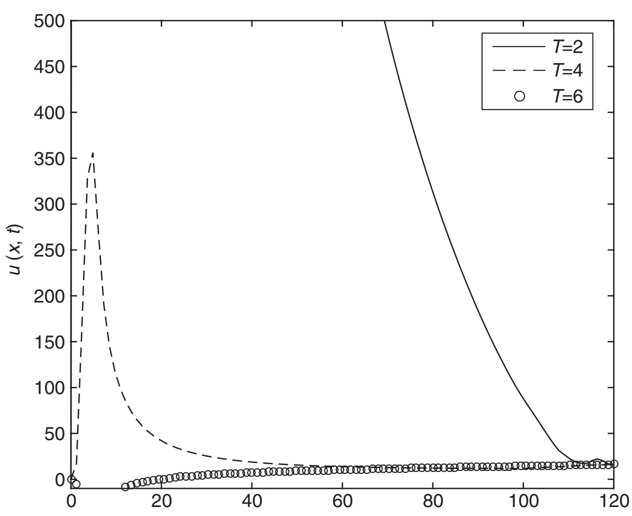

Figure 5: Numerical solution for the deflection $u(x, t)$ with various values of the time T with $\delta_1=1$, $F_0=1.5$ for clamped end boundary conditions where $x\in[0, 120]$.

differential function $f\in\mathcal{A}$, where $\mathcal{A}$ is the space of differential functions.

A Lie-Bäcklund operator in infinite formal sum is given by

$$X=\xi^i\frac{\partial}{\partial x^i}+\eta^\alpha\frac{\partial}{\partial u^\alpha}+\sum_{s\geq 1}\zeta^\alpha_{i_1i_2\ldots i_s}\frac{\partial}{\partial u^\alpha_{i_1i_2\ldots i_s}},\tag{4}$$

where ξ^i, η^α $\alpha\in\mathcal{A}$. The additional coefficients $\zeta^\alpha_{i_1i_2\ldots i_s}$ are determined uniquely by the prolongation formulae:

$$\begin{aligned}\zeta^\alpha_i&=D_i(W^\alpha)+\xi^j u^\alpha_{ij},\\ \zeta^\alpha_{i_1\ldots i_s}&=D_{i_1}\ldots D_{i_s}(W^\alpha)+\xi^j u^\alpha_{ji_1\ldots i_s},\quad s>1,\end{aligned}\tag{5}$$

in which $W^\alpha=\eta^\alpha-\xi^j u^\alpha_j$ is the Lie characteristic function. Usually, the prolongation symbol is used to denote the prolonged operator when the additional coefficients are inserted, as that of Olver [8].

The n-tuple vector $T=(T^1, T^2, \ldots, T^n)$, $T^j\in\mathcal{A}$, $j=1, \ldots, n$, is a conserved vector of (2) if T^i satisfies

$$D_iT^i|_{(2)}=0.\tag{6}$$

Equation (6) is called a local conservation law of system (2).

The Euler-Lagrange operator, $\delta/\delta u^\alpha$, is given by

$$\frac{\delta}{\delta u^\alpha}=\frac{\partial}{\partial u^\alpha}+\sum_{s\geq 1}(-1)^s D_{i_1}\ldots D_{i_s}\frac{\partial}{\partial u^\alpha_{i_1i_2\ldots i_s}},\quad \alpha=1, \ldots, m.\tag{7}$$

Let us write the operator in (4) in the following simple point form given by

$$X=\xi^i(t, x, u)\frac{\partial}{\partial x^i}+\eta^\alpha(t, x, u)\frac{\partial}{\partial u^\alpha}+\sum_{s\geq 1}\zeta^\alpha_{i_1i_2\ldots i_s}\frac{\partial}{\partial u^\alpha_{i_1i_2\ldots i_s}}.\tag{8}$$

Here we have a subclass of the Lie-Bäcklund operators, viz, point symmetry operators. The operator X in (8) is called a Noether point symmetry generator corresponding to a Lagrangian $L=L(x, u, u_{(1)}, \ldots, u_{(l)})\in\mathcal{A}$, $l\leq k$ of (2) if it can be determined from

$$X(L)+LD_i(\xi^i)=D_i(B^i),\tag{9}$$

for some vector $B=(B^1, B^2, \ldots, B^n)$, $B^i\in\mathcal{A}$.

Noether Theorem 2.1 (see, e.g. [10]). If the operator X as in (8) is a Noether symmetry generator of a Lagrangian L corresponding to a system of (2), then the components T^i of the conserved vector T can be constructed by the formula

$$T^i=B^i-\xi^i L-W^\alpha\frac{\delta L}{\delta u^\alpha_i}-\sum_{s\geq 1}D_{i_1}\ldots D_{i_s}(W^\alpha)\frac{\delta L}{\delta u^\alpha_{ii_1i_2\ldots i_s}},\quad i=1, \ldots, n.\tag{10}$$

The following results are taken from Kara and Mahomed [11].

A Lie-Bäcklund generator X is associated [11] with a conserved vector $T=(T^1, \ldots, T^n)$ of (2) if X and T^i satisfy

$$X(T^i)+T^iD_j(\xi^j)-T^jD_j(\xi^i)=0,\quad i=1, \ldots, n.\tag{11}$$

If X is any Lie-Bäcklund symmetry generator of (2) and T^i, $i=1, \ldots, n$, are the components of a conserved vector of (2), then [11]

$$T_*^i = X(T^i)+T^iD_j(\xi^j)-T^jD_j(\xi^i), \quad i=1, \ldots, n, \tag{12}$$

are components of a conserved vector of (2), which could be trivial or nontrivial.

Suppose that $D_iT^i=0$ is a conservation law of (2). Then by means of the transformation $\bar{x}^i=f^i(x)$, there exist transformed conserved components [12] $\bar{T}^i$ such that $JD_iT^i=\bar{D}_i\bar{T}^i$, where $\bar{T}^i$ are given as

$$\begin{pmatrix} \bar{T}^1 \\ \bar{T}^2 \\ \vdots \\ \bar{T}^n \end{pmatrix} = J(A^{-1})^T \begin{pmatrix} T^1 \\ T^2 \\ \vdots \\ T^n \end{pmatrix} \tag{13}$$

in which

$$A^{-1}=\begin{pmatrix} D_1\bar{x}^1 & D_1\bar{x}^2 & \cdots & D_1\bar{x}^n \\ D_2\bar{x}^1 & D_2\bar{x}^2 & \cdots & D_2x^n \\ \vdots & \vdots & \cdots & \cdots \\ D_n\bar{x}^1 & D_n\bar{x}^2 & \cdots & D_n\bar{x}^n \end{pmatrix} \tag{14}$$

and $J=\det A$.

Any symmetry (see [12]) X of the conservation law $D_iT^i=0$ of (2) can be transformed under the similarity transformation of the symmetry for the PDE to the symmetry $\bar{X}$ for the PDE $\bar{D}_i\bar{T}i=0$.

The result shows when one can obtain a basis [11] of conservation laws for a system (2) using the adjoint action of the Lie algebra.

Theorem 2.2 (see [11]). If X, Y, and Z are Noether symmetry generators associated with a given Lagrangian L of (2) (that is assumed variational), which satisfy ad $Y(X)=[X, Y]=Z$, where $Z\neq 0$ and $Z\neq bY$, such that Y is associated with the conserved vector T, then the Noether conserved vector T^*, with components T_*^i given by (12), is associated with (or multiple of) Z. Further, T_* is a nontrivial conserved vector that is different from T.

3 Noether Symmetry Classification of (1)

Equation (1) has the usual Lagrangian $L=\frac{1}{2}u_{xx}^2-\frac{1}{2}u_t^2-\int f(u)\mathrm{d}u$ whose Noether point symmetry generator $X=\xi^1(t, x, u)\partial/\partial t+\xi^2(t, x, u)\partial/\partial x+\eta(t, x, u)\partial/\partial u$ satisfies (9).

The coefficient functions ξ^1, ξ^2, and η are independent of the derivatives of u as we assume a subclass of the Lie-Bäcklund operators, which are of point type. Just to point out if we consider higher symmetries, then we need to make an ansatz on the form of the coefficient functions ξ^1, ξ^2, and η to arrive at a split system of determining equations for these functions. Here we invoke point transformations that yield many cases, which are as presented in the following paragraphs. It will be worthwhile in the future to investigate higher symmetries after this study of point symmetries.

Thus, equating (because our operator is of point type) the coefficients of like derivatives of u in determining (9) yields the following overdetermined system of linear PDEs:

$$\xi^1=\xi^1(t),\ \xi^2=\xi^2(x),\ \eta_{uu}=0,\ \eta_{xx}=0, \tag{15}$$

$$2\eta_{xu}-\xi_{xx}^2=0,\ \eta_u+\frac{1}{2}\xi_t^1-\frac{3}{2}\xi_x^2=0,\ -\eta_u+\frac{1}{2}\xi_t^1-\frac{1}{2}\xi_x^2=0, \tag{16}$$

$$B_u^1=-\eta_t,\ B_u^2=0, \tag{17}$$

$$-f(u)\eta-\xi_t^1\int f(u)\mathrm{d}u-\xi_x^2\int f(u)\mathrm{d}u=B_t^1+B_x^2. \tag{18}$$

Solving (15)–(18) for ξ^1, ξ^2, and η, we obtain

$$\begin{aligned}&\xi^1=2c_1t+c_2,\ \xi^2=c_1x+c_3,\ \eta=\frac{c_1}{2}u+g(t)x+h(t),\\ &B^1=(-g'x-h')u+i(t, x),\ B^2=j(t, x)\end{aligned} \tag{19}$$

with the classifying relation given by

$$f(u)\left[\frac{c_1}{2}u+g(t)x+h(t)\right]+3c_1\int f(u)\mathrm{d}u=(g''x+h'')u-i_t-j_x. \tag{20}$$

In (19) and (20), c_1 to c_3 are constants and $g(t)$ and $h(t)$ are arbitrary functions of t; moreover, the prime denotes differentiation with respect to the variable t.

Differentiating (20) with respect to u, x gives rise to the following equations:

$$f'(u)\left[\frac{c_1}{2}u+g(t)x+h(t)\right]+\frac{7c_1}{2}f(u)=g''x+h'', \tag{21}$$

and

$$f'(u)g(t)=g''. \tag{22}$$

Case 1: $f(u)$ is arbitrary

From (21), we get $c_1=0$ and $g=0=h$. Hence, from (19), we have, $\xi^1=c_2$, $\xi^2=c_3$, $\eta=0$, $B^1=0$, and $B^2=0$. Thus, for arbitrary $f(u)$, (1) has the following Noether symmetries:

$$X_1=\frac{\partial}{\partial t},\quad X_2=\frac{\partial}{\partial x} \tag{23}$$

which is the principal Noether algebra of (1).

Case 2: $f(u)=A_0u+A_1$ with arbitrary constants A_0 and A_1, and $g(t)\neq 0$

Then from (22), we have

$$g''-A_0g=0. \tag{24}$$

Case 2.1: $A_0\neq 0$

(a) $A_0>0$. In this case, the general solution of (24) takes the form $g(t)=c_4\exp(\sqrt{A_0}t)+c_5\exp(-\sqrt{A_0}t)$, where c_4 and c_5 are constants. Substitution of the values of f and g into (20) and then splitting it into powers of u give rise to $c_1=0$ and

$$h''-A_0h=0. \tag{25}$$

The solution of (25) is $h(t)=c_6\exp(\sqrt{A_0}t)+c_7\exp(-\sqrt{A_0}t)$, where c_6 and c_7 are constants. Now by inserting the values of f, g, and c_1 into (20), it becomes

$$(A_0u+A_1)(gx+h)=(g''x+h'')u-i_t-j_x. \tag{26}$$

We set $i=0=j$ and split (26) with respect to powers of u to find $A_1g=0$ and $A_1h=0$; the first equation implies $A_1=0$ as $g(t)\neq 0$. Here the vector functions take the form $B^1=(-c_3\sqrt{A_0}\exp(\sqrt{A_0}t)x+c_4\sqrt{A_0}\exp(-\sqrt{A_0}t)x-c_5\sqrt{A_0}\exp(\sqrt{A_0}t)+c_6\sqrt{A_0}\exp(-\sqrt{A_0}t))u$ and $B^2=0$. Therefore, the principal Noether algebra extends by four Noether symmetry generators. The Noether algebra is then spanned by (23) and

$$X_3=x\exp(\sqrt{A_0}t)\frac{\partial}{\partial u},\ X_4=x\exp(-\sqrt{A_0}t)\frac{\partial}{\partial u},\quad X_5=\exp(\sqrt{A_0}t)\frac{\partial}{\partial u},\ X_6=\exp(-\sqrt{A_0}t)\frac{\partial}{\partial u}. \tag{27}$$

(b) $A_0<0$. Here we obtain the general solutions of (24) and (25) as $g(t)=c_4\sin(\sqrt{-A_0}t)+c_5\cos(\sqrt{-A_0}t)$ and $h(t)=c_6\sin(\sqrt{-A_0}t)+c_7\cos(\sqrt{-A_0}t)$, where c_4 to c_7 are constants. The calculations are similar to item (a), so that we just list the results. The vector functions are obtained as $B^1=(-c_3\sqrt{-A_0}x\cos(\sqrt{-A_0}t)+c_4\sqrt{-A_0}x\sin(\sqrt{-A_0}t)-c_5\sqrt{-A_0}\cos(\sqrt{-A_0}t)+c_6\sqrt{-A_0}\sin(\sqrt{-A_0}t))u$ and $B^2=0$. The Noether algebra is now spanned by (23) and

$$X_3=x\sin(\sqrt{-A_0}t)\frac{\partial}{\partial u},\ X_4=x\cos(\sqrt{-A_0}t)\frac{\partial}{\partial u},\quad X_5=\sin(\sqrt{-A_0}t)\frac{\partial}{\partial u},\ X_6=\cos(\sqrt{-A_0}t)\frac{\partial}{\partial u}. \tag{28}$$

Case 2.2: $A_0=0$

From (22) and (24), we get $f(u)=A_1$ and $g(t)=c_4+c_5t$, where c_4 and c_5 are constants. Now from (21), we determine $h(t)=(7c_1A_1/4)t^2+c_6t+c_7$, where c_6 and c_7 are constants. Substituting the values of g and h into (20) and setting $i=0=j$, we derive

$$A_1\left[(c_4+c_5t)x+\left(\frac{7c_1A_1}{4}t^2+c_6+tc_7\right)\right]=0. \tag{29}$$

Case 2.2.1: $A_1\neq 0$

Now splitting (29) by the powers of x, we find that the constants c_1, c_4, c_5, c_6, and c_7 are zero. Thus, the infinitesimal coefficients and vector functions in (19) take the form $\xi^1=c_2$, $\xi^2=c_3$, $\eta=0$, $B^1=0$, and $B^2=0$, which is the same as Case 1.

Case 2.2.2: $A_1=0$

Then we deduce from (20) $g(t)=c_4+c_5t$ and $h(t)=c_6+c_7t$. Therefore, (19) becomes $\xi^1=2c_1t+c_2$, $\xi^2=c_1x+c_3$, $\eta=(c_1/2)u+c_4x+c_5tx+c_6+c_7t$, $B^1=(-c_6x-c_7)u$, and $B^2=0$. Thus, the Noether algebra is spanned by (23) and

$$X_3=\frac{\partial}{\partial u},\ X_4=t\frac{\partial}{\partial u},\ X_5=x\frac{\partial}{\partial u},\ X_6=tx\frac{\partial}{\partial u}\quad X_7=2t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}+\frac{u}{2}\frac{\partial}{\partial u}. \tag{30}$$

Case 3: $f(u)\neq A_0u+A_1$, where A_0 and A_1 are arbitrary constants, and $g(t)=0$

Differentiating (21) with respect to u twice and considering that $f''(u)\neq 0$, we obtain

$$c_1\left[\frac{1}{2}+4\left(\frac{f'}{f''}\right)'\right]=0. \tag{31}$$

Case 3.1: $c_1=0,\ \left(\frac{f'}{f''}\right)'\neq-\frac{1}{8}$

For this case, from (21), we arrive at $h(t)=0$. Thus, we find that this results in Case 1 again.

Case 3.2: $c_1\neq 0,\ \left(\frac{f'}{f''}\right)'\neq-\frac{1}{8}$

In this case, our computations result in

$$f(u)=\frac{8A_2}{7}\left(A_1-\frac{u}{8}\right)^{-7},\ h(t)=-4c_1A_1,$$

where A_1 and A_2 are arbitrary constants. The infinitesimal coefficients and vector functions in (19) are now given by

$\xi^1=2c_1t+c_2$, $\xi^2=c_1x+c_3$, $\eta=(c_1/2)u-4c_1A_1$, $B^1=0$, and $B^2=0$. Thus, the Noether algebra, in this case, is spanned by (23) and

$$X_3=2t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}+\left(\frac{u}{2}-4A_1\right)\frac{\partial}{\partial u}. \tag{32}$$

Remark: A complete Lie symmetry group classification of the PDE (1) was obtained by Bokhari et al. [5]. In this present work, the Noether symmetry classification of (1) is performed with respect to the usual Lagrangian. The Noether symmetries presented in Case 1 are exactly the same as the Lie point symmetries obtained by Bokhari et al. [5] for arbitrary $f(u)$. Case 2, where f is linear in u, corresponds to the Case I of Bokhari et al. [5]. Moreover, the Noether symmetries obtained in Case 2.1 form a subalgebra of the Case I (iii) of Bokhari et al. [5], and the infinitesimal coefficients of the Noether symmetries satisfy the linear PDE $\alpha_{tt}-A_0\alpha=0$. The Noether symmetries derived in Case 2.2.1 with $A_1=1$ relates to the Case I (ii) of Bokhari et al. [5] and is a subalgebra spanned by the symmetry generators $\partial/\partial t$ and $\partial/\partial x$. The Noether symmetries obtained in Case 2.2.2 are the same as in Case I (i) of Bokhari et al. [5], where in this case the infinitesimal coefficients of the infinite symmetry generators satisfy the linear PDE $\alpha_{tt}+\alpha_{4x}=0$. Case 3.2 is a nonlinear one, which is equivalent to the Case II of Bokhari et al. [5]. However, the Noether case pick out the special value of $\sigma=-7$. The Noether symmetry algebra is three-dimensional and is isomorphic to that of the Case II of Bokhari et al. [5].

4 Conservation Laws of (1)

In this section, we determine the conservation laws and also obtain a basis of conservation laws for the PDE (1) with respect to the usual Lagrangian formulation.

For Case 1, the Noether symmetries are translations in t and x, which span the principal algebra of (2). Noether Theorem 2.1 results in the following conserved vector components:

$$T_1^1=\int f(u)\mathrm{d}u-\frac{1}{2}u_{xx}^2-\frac{1}{2}u_t^2,\; T_1^2=u_{tx}u_{xx}-u_tu_{xxx}, \tag{33}$$

$$T_2^1=-u_tu_x,\; T_2^2=\frac{1}{2}u_{xx}^2+\frac{1}{2}u_t^2+\int f(u)\mathrm{d}u-u_xu_{xxx}. \tag{34}$$

One can easily verify that the conservation relation $D_iT^i=0$ is satisfied on the solution to the PDE (2). These conservation laws according to the Theorem 2.2 comprise a basis as $[X_1, X_2]=0$.

We now consider Case 2. Here there are subcases. For 2.1a, if $f(u)=A_0u$, $A_0>0$, we have that the nonzero commutation relations of the Noether algebra are

$$[X_1, X_3]=\sqrt{A_0}X_3,\; [X_1, X_4]=-\sqrt{A_0}X_4,\; [X_1, X_5]=\sqrt{A_0}X_5,$$
$$[X_1, X_6]=-\sqrt{A_0}X_6,\; [X_2, X_3]=X_5,\; [X_2, X_4]=X_6. \tag{35}$$

Thus, because of the last two commutation relations, according to Theorem 2.2, a basis of conservation laws is given by (33) and (34) with $f(u)=A_0u$, $A_0>0$ and

$$T_1^3=(xu_t-xu\sqrt{A_0})\exp(\sqrt{A_0}t),\; T_3^2=(xu_{xxx}-u_{xx})\exp(\sqrt{A_0}t), \tag{36}$$

$$T_1^4=(xu_t+xu\sqrt{A_0})\exp(-\sqrt{A_0}t),\; T_3^2=(xu_{xxx}-u_{xx})\exp(-\sqrt{A_0}t). \tag{37}$$

These are generated by the subalgebra $\{X_1, X_2, X_3, X_4\}$. We show how one can generate the conservation law associated with X_5. From (35), we have $[X_2, X_3]=X_5$. Thus, using (12) (note X_3 is prolonged to the derivatives),

$$T_5^1=X_3T_2^1,\; T_5^2=X_3T_2^2$$

which gives

$$T_5^1=(-\sqrt{A_0}xu_x-u_t)\exp(\sqrt{A_0}t),$$
$$T_5^2=(u_tx\sqrt{A_0}+A_0xu-u_{xxx})\exp(\sqrt{A_0}t), \tag{38}$$

In all cases, one can check that $D_iT^i=0$ holds on the solutions of the equation.

For the subcase 2.1b, when $f(u)=A_0u$, $A_0<0$, the nonzero commutation relations of the Noether algebra are

$$[X_1, X_3]=\sqrt{-A_0}X_4,\; [X_1, X_4]=-\sqrt{-A_0}X_3,\; [X_1, X_5]=\sqrt{-A_0}X_6,$$
$$[X_1, X_6]=-\sqrt{-A_0}X_5,\; [X_2, X_3]=X_5,\; [X_2, X_4]=X_6. \tag{39}$$

Thus, by Theorem 2.2, a basis of conservation laws is generated by the subalgebra $\{X_1, X_2, X_3, X_4\}$. A basis of conservation laws is given by (33) and (34) with $f=A_0u$, $A_0<0$ and

$$T_3^1=xu_t\sin(\sqrt{-A_0}t)-xu\sqrt{-A_0}\cos(\sqrt{-A_0}t),$$
$$T_3^2=xu_{xxx}\sin(\sqrt{-A_0}t)-u_{xx}\sin(\sqrt{-A_0}t), \tag{40}$$

$$T_1^4=xu_t\cos(\sqrt{-A_0}t)+xu\sqrt{-A_0}\sin(\sqrt{-A_0}t),$$
$$T_3^2=xu_{xxx}\cos(\sqrt{-A_0}t)-u_{xx}\cos(\sqrt{-A_0}t). \tag{41}$$

For the subcase 2.2.2, when $A_0=A_1=0$, which is the wave equation $u_{tt}+u_{xxxx}=0$, we have that the nonzero commutation relations of the Noether algebra are

$$[X_1, X_4]=X_3,\ [X_1, X_6]=X_5,\ [X_1, X_7]=2X_1,$$
$$[X_2, X_5]=X_3,\ [X_2, X_6]=X_4,\ [X_2, X_7]=X_2,\ [X_3, X_7]=\frac{1}{2}X_3,$$
$$[X_4, X_7]=-\frac{3}{2}X_4,\ [X_5, X_7]=-\frac{1}{2}X_5,\ [X_6, X_7]=-\frac{5}{2}X_6. \quad (42)$$

Therefore, by Theorem 2.2, a basis of conservation laws is spanned by the subalgebra $\{X_1, X_2, X_6, X_7\}$. A basis of conservation laws corresponding to X_1 and X_2 with $f=0$ are as in (33) and (34), and the other generators X_6 and X_7 yield

$$T_6^1=txu_t-xu,\ T_6^2=txu_{xxx}-tu_{xx}, \quad (43)$$

$$T_7^1=uu_t-2tu_{xx}^2-2tu_t^2-2xu_tu_x,$$
$$T_7^2=xu_{xx}^2+xu_t^2+uu_{xxx}-4tu_tu_{xxx}-2xu_xu_{xxx}+u_xu_{xx}+4tu_{tx}u_{xx}. \quad (44)$$

In Case 3.2, which corresponds to the power law of the load function in Section 3, we have the following. To simplify the computations, we take $A_2=42/(8^8)$. Thus, the function f becomes $f(u)=6(8A_1-u)^{-7}$. Thus, the PDE (1) becomes

$$\frac{\partial^4 u}{\partial x^4}+\frac{\partial^2 u}{\partial t^2}-6(8A_1-u)^{-7}=0. \quad (45)$$

For this case, the Noether symmetry algebra is spanned by the symmetry generators (23) and (32). Now invoking the Noether Theorem 2.1, we obtain the following conserved vectors for (45):

(i) $X_1=\partial/\partial t,\ B_1^1=0,\ B_1^2=0.$

$$T_1^1=-\frac{1}{2}u_t^2-\frac{1}{2}u_{xx}^2+(8A_1-u)^{-6},\ T_1^2=-u_tu_{xxx}+u_{tx}u_{xx}.$$

It can again be verified that the components of the conserved vector satisfy the equation $D_iT_1^i|_{(45)}=0.$

(ii) $X_2=\partial/\partial x,\ B_2^1=0,\ B_2^2=0.$

$$T_2^1=-u_tu_x,\ T_2^2=\frac{1}{2}u_t^2+\frac{1}{2}u_{xx}^2-u_xu_{xxx}+(8A_1-u)^{-6}.$$

One can show that the components of the conserved vector satisfy the equation $D_iT_2^i|_{(45)}=0.$

(iii) $X_3=2t\partial/\partial t+x\partial/\partial x+(u/2-4A_1)\partial/\partial u,\ B_3^1=0,\ B_3^2=0.$

$$T_3^1=-tu_t^2-tu_{xx}^2+\left(\frac{u}{2}-4A_1\right)u_t-xu_tu_x+2t(8A_1-u)^{-6},$$
$$T_3^2=\frac{1}{2}xu_t^2+\frac{1}{2}xu_{xx}^2+x(8A_1-u)^{-6}-2tu_tu_{xxx}-xu_xu_{xxx}$$
$$+\left(\frac{u}{2}-4A_1\right)u_{xxx}+\frac{1}{2}u_xu_{xx}+2tu_{xt}u_{xx}.$$

In the above-mentioned case, the nonzero commutation relations are $[X_1, X_3]=2X_1$ and $[X_2, X_3]=X_2$. Thus, according to Theorem 2.2, a basis of conservation laws is spanned by the Noether algebra itself and consists of the conserved vectors T_1, T_2, and T_3 as in items i to iii.

By using the Noether symmetries that have the conservation laws of Case 3.2 as associated symmetries, we look at a Noether reduction of our equation with power law u^{-7}. An easy translation and rescaling of (45) is (this puts the PDE in the form considered by Bokhari et al. [5])

$$\frac{\partial^4 u}{\partial x^4}+\frac{\partial^2 u}{\partial t^2}=\delta_1 u^{-7}, \quad (46)$$

where δ_1 is a constant. We present the initial conditions to be used, which are

$$u(x, 0)=g(x),\quad u_t(x, 0)=h(x), \quad (47)$$

where g and h are determined by the invariance of (47) under the Noether generators.

The boundary conditions are of four types, as listed and deduced by Han et al. [2]. They arise from (see (12) of Han et al. [2] and note that nondimensional quantities are used)

$$\frac{\partial^2 u}{\partial x^2}\delta\left(\frac{\partial u}{\partial x}\right)\Bigg|_0^1=0,\quad \frac{\partial^3 u}{\partial x^3}\delta u\Bigg|_0^1=0, \quad (48)$$

where δ is the variation so that $\delta u=0$ or $\delta\left(\frac{\partial u}{\partial x}\right)=0$ means that the displacement or the slope is zero. It is shown by Han et al. [2] that as a consequence of (48), four combinations of end conditions are possible. These correspond to the hinged end given by

$$u(0, t)=0,\quad u_{xx}(0, t)=0,$$

the clamped end by

$$u(0, t)=0,\quad u_x(0, t)=0,$$

the free end by

$$u_{xx}(0, t)=0,\quad u_{xxx}(0, t)=0,$$

and the sliding end by

$$u_x(0, t)=0,\quad u_{xxx}(0, t)=0.$$

For the general power law Case 3.2, a linear combination of the Noether operators X_i, viz, $Y=a_1X_1+a_2X_2+a_3X_3$, leaves invariant $t=0$ and $x=0$ provided $a_1=a_2=0$. This is the same as in the Lie case [5]. Thus, we have the scaling Noether generator $X_3=2t\partial/\partial t+x\partial/\partial x+\frac{1}{2}u\partial/\partial u.$ The invariant solution corresponding to this operator is

$$u = x^{\frac{1}{2}} F\left(\frac{x}{\sqrt{t}}\right). \tag{49}$$

The substitution of (49) into the PDE for Case 3.2, viz, (46), results in the variable coefficient nonlinear fourth-order ODE:

$$\alpha^4 F^{(4)} + 2\alpha^3 F''' - \frac{3}{2}\alpha^2 F'' + \frac{1}{4}\alpha^6 F'' + \frac{3}{2}\alpha F' + \frac{3}{4} F'\alpha^5 - \frac{15}{16} F = \delta_1 F^{-7}, \tag{50}$$

where $\alpha = \frac{x}{\sqrt{t}}$ is the similarity variable, and here prime implies differentiation with respect to the variable α. This is exactly the same as in Bokhari et al. [5] obtained by setting $\sigma = -7$, which gives the Noether condition for conservation laws for this case. However, here the special value −7 is picked out.

Now for the PDE (46), the conserved components adjust as follows:

$$\begin{aligned} T_3^1 &= -tu_t^2 - tu_{xx}^2 + \frac{u}{2}u_t - xu_t u_x - \frac{1}{3}t\delta_1 u^{-6}, \\ T_3^2 &= \frac{1}{2}xu_t^2 + \frac{1}{2}xu_{xx}^2 - \frac{1}{6}\delta_1 xu^{-6} - 2tu_t u_{xxx} - xu_x u_{xxx} + \frac{u}{2}u_{xxx} \\ &\quad + \frac{1}{2}u_x u_{xx} + 2tu_{xt}u_{xx}. \end{aligned} \tag{51}$$

The symmetry X_3 is associated with these conserved components (51). This is obtained by (11). We have

$$\begin{aligned} X_3^{[2]} T_3^1 + T_3^1 &= 0, \\ X_3^{[3]} T_3^2 + 2T_3^2 &= 0. \end{aligned} \tag{52}$$

The symmetry X_3 can then be used to effect reduction (see [12], and Section 2) of the conserved vector (T_3^1, T_3^2) in (51). The invariant corresponding to X_3, viz, $u = x^{1/2}F(\alpha)$, after substitution in the conserved vector, using the method of [12], results in the first integral

$$\begin{aligned} &\frac{1}{4}F'^2 - \frac{1}{16}F^2\alpha^{-2} - F''^2\alpha^2 + \frac{1}{2}FF'\alpha^{-1} + \frac{1}{2}FF'' \\ &-2FF''\alpha - F'^2 - \frac{1}{3}\delta_1 F^{-6}\alpha^{-6} = C, \end{aligned} \tag{53}$$

where C is an arbitrary constant. This first integral plays a vital physical role in the numerical solution in the next section.

The initial and boundary conditions on (49) for a clamped end beam give rise to the ODE (50) to be solved subject to the boundary conditions

$$F(\infty) = F_0,\ F'(\infty) = 0,\ F(0) = 0,\ F'(0) = 0, \tag{54}$$

where $g(x) = F_0 x^{\frac{1}{2}}$ [as well as preserving the integral (53)].

For the hinged end beam, the ODE (50) should be solved subject to the boundary conditions (54) as well as in addition $F''(0) = 0$. In the case of the free and sliding end beam, one has to add also $F'''(0) = 0$.

We focus on the numerical solution of the PDE (46) using the similarity reduction (49) with F satisfying the nonlinear ODE (50) with boundary conditions (54) (the clamped end beam). Moreover, we have that the conservation law needs to be satisfied for this case. This is in the form of the reduced first integral (53), which we obtained as a consequence of the conservation law that has this scaling symmetry.

5 Numerical Solutions

Given the structure of (50) (subject to the first integral derived) in the previous section, we will attempt to use numerical techniques as a means of obtaining solutions for the varied boundary conditions. We use an in-built function in MATLAB called `bvp4c`, which is a boundary value solver for differential equations. It is a finite difference code that implements the three-stage Lobatto IIIa formula, a collocation formula, such that the collocation polynomial provides a C^1-continuous solution that is fourth-order accurate uniformly on the relevant interval. The initial guess provided for all calculations is $F = 1$, $F' = 0$, $F'' = 0$, and $F''' = 0$.

To implement this in-built function, we structure (50) as a system where we have defined $F(\alpha) = F_1(\alpha)$:

$$\begin{aligned} F_1' &= F_2, \\ F_2' &= F_3, \\ F_3' &= F_4, \\ F_4' &= -2\alpha^3 F_4 + \left(\frac{3}{2}\alpha^2 - \frac{1}{4}\alpha^6\right)F_3 - \left(\frac{3}{2}\alpha + \frac{3}{4}\alpha^5\right)F_2 + \frac{15}{16}F_1 + \delta_1 F_1^{-7} \end{aligned} \tag{55}$$

where we have defined $F' = F_2$, $F'' = F_3$, and $F''' = F_4$, giving $F^{(4)} = F_4'$. When considering the initial and boundary conditions for a clamped end beam, we consider

$$\begin{aligned} &F_1(0) = 0, \quad F_2(0) = 0, \\ &F_1(\infty) = F_0, \quad F_2(\infty) = 0. \end{aligned} \tag{56}$$

The solutions obtained in the figures provided indicate that there is decay for F whereas the deflection seems to increase for u; this can be observed as the interval upon which we consider α is enlarged (see Figure 1). Upon obtaining a solution, as shown in the figures below, we find that the numerical method has difficulty in enforcing the boundary conditions to a high level of accuracy. Given this, it becomes important to have another mechanism for verifying the solution obtained. The manner in which we attempt to do so is via the consideration of the first integral obtained as per (53), which is structured as follows:

$$FI(\alpha)=C,\quad \text{with } C \text{ a constant and}$$
$$FI(\alpha)=\frac{1}{4}(F')^2\alpha^4-\frac{1}{16}F^2\alpha^{-2}-(F'')^2\alpha^2$$
$$+\frac{1}{2}FF'\alpha^{-1}+\frac{1}{2}FF''-2FF''\alpha-(F')^2-\frac{1}{3}\delta_1F^{-6}\alpha^{-2}. \tag{57}$$

All that is required is the numerical verification that in fact $FI(\alpha)$ remains constant at some C across all values of α.

Upon incorporating this requirement, we find that in fact our solution has difficulty in preserving the first integral (57), specifically at the boundaries (see Figure 2). It is worth noting that we are unable to verify (57) at the boundary where $\alpha=0$ because this leads to $F(0)=0$, leading to a singularity in the first integral. This inability to validate $FI(\alpha)=C$ for all values of α immediately leads us to question the validity of the numerical solution obtained at the boundary. As such, we investigate the influence of the initial and boundary value choices on the behavior of the solution.

In doing so, we first discover that the initial guess for F needs to be nonzero; if chosen to be zero, the function is unable to solve the collocation equations because of a singular Jacobian. Furthermore, we find that the guess for F''' does not elicit the same sensitive behavior that the other initial guesses do. The solutions are seen to change depending on the initial guesses chosen for F, F', and F''. Similar solutions required the same choices for F, F', and F''; however, the choice of F''' could vary. This leads one to conclude that the choice for the former heavily influences the in-built function's ability to obtain a solution and the nature of the said solution.

Further investigation indicates that the solution exhibits unexpected behavior at the boundaries. The required accuracy with regard to the specified boundary conditions is not only inconsistent but also results in exaggerated numerical values. For instance, $F''(0)$ has been obtained numerically as $3.6542e+04$, which contradicts the requirement that it should be zero. This confirms that the numerical method used is not able to adhere to the stipulated boundary conditions. This, in and of itself, could explain why the first integral is not consistently preserved.

Given the discussion in the previous paragraph, we surmise that when solving the hinged end beam problem, requiring $F''(0)=0$ in addition to the boundary conditions given by (51), maintaining the required boundary conditions would prove difficult. In the case of the free and sliding end beam, one has to include the condition $F'''(0)=0$. It so happens that this boundary condition does not elicit the same sensitive behavior that the other initial guesses do; however, the degree to which one could numerically implement the condition is not certain.

6 Concluding Remarks

We have performed the Noether symmetry classification of the fourth-order dynamic Euler-Bernoulli beam (1) and determined the possible functional forms of the applied load function $f(u)$ by using the determining equation for the Noether symmetries of the PDE (1). We found that for the arbitrary values of the load function $f(u)$, the principle Noether symmetry algebra is spanned by two Noether symmetries, viz, translations in time and space. We have shown that this principle Noether algebra extends in two cases, viz, the linear and the power law cases. For each of these extended cases, we obtained the Noether symmetries as well. We made a comparison of the results obtained in this work with that of the Lie symmetry group classification of the PDE (1) derived in Bokhari et al. [5]. For each of the cases, including where $f(u)$ corresponds to the power law, for each of the Noether symmetries associated with the PDE (1), we constructed conservation laws for the equation via the Noether theorem. Moreover, we obtained a basis of conservation laws generated by the adjoint action of a subalgebra of the Noether algebra. The Noether classification points to the special case of the exponent of the power law, viz, –7, which we focus on for symmetry reduction and numerical solutions. The reduction and physical solutions are dictated by the conservation law, which corresponds to the scaling symmetry and which results in the first integral that is approximately preserved by the numerical approach inside the boundary point. The numerical solutions are graphically presented in Figures 1–5. These are achieved by using an in-built function in MATLAB called `bvp4c`, which is a boundary value solver for differential equations. In Figure 1, we see that $F(\alpha)$ is approximately 0 except at the boundary point. In Figure 2, we observe that the deflection u has spikes for different δ_1 values in the range 1–2.5 for fixed $F_0=1.5$ and time $T=8$ when the length $x<30$. For the length $x\geq 30$, the deflection decays to a value above zero. Furthermore, in Figure 3, it is noticed that the deflection u of the beam is unstable for values of the length $x<100$ for $\delta_1\leq 0.5$. Thus, the deflection appears sensitive for small δ_1. The deflection u of the beam is also sensitive to changes in the initial condition value F_0 as we see in Figure 4. In Figure 5, we observe that for time $T \leq 2$, the deflection u is quite pronounced for length $x < 110$.

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