Abdelmalek Boumali* and Hassan Hassanabadi

Exact Solutions of the (2+1)-Dimensional Dirac Oscillator under a Magnetic Field in the Presence of a Minimal Length in the Non-commutative Phase Space

DOI 10.1515/zna-2015-0140
Received March 26, 2015; accepted May 12, 2015; previously published online June 11, 2015

Abstract: We consider a two-dimensional Dirac oscillator in the presence of a magnetic field in non-commutative phase space in the framework of relativistic quantum mechanics with minimal length. The problem in question is identified with a Poschl–Teller potential. The eigenvalues are found, and the corresponding wave functions are calculated in terms of hypergeometric functions.

Keywords: Dirac Oscillator; Minimal Length; Non-commutative Phase Space.

PACS Number: 03.65.Ge.

1 Introduction

The Dirac relativistic oscillator is an important potential for both theory and application. For the first time, it was studied by Ito et al. [1]. They considered a Dirac equation in which the momentum $\vec{p}$ is replaced by $\vec{p}-\mathrm{i}m\beta\omega\vec{r}$, with $\vec{r}$ being the position vector, m the mass of the particle, and ω the frequency of the oscillator. The interest in the problem was revived by Moshinsky and Szczepaniak [2], who gave it the name of Dirac oscillator (DO) because, in the non-relativistic limit, it becomes a harmonic oscillator with a very strong spin–orbit coupling term. Physically, it can be shown that the DO interaction is a physical system that can be interpreted as the interaction of the anomalous magnetic moment with a linear electric field [3, 4]. The electromagnetic potential associated with the DO has been found by Benitez et al. [5]. The DO has attracted a lot of interest because not only it provides one of the examples of the Dirac equation exact solvability but also for its numerous physical applications [6–10]. Recently, Franco-Villafane et al. [11] exposed the proposal of the first experimental microwave realisation of the one-dimensional DO. The experiment relies on a relation of the DO to a corresponding tight-binding system. The experimental results obtained, concerning the spectrum of the one-dimensional DO with and without the mass term, are in good agreement with those obtained in the theory. In addition, Quimbay and Strange [12, 13] showed that the DO can describe a naturally occurring physical system. Specifically, the case of a two-dimensional DO can be used to describe the dynamics of the charge carriers in graphene, and hence its electronic properties. Also, the exact mapping of the DO in the presence of a magnetic field with a quantum optics leads to regarding the DO as a theory of an open quantum systems coupled to a thermal bath [6].

The unification between the general theory of relativity and the quantum mechanics is one of the most important problems in theoretical physics. This unification predicts the existence of a minimal measurable length on the order of the Planck length. All approaches of quantum gravity show the idea that near the Planck scale, the standard Heisenberg uncertainty principle should be reformulated. The minimal length uncertainty relation appears in the context of the string theory, as a consequence of the fact that the string cannot probe distances smaller than the string scale $\hbar\sqrt{\beta}$ where β is a small positive parameter called the deformation parameter. This minimal length can be introduced as an additional uncertainty in position measurement, so that the usual canonical commutation relation between position and momentum operators becomes $[\hat{x},\hat{p}]=i\hbar(1+\beta p^2)$. This commutation relation leads to the standard Heisenberg uncertainty relation $\Delta\hat{x}\Delta\hat{p}\geq i\hbar(1+\beta(\Delta p)^2)$, which clearly implies the existence of a non-zero minimal length $\Delta x_{\min}=\hbar\sqrt{\beta}$. This

***Corresponding author: Abdelmalek Boumali,** Laboratoire de Physique Appliquée et Théorique,Université de Larbi-Tébessi, 12000, W. Tébessa, Algeria, E-mail: boumali.abdelmalek@gmail.com; boumali.abdelmalek@univ-tebessa.dz
Hassan Hassanabadi: Physics Department, University of Shahrood, Shahrood, Iran, E-mail: h.hasanabadi@shahroodu.ac.ir

modification of the uncertainty relation is usually termed the generalised uncertainty principle (GUP) or the minimal length uncertainty principle [14–17]. Investigating the influence of the minimal length assumption on the energy spectrum of quantum systems has become an interesting issue primarily for two reasons. First, this may help to set some upper bounds on the value of the minimal length. In this context, we can cite some studies of the hydrogen atom and a two-dimensional Dirac equation in an external magnetic field. Moreover, the classical limit has also provided some interesting insights into some cosmological problems. Second, it has been argued that quantum mechanics with a minimal length may also be useful to describe non-point-like particles, such as quasi-particles and various collective excitations in solids, or composite particles (see [18] and references therein).

Nowadays, the reconsideration of the relativistic quantum mechanics in the presence of a minimal measurable length has been studied extensively. In this context, many papers were published where a different quantum system in space with the Heisenberg algebra was studied. They are the Abelian Higgs model [19], the thermostatics with minimal length [20], the one-dimensional hydrogen atom [21], the casimir effect in minimal length theories [22], the effect of minimal lengths on electron magnetism [23], the DO in one and three dimensions [24–28], the non-commutative (NC) (2+1)-dimensional DO and quantum phase transition [10], the solutions of a two-dimensional Dirac equation in the presence of an external magnetic field [29], the NC phase space Schrödinger equation [30], and the Schrödinger equation with harmonic potential in the presence of a magnetic field [31].

The study of NC spaces and their implications in physics is an extremely active area of research. It has been argued in various instances that non-commutativity should be considered as a fundamental feature of space time at the Planck scale. On the other side, the study of quantum systems in an NC space has been the subject of much interest in past years, assuming that non-commutativity may be, in fact, a result of quantum gravity effects. In these studies, some attention has been paid to the models of NC quantum mechanics (NCQM). The interest in this approach lies on the fact that NCQM is a fruitful theoretical laboratory where we can get some insight on the consequences of non-commutativity in field theory by using standard calculation techniques of quantum mechanics. Various NC field theory models have been discussed as well as many extensions of quantum mechanics. Of particular interest is the so-called phase space non-commutativity, which has been investigated in the context of quantum cosmology, black holes physics, and the singularity problem. This specific formulation is necessary to implement the Bose–Einstein statistics in the context of NCQM (see [32–36]).

The purpose of this work is to investigate the formulation of a two-dimensional DO in the presence of a magnetic field by solving fundamental equations in the framework of relativistic quantum mechanics with minimal length in the NC phase space. To do this, we first mapped the problem in question into a commutative space by using an appropriate transformations. Then, we solved it in the presence of a minimal length. We would like to mention here that the origin of relativistic Landau problem and the DO is entirely different. In the former case, the magnetic field is introduced via minimal coupling, whereas in the latter case, the interaction is introduced via non-minimal coupling and can be viewed as anomalous magnetic interaction [37, 38].

The article is organised as follows. In Section 2, we solve the DO in the presence of magnetic field in NC phase space. Then, in Section 3, we study this problem in the framework of relativistic quantum mechanics with minimal length. Finally, in Section 4, we present the conclusion.

2 The Solutions in Non-commutative Phase Space

To begin with, we note that the NC phase space is characterised by the fact that their coordinate operators satisfy the following equation [32–36]:

$$\begin{aligned}&[x_{\nu}^{(\mathrm{NC})}, x_{\mu}^{(\mathrm{NC})}] = i\tilde{\theta}_{\mu\nu},\ [p_{\mu}^{(\mathrm{NC})}, p_{\nu}^{(\mathrm{NC})}] = i\bar{\theta}_{\mu\nu},\\ &[x_{\mu}^{(\mathrm{NC})}, p_{\nu}^{(\mathrm{NC})}] = i\hbar\delta_{\mu\nu},\end{aligned} \tag{1}$$

where $\tilde{\theta}_{\mu\nu}$ and $i\bar{\theta}_{\mu\nu}$ are antisymmetric tensors of space dimension. In order to obtain a theory that includes the aspects of being unitary and causal, we choose $\tilde{\theta}_{0\nu}=0$ and $\bar{\theta}_{0\nu}=0$, which implies that the time remains as a parameter and the non-commutativity affects only the physical space. By replacing the normal product with star product, the Dirac equation in commuting space will change into the Dirac equation in NC space:

$$\hat{H}_D(p, x)\star\psi_D(x) = E\psi_D(x), \tag{2}$$

where the $\star$–product Moyal between two functions is defined by

$$(f\star g)(x) = \exp\left[\frac{i}{2}\tilde{\theta}_{ab}\partial_{x_a}\partial_{x_b}\right] f(x)g(y)|_{x=y}. \tag{3}$$

Since the system in which we study is two dimensional, we limit our analysis to the *xy* plane, where the NC algebra is written by

$$[x_i^{(\mathrm{NC})}, x_j^{(\mathrm{NC})}] = i\tilde{\theta}\epsilon_{ij},\ [p_i^{(\mathrm{NC})}, p_j^{(\mathrm{NC})}] = i\bar{\theta}\epsilon_{ij},\\ [x_i^{(\mathrm{NC})}, p_j^{(\mathrm{NC})}] = i\hbar\delta_{ij},\ (i, j = 1,\ 2). \tag{4}$$

with ϵ_{ij} is two-dimensional Levi-civita tensor.

Instead of solving the NC Dirac equation by using the star product procedure, we use Bopp's shift method, that is, we replace the star product by the usual product by making a Bopp's shift

$$x_i^{(\mathrm{NC})} = x_i - \frac{1}{2\hbar}\tilde{\theta}\epsilon_{ij}p_j,\ p_i^{(\mathrm{NC})} = p_i + \frac{1}{2\hbar}\bar{\theta}\epsilon_{ij}x_j. \tag{5}$$

Hence, in the two-dimensional NC phase space, (5) becomes

$$x^{(\mathrm{NC})} = x - \frac{\tilde{\theta}}{2\hbar}p_y,\ y^{(\mathrm{NC})} = y + \frac{\tilde{\theta}}{2\hbar}p_x,\ p_x^{(\mathrm{NC})} = p_x + \frac{\bar{\theta}}{2\hbar}y,\\ p_y^{(\mathrm{NC})} = p_y - \frac{\bar{\theta}}{2\hbar}x. \tag{6}$$

In this case, the two-dimensional DO equation, in commutative space, which is written by

$$\{c\alpha_x(p_x - \mathrm{i}m_0\omega\tilde{\beta}x) + c\alpha_y(p_y - \mathrm{i}m_0\omega\tilde{\beta}y) + \tilde{\beta}m_0c^2\}\psi_D = E\psi_D,$$

is modified and transformed into

$$\{c\alpha_x(p_x^{(NC)} - \mathrm{i}m_0\omega\tilde{\beta}x^{(NC)}) + c\alpha_y(p_y^{(NC)} - \mathrm{i}m_0\omega\tilde{\beta}y^{(NC)})\\ + \tilde{\beta}m_0c^2\}\psi_D = E_{NC}\psi_D. \tag{7}$$

Using the following representation of Dirac matrices,

$$\alpha_x = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \alpha_y = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \tilde{\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{8}$$

and with $\psi_D = (\psi_1, \psi_2)^T$, (7) becomes

$$\begin{pmatrix} m_0c^2 & cp_- \\ cp_+ & -m_0c^2 \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = E_{\mathrm{NC}}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{9}$$

or

$$m_0c^2\psi_1 + cp_-\psi_2 = E_{\mathrm{NC}}\psi_1, \tag{10}$$

$$cp_+\psi_1 - m_0c^2\psi_2 = E_{\mathrm{NC}}\psi_2, \tag{11}$$

where

$$p_- = p_x^{(\mathrm{NC})} - ip_y^{(\mathrm{NC})} + \mathrm{i}m_0\omega(x^{(\mathrm{NC})} - iy^{(\mathrm{NC})}) = \varrho_1(p_x - ip_y)\\ + \mathrm{i}m_0\omega\varrho_2(x - iy), \tag{12}$$

$$p_+ = p_x^{(\mathrm{NC})} + ip_y^{(\mathrm{NC})} - \mathrm{i}m_0\omega(x^{(\mathrm{NC})} + iy^{(\mathrm{NC})}) = \varrho_1(p_x + ip_y)\\ - \mathrm{i}m_0\omega\varrho_2(x + iy), \tag{13}$$

and where

$$\varrho_1 = 1 + \frac{m_0\omega}{2\hbar}\tilde{\theta},\ \varrho_2 = 1 + \frac{\bar{\theta}}{2m_0\omega\hbar}. \tag{14}$$

From (10) and (11), we have

$$\{c^2p_-p_+ - (E_{\mathrm{NC}}^2 - m_0^2c^4)\}\psi_1 = 0. \tag{15}$$

Now, in order to solve the last equation, and for the sake of simplicity, we bring the problem into the momentum space.

Recalling that

$$x = i\hbar\frac{\partial}{\partial p_x},\ y = i\hbar\frac{\partial}{\partial p_x},\ \hat{p}_x = p_x,\ \hat{p}_y = p_y, \tag{16}$$

and passing onto polar coordinates with the following definition [29]:

$$p_x = p\cos\theta,\ \ p_y = p\sin\theta,\ p^2 = p_x^2 + p_y^2, \tag{17}$$

$$\hat{x} = i\hbar\frac{\partial}{\partial p_x} = i\hbar\left(\cos\theta\frac{\partial}{\partial p} - \frac{\sin\theta}{p}\frac{\partial}{\partial\theta}\right), \tag{18}$$

$$\hat{y} = i\hbar\frac{\partial}{\partial p_y} = i\hbar\left(\sin\theta\frac{\partial}{\partial p} + \frac{\cos\theta}{p}\frac{\partial}{\partial\theta}\right), \tag{19}$$

Equations (12) and (13) transform into

$$p_- = e^{-i\theta}\left\{\varrho_1 p - \lambda\left(\frac{\partial}{\partial p} - \frac{i}{p}\frac{\partial}{\partial\theta}\right)\right\}, \tag{20}$$

$$p_+ = e^{i\theta}\left\{\varrho_1 p + \lambda\left(\frac{\partial}{\partial p} + \frac{i}{p}\frac{\partial}{\partial\theta}\right)\right\}, \tag{21}$$

where

$$\lambda = \left(1 + \frac{\bar{\theta}}{2m_0\omega\hbar}\right)m_0\hbar\omega. \tag{22}$$

With the aid of these expressions, the p_-p_+ term, appears in (15), can be written by

$$p_-p_+ = \varrho_1^2p^2 - 2\varrho_1\lambda - \lambda^2\frac{\partial^2}{\partial p^2} - \frac{\lambda^2}{p^2}\frac{\partial^2}{\partial\theta^2} - \frac{\lambda^2}{p}\frac{\partial}{\partial p} + 2i\lambda\varrho_1\frac{\partial}{\partial\theta}. \tag{23}$$

So, (15) becomes

$$\left\{\varrho_1^2p^2 - \lambda^2\left(\frac{\partial^2}{\partial p^2} + \frac{1}{p}\frac{\partial}{\partial p} + \frac{1}{p^2}\frac{\partial^2}{\partial\theta^2}\right) + 2i\lambda\varrho_1\frac{\partial}{\partial\theta} - 2\lambda\varrho_1 - \zeta\right\}\psi_1 = 0, \tag{24}$$

with

$$\zeta = \frac{E_{\mathrm{NC}}^2 - m_0^2c^4}{c^2}. \tag{25}$$

With the help of the following relation [39]

$$\psi_1(p,\theta)=f(p)e^{\mathrm{i}m\theta}, \tag{26}$$

Equation (24) is modified and transforms into

$$\left(\frac{\mathrm{d}^2 f(p)}{\mathrm{d}p^2}+\frac{1}{p}\frac{\mathrm{d}f(p)}{\mathrm{d}p}-\frac{m^2}{p^2}f(p)\right)+(\kappa^2-k^2p^2)f(p)=0, \tag{27}$$

with

$$\kappa^2=\frac{2\lambda\varrho_1(m+1)+\zeta}{\lambda^2},\ k^2=\frac{\varrho_1^2}{\lambda^2}. \tag{28}$$

Putting that

$$f(p)=p^m e^{-\frac{k^2}{2}p^2}F(p), \tag{29}$$

then, the differential equation

$$F''+\left(\frac{2m+1}{p}-2kp\right)F'-[2k(m+1)-\kappa^2]F=0, \tag{30}$$

is obtained for $F(p)$ which by using, instead of p, the variable $xt=kp^2$, is transformed into the Kummer equation:

$$t\frac{\mathrm{d}^2F}{\mathrm{d}t^2}+\{m+1-t\}\frac{\mathrm{d}F}{\mathrm{d}t}-\frac{1}{2}\left\{m+1-\frac{\kappa^2}{4k}\right\}F=0, \tag{31}$$

whose solution is the confluent series ${}_1F_1(a;\, m+1;\, t)$, with

$$a=\frac{1}{2}(m+1)-\frac{\kappa^2}{4k}. \tag{32}$$

The confluent series becomes a polynomial if and only if $a=-n,(n=0,\,1,\,2,)$. Thus, we have [40]

$$\psi_1(p,\theta)=C_{n,m}p^m e^{-\frac{k}{2}p^2}{}_1F_1(-n;|m|+1;\,kp^2)e^{\mathrm{i}m\theta}, \tag{33}$$

$$(E_{\mathrm{NC}})_n=\pm m_0c^2\sqrt{1+4\left(1+\frac{m_0\omega}{2\hbar}\tilde{\theta}\right)\left(1+\frac{\overline{\theta}}{2m_0\omega\hbar}\right)n}. \tag{34}$$

The total associated wave function is

$$\psi_{n,m}(p,\theta)=\begin{pmatrix}1\\ \dfrac{cp_+}{E_{\mathrm{NC}}+m_0c^2}\end{pmatrix}\psi_1. \tag{35}$$

Now, in the presence of an external magnetic field, (7) is transformed into

$$\left\{c\alpha_x\left[\left(p_x^{(\mathrm{NC})}+\frac{eBy^{(\mathrm{NC})}}{2c}\right)-\mathrm{i}m_0\omega\beta x^{(\mathrm{NC})}\right]+c\alpha_y\left[\left(p_y^{(\mathrm{NC})}-\frac{eBx^{(\mathrm{NC})}}{2c}\right)-\mathrm{i}m_0\omega\beta y^{(\mathrm{NC})}\right]+\beta m_0c^2\right\}\psi_D=\epsilon\psi_D, \tag{36}$$

where ϵ is the eigenvalue of the system. Here, the potential vectors is chosen as

$$\vec{A}=\left(-\frac{By^{(\mathrm{NC})}}{2},\ \frac{Bx^{(\mathrm{NC})}}{2},\ 0\right), \tag{37}$$

and (36) can be cast into a detail form as follows:

$$\begin{pmatrix}m_0c^2 & c\tilde{p}_-\\ c\tilde{p}_+ & -m_0c^2\end{pmatrix}\begin{pmatrix}\tilde{\psi}_1\\ \tilde{\psi}_2\end{pmatrix}=\epsilon\begin{pmatrix}\tilde{\psi}_1\\ \tilde{\psi}_2\end{pmatrix}. \tag{38}$$

with

$$\tilde{p}_-=\left(p_x^{(\mathrm{NC})}+\frac{eBy^{(\mathrm{NC})}}{2c}\right)+\mathrm{i}m_0\omega x^{(\mathrm{NC})}-i\left(p_y^{(\mathrm{NC})}-\frac{eBx^{(\mathrm{NC})}}{2c}\right)+m_0\omega y^{(\mathrm{NC})}, \tag{39}$$

$$\tilde{p}_+=\left(p_x^{(\mathrm{NC})}+\frac{eBy^{(\mathrm{NC})}}{2c}\right)-\mathrm{i}m_0\omega x^{(\mathrm{NC})}+i\left(p_y^{(\mathrm{NC})}-\frac{eBx^{(\mathrm{NC})}}{2c}\right)+m_0\omega y^{(\mathrm{NC})} \tag{40}$$

or

$$\tilde{p}_-=p_x^{(\mathrm{NC})}-ip_y^{(\mathrm{NC})}+\mathrm{i}m_0\tilde{\omega}(x^{(\mathrm{NC})}-iy^{(\mathrm{NC})}), \tag{41}$$

$$\tilde{p}_+=p_x^{(\mathrm{NC})}+ip_y^{(\mathrm{NC})}-\mathrm{i}m_0\tilde{\omega}(x^{(\mathrm{NC})}+iy^{(\mathrm{NC})}), \tag{42}$$

where

$$\tilde{\omega}=\omega-\frac{\omega_c}{2},\ \omega_c=\frac{|e|B}{m_0c}. \tag{43}$$

is a cyclotron frequency.

Thus, the (2+1)-dimensional DO in a magnetic field is mapped onto the one with reduced angular frequency $\tilde{\omega}$ in the absence of magnetic field. Hence, the only role of a magnetic field consists in reducing the angular frequency, and the entire dynamics remains unchanged.

Using the mapping defined by (6), the systems of equations become

$$\tilde{p}_-=\varrho_1(p_x-ip_y)+\mathrm{i}m_0\tilde{\omega}\varrho_2(x-iy), \tag{44}$$

$$\tilde{p}_+=\varrho_1(p_x+ip_y)-\mathrm{i}m_0\tilde{\omega}\varrho_2(x+iy). \tag{45}$$

By the same way used above, we obtain

$$\tilde{\psi}_1(p,\theta)=\tilde{C}_{n,m}p^m e^{-\frac{k}{2}p^2}{}_1F_1(-n;\ |m|+1;\ kp^2)e^{\mathrm{i}m\theta}, \tag{46}$$

$$\epsilon_n=\pm m_0c^2\sqrt{1+4\left(1+\frac{m_0\tilde{\omega}}{2\hbar}\tilde{\theta}\right)\left(1+\frac{\overline{\theta}}{2m_0\tilde{\omega}\hbar}\right)n}. \tag{47}$$

The last equation concerning the eigenvalue is in a good agreement with the one obtained in the literature [6].

The corresponding total eigenfunction is given by

$$\psi_{n,m}(p,\theta)=\begin{pmatrix}1\\ \dfrac{c\tilde{p}_{+}}{\epsilon+m_0c^2}\end{pmatrix}\tilde{\psi}_1. \quad (48)$$

In this section, we have studied the solutions of the two-dimensional DO with or without an external magnetic field by using the same way described in [29]: the authors work within a momentum space representation of the Heisenberg algebra, and by an appropriate transformation, the problem is identified as a Kummer differential equation where the solutions are well known. The solutions that we have found are in wellagreement with those obtained in the literature. This agreement allows us to extend this method by introducing the concept of the minimal length.

3 The Problem with a Minimal Length

3.1 The Solutions without a Magnetic Field

In the minimal length formalism, the Heisenberg algebra is given by

$$[\hat{x}_i,\hat{p}_i]=i\hbar\delta_{ij}(1+\beta p^2), \quad (49)$$

where $\beta>0$ is the minimal length parameter. A representation of $\hat{x}_i$ and $\hat{p}_i$, which satisfies (49), may be taken as

$$\hat{x}_i=i\hbar(1+\beta p^2)\frac{\mathrm{d}}{\mathrm{d}p_i},\ \hat{p}_i=p_i \quad (50)$$

or

$$\hat{x}=i\hbar(1+\beta p^2)\frac{\mathrm{d}}{\mathrm{d}p_x},\ \hat{y}=i\hbar(1+\beta p^2)\frac{\mathrm{d}}{\mathrm{d}p_y}, \quad (51)$$

$$\hat{p}_x=p_x,\ \hat{p}_y=p_y. \quad (52)$$

In this case, (15) is modified and becomes

$$\{c^2P_-P_+-(\varepsilon^2-m_0^2c^4)\}\psi_1=0, \quad (53)$$

with

$$P_-=\varrho_1(p_x-ip_y)-\lambda(1+\beta p^2)\left(\frac{\partial}{\partial p_x}-i\frac{\partial}{\partial p_y}\right), \quad (54)$$

$$P_+=\varrho_1(p_x-ip_y)+\lambda(1+\beta p^2)\left(\frac{\partial}{\partial p_x}+i\frac{\partial}{\partial p_y}\right). \quad (55)$$

In the polar coordinates, (54) and (55) can be written as

$$P_-=e^{-i\theta}\left\{\varrho_1 p-\lambda(1+\beta p^2)\left(\frac{\partial}{\partial p}-\frac{i}{p}\frac{\partial}{\partial\theta}\right)\right\}, \quad (56)$$

$$P_+=e^{i\theta}\left\{\varrho_2 p+\lambda(1+\beta p^2)\left(\frac{\partial}{\partial p}+\frac{i}{p}\frac{\partial}{\partial\theta}\right)\right\}. \quad (57)$$

When we evaluate the P_-P_+ term, we get

$$\begin{aligned}P_-P_+&=\varrho_1^2p^2+2(1+\beta p^2)\left\{\lambda\varrho_1\left(i\frac{\partial}{\partial\theta}-1\right)-\beta\lambda^2\left(p\frac{\partial}{\partial p}+i\frac{\partial}{\partial\theta}\right)\right\}\\&\quad-\lambda^2(1+\beta p^2)^2\left(\frac{\partial^2}{\partial p^2}+\frac{1}{p}\frac{\partial}{\partial p}+\frac{1}{p^2}\frac{\partial^2}{\partial\theta^2}\right),\end{aligned} \quad (58)$$

and then we have

$$\left[\varrho_1^2p^2+2(1+\beta p^2)\left\{\lambda\varrho_1\left(i\frac{\partial}{\partial\theta}-1\right)-\beta\lambda^2\left(p\frac{\partial}{\partial p}+i\frac{\partial}{\partial\theta}\right)\right\}-\lambda^2(1+\beta p^2)^2\left(\frac{\partial^2}{\partial p^2}+\frac{1}{p}\frac{\partial}{\partial p}+\frac{1}{p^2}\frac{\partial^2}{\partial\theta^2}\right)-\xi^2\right]\psi_1=0, \quad (59)$$

with

$$\xi^2=\frac{\varepsilon^2-m_0^2c^4}{c^2}. \quad (60)$$

Putting that

$$\psi_1=h(p)e^{\mathrm{i}m\theta}, \quad (61)$$

Equation (60) reads

$$\left[\varrho_1^2p^2-2(1+\beta p^2)\left\{\lambda\varrho_1(m+1)+\beta\lambda^2\left(p\frac{\mathrm{d}}{\mathrm{d}p}-m\right)\right\}-\lambda^2(1+\beta p^2)^2\left(\frac{\mathrm{d}^2}{\mathrm{d}p^2}+\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}p}-\frac{m^2}{p^2}\right)-\xi^2\right]h(p)=0. \quad (62)$$

This equation can be written in another form as follows:

$$\left\{-a(p)\frac{\mathrm{d}^2}{\mathrm{d}p^2}+b(p)\frac{\mathrm{d}}{\mathrm{d}p}+c(p)-\xi^2\right\}h(p)=0, \quad (63)$$

where

$$a(p)=\lambda^2(1+\beta p^2)^2, \quad (64)$$

$$b(p)=-2\beta\lambda^2(1+\beta p^2)p-\frac{\lambda^2(1+\beta p^2)^2}{p}, \quad (65)$$

$$c(p)=\varrho_1^2p^2-2\lambda\varrho_1(m+1)(1+\beta p^2)+2\beta\lambda^2m(1+\beta p^2)+\frac{\lambda^2(1+\beta p^2)^2m^2}{p^2}. \quad (66)$$

The solutions of (63) can be found by using the following transformations [41]:

$$h(p)=\rho(p)\varphi(p),\ q=\int\frac{1}{\sqrt{a(p)}}\mathrm{d}p, \tag{67}$$

with

$$\rho(p)=e^{\int\chi(p)\mathrm{d}p}. \tag{68}$$

Using these transformations, we obtain a form similar to the Schrödinger differential equation, so

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}q^2}+V(q)\right)\varphi(p)=\xi\varphi(p), \tag{69}$$

where

$$\chi(p)=\frac{2b+a'}{4a}=-\frac{1}{2p}, \tag{70}$$

and

$$V(p)=\varrho_1^2p^2-2\lambda\varrho_1(m+1)(1+\beta p^2)+2\beta\lambda^2m(1+\beta p^2)+\beta\lambda^2(1+\beta p^2)+\frac{\lambda^2(1+\beta p^2)^2}{p^2}\left(m^2-\frac{1}{4}\right). \tag{71}$$

Using that

$$p=\frac{1}{\sqrt{\beta}}\tan(q\lambda\sqrt{\beta}), \tag{72}$$

we get

$$V(p)=-\frac{1}{\beta}+\beta\lambda^2\left\{\frac{\zeta_1(\zeta_1-1)}{\sin^2(q\lambda\sqrt{\beta})}+\frac{\zeta_2(\zeta_2-1)}{\cos^2(q\lambda\sqrt{\beta})}\right\}, \tag{73}$$

with

$$\zeta_1(\zeta_1-1)=m^2-\frac{1}{4}, \tag{74}$$

$$\zeta_2(\zeta_2-1)=\left(m-\frac{\varrho_1}{\beta\lambda}+\frac{1}{2}\right)\left(m-\frac{\varrho_1}{\beta\lambda}+\frac{3}{2}\right). \tag{75}$$

Thus, we have

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}q^2}+\frac{1}{2}U_0\left\{\frac{\zeta_1(\zeta_1-1)}{\sin^2(\alpha q)}+\frac{\zeta_2(\zeta_2-1)}{\cos^2(\alpha q)}\right\}\right)\varphi(p)=\bar{\xi}^2\varphi(p) \tag{76}$$

with $\bar{\xi}^2=\xi^2+(\alpha_1^2/\beta)$ and $U_0=u^2$ with $u=\lambda\sqrt{\beta}$.

The last equation is the well-known Schrödinger equation in a Poschl–Teller potential where [39]

$$U=\frac{1}{2}U_0\left\{\frac{\zeta_1(\zeta_1-1)}{\sin^2(uq)}+\frac{\zeta_2(\zeta_2-1)}{\cos^2(uq)}\right\} \tag{77}$$

with $\zeta_1>1$ and $\zeta_2>1$. Thus, following (74) and (75), we have

$$\zeta_1=m\pm\frac{1}{2}, \tag{78}$$

$$\zeta_2=\frac{1}{2}\pm\left(m+1-\frac{\varrho_1}{\beta\lambda}\right). \tag{79}$$

Introducing the new variable

$$z=\sin^2(uq), \tag{80}$$

the Schrödinger equation is transformed into

$$z(1-z)\varphi''+\left(\frac{1}{2}-z\right)\varphi'+\frac{1}{4}\left\{\frac{\bar{\xi}^2}{u^2}-\frac{\zeta_1(\zeta_1-1)}{z}-\frac{\zeta_2(\zeta_2-1)}{1-z}\right\}\varphi=0. \tag{81}$$

Now, putting

$$\varphi=z^{\frac{\zeta_1}{2}}(1-z)^{\frac{\zeta_2}{2}}\Psi(z), \tag{82}$$

we arrive at

$$z(1-z)\Psi''+\left[\left(\zeta_1+\frac{1}{2}\right)-z(\zeta_1+\zeta_2+1)\right]\Psi'+\frac{1}{4}\left\{\frac{\bar{\xi}^2}{u^2}-(\zeta_1+\zeta_2)^2\right\}\Psi=0. \tag{83}$$

The general solutions of this equation are [40, 42]

$$\Psi=C_1\,{}_2F_1(a;\,b;\,c;\,z)+C_2z^{1-c}\,{}_2F_1(a+1-c;\,b+1-c;\,2-c;\,z), \tag{84}$$

where

$$a=\frac{1}{2}\left(\zeta_1+\zeta_2+\frac{\bar{\xi}}{u^2}\right),\ b=\frac{1}{2}\left(\zeta_1+\zeta_2-\frac{\bar{\xi}}{u^2}\right),\ c=\zeta_1+\frac{1}{2}. \tag{85}$$

With the condition $a=-n$, we obtain

$$\bar{\xi}^2=u^2(\zeta_1+\zeta_2+2n)^2. \tag{86}$$

In order to obtain the energy spectrum, it should be noted that in the limit $\beta\to 0$, the energy of spectrum should be covert to no-GUP result. Thus, we choose

$$\zeta_1=m+\frac{1}{2}, \tag{87}$$

$$\zeta_2=\frac{1}{2}-\left(m+1-\frac{\varrho_1}{\beta\lambda}\right). \tag{88}$$

Following this, we obtain

$$\epsilon_n=\pm\sqrt{m_0^2c^4+4c^2\left(1+\frac{m_0\omega}{2\hbar}\tilde{\theta}\right)\left(1+\frac{\bar{\theta}}{2m_0\omega\hbar}\right)n+4c^2\beta\left(1+\frac{\bar{\theta}}{2m_0\omega\hbar}\right)^2n^2}, \tag{89}$$

where

$$\beta<\beta_0,\beta_0=\frac{1}{m+\frac{3}{2}}\frac{\left(1+\frac{m_0\omega}{2\hbar}\tilde{\theta}\right)}{\left(1+\frac{\bar{\theta}}{2m_0\omega\hbar}\right)\hbar\omega m_0},\text{ with } m>0. \quad (90)$$

So, the non-zero minimal length is

$$\Delta x_{\min}=\hbar\sqrt{\beta}<(\Delta x_{\min})_0=\sqrt{\frac{1}{m+\frac{3}{2}}\frac{\left(1+\frac{m_0\omega}{2\hbar}\tilde{\theta}\right)}{\left(1+\frac{\bar{\theta}}{2m_0\omega\hbar}\right)}}l_{\min}, \quad (91)$$

with $l_{\min}=\sqrt{\frac{\hbar}{m_0\omega}}$ is the characteristic length of the DO, and $(\Delta x_{\min})_0$ is the admissible length above which the physics becomes experimentally inaccessible. We can see that the influence of the NC parameters on $(\Delta x_{\min})_0$ is very clear. Now, expanding to first order in terms of the variable β we have [24]

$$\epsilon_n\simeq\pm m_0c^2\sqrt{1+\frac{4}{m_0^2c^2}\left(1+\frac{m_0\omega}{2\hbar}\tilde{\theta}\right)\left(1+\frac{\bar{\theta}}{2m_0\omega\hbar}\right)n}\times\left(1+\frac{2\beta\left(1+\frac{\bar{\theta}}{2m_0\omega\hbar}\right)}{m_0^2c^2}\frac{n^2}{1+\frac{4}{m_0^2c^2}\left(1+\frac{m_0\omega}{2\hbar}\tilde{\theta}\right)\left(1+\frac{\bar{\theta}}{2m_0\omega\hbar}\right)n}\right). \quad (92)$$

The first term is the energy spectrum of the usual two-dimensional DO, and the second term represents the correction due to the presence of the minimal length. As mentioned in [23], we note the dependence on n^2, which is a feature of hard confinement. For a large values of n, we have

$$\epsilon_n=\hbar\bar{\omega}n, \quad (93)$$

which means the energy continuum for large n for the DO without the minimal length disappears in the presence of the minimal length, and consequently the behaviour of the DO can be described by a non-relativistic harmonic oscillator with a frequency of $\bar{\omega}=\frac{2c\sqrt{\beta}}{\hbar}\left(1+\frac{\bar{\theta}}{2m_0\omega\hbar}\right)$.

According to (85) and (87), we can see that the parameter $c=m+1$ is an integer; thus, either the two solutions of (84) coincide or one of the solutions will blow up. Now, when c is an integer greater than 1, which is our case, the second solution diverges. Thus, the component ψ_1 will has the following form:

$$(\psi_1)_{n,m}(p,\theta,z)=(C_1)_{n,m}p^{-\frac{1}{2}}e^{im\theta}z^{\frac{\xi_1}{2}}(1-z)^{\frac{\xi_2}{2}}\,{}_2F_1(-n;\,b,\,|m|+1;\,z). \quad (94)$$

Finally, the total associated eigenfunction is determined by

$$\psi_{n,m}(p,\theta,z)=\begin{pmatrix}1\\ \frac{cP_+}{\epsilon+m_0c^2}\end{pmatrix}\psi_1. \quad (95)$$

3.2 The Solutions in the Presence of a Magnetic Field

Now, in the presence of a uniform magnetic field, (7) is transformed into

$$\left\{c\alpha_x\left[\left(p_x^{(NC)}+\frac{eBy^{(NC)}}{2c}\right)-im_0\omega\tilde{\beta}x^{(NC)}\right]+c\alpha_y\left[\left(p_y^{(NC)}-\frac{eBx^{(NC)}}{2c}\right)-im_0\omega\tilde{\beta}y^{(NC)}\right]+\tilde{\beta}m_0c^2\right\}\psi_D=\bar{\epsilon}\psi_D. \quad (96)$$

In this case, (9) takes the following form:

$$\begin{pmatrix}m_0c^2 & c\tilde{P}_-\\ c\tilde{P}_+ & -m_0c^2\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=\bar{\epsilon}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \quad (97)$$

with

$$\tilde{P}_-=\left(p_x^{(NC)}+\frac{eBy^{(NC)}}{2c}\right)+im_0\omega x^{(NC)}-i\left(p_y^{(NC)}-\frac{eBx^{(NC)}}{2c}\right)+m_0\omega y^{(NC)}, \quad (98)$$

$$\tilde{P}_+=\left(p_x^{(NC)}+\frac{eBy^{(NC)}}{2c}\right)-im_0\omega x^{(NC)}+i\left(p_y^{(NC)}-\frac{eBx^{(NC)}}{2c}\right)+m_0\omega y^{(NC)}, \quad (99)$$

or

$$\tilde{P}_-=\varrho_1(p_x-ip_y)+im_0\tilde{\omega}\varrho_2(x-iy), \quad (100)$$

$$\tilde{P}_+=\varrho_1(p_x+ip_y)+im_0\tilde{\omega}\varrho_2(x+iy), \quad (101)$$

where

$$\tilde{\omega}=\omega-\frac{\omega_c}{2},\ \omega_c=\frac{|e|B}{m_0c}, \quad (102)$$

and where ω_c is a cyclotron frequency. According to the above case, the eigen solutions are given by

$$(\psi_1)_{n,m}(p,\theta,z)=(\tilde{C}_1)_{n,m}p^{-\frac{1}{2}}e^{im\theta}z^{\frac{\xi_1}{2}}(1-z)^{\frac{\xi_2}{2}}\,{}_2F_1(-n;\,b,\,|m|+1;\,z), \quad (103)$$

$$\bar{\epsilon}_n = \pm\sqrt{\begin{array}{l} m_0^2c^4 + 4c^2\left(1+\frac{m_0\tilde{\omega}}{2\hbar}\tilde{\theta}\right)\left(1+\frac{\bar{\theta}}{2m_0\tilde{\omega}\hbar}\right)n \\ + 4c^2\beta\left(1+\frac{\bar{\theta}}{2m_0\tilde{\omega}\hbar}\right)^2 n^2 \end{array}},$$

where the total wavefunction is given by

$$\psi_{n,m}(p, \theta, z) = \begin{pmatrix} 1 \\ \frac{c\tilde{P}_+}{\bar{\epsilon} + m_0c^2} \end{pmatrix}\psi_1. \tag{104}$$

4 Results and Discussions

Here, we have obtained exact solutions of the two-dimensional DO in NC phase space with the presence of minimal length. Firstly, by adopting the same procedure that used by Menculini et al. [29], we have solved the problem only in the case of NC space. The results found are in well agreement with those obtained in the literature. After that, we have introduced the minimal length in the problem in question. This introduction has been made as follows: (i) we write the coordinates of the NC space with those in commutative space by using the Bopp shift approximation, and (ii) then we introduce the minimal length in our equation. By these, the problem in question is identified with a Poschl–Teller potential. Also, when θ and $\bar{\theta}$ tend to zero, we recover exactly the same results of [43].

Finally, let us note that the non-relativistic harmonic oscillator is used as a model for describing the quark's confinement in mesons and baryons, while the DO is expected to give a good description of the confinement in heavy quark systems. Quimby and Strange suggested that the two-dimensional DO model can describe some properties of electrons in graphene. This model explains the origin of the left-handed chirality observed for charge carriers in monolayer and bilayer graphene. They have shown that the change in the strength of a magnetic field leads to the existence of a quantum phase transition in the chirality of the systems. In addition, in a recent paper, it has been shown that we can modulate the system of graphene under a magnetic field with a model based on a DO. With this, the author has determined all thermodynamic properties of this system by using the thermal zeta function [44, 45].

In our case, a possible application is the determination of the upper limit of the length in comparison with the data found experimentally for the case of graphene: this idea has been used by Menculini et al. [29] in order to obtain an upper bound on the minimal length appearing in the framework of GUP.

5 Conclusion

In this article, we have exactly solved the DO in two dimensions in the presence of an external magnetic field in the framework of relativistic quantum mechanics with minimal length and in the NC phasespace. Firstly, the eigensolutions of the problem in question are obtained in NC space. Then, we extend our study in the presence of a minimal length. The energy levels, for both cases, show a dependence on n^2 in the presence of the minimal length, which describe a hard confinement. For the large values of n, our DO becomes like a non-relativistic harmonic oscillator. The dependence of the non-zero minimum length on the noncommutativite parameters is very clear. In the limit where $\beta \to 0$, and where θ and $\bar{\theta}$ tend to zero, we recover the results obtained in the literature.

Acknowledgments: We wish to give our sincere gratitude to the referees for their instructive comments and careful reading of the article.

References

[1] D. Itô, K. Mori, and E. Carriere, Nuovo Cimento. A, **51**, 1119 (1967).
[2] M. Moshinsky and A. Szczepaniak, J. Phys. A Math. Gen. **22**, L817 (1989).
[3] R. P. Martinez-y-Romero and A. L. Salas-Brito, J. Math. Phys. **33**, 1831 (1992).
[4] M. Moreno and A. Zentella, J. Phys. A Math. Gen. **22**, L821 (1989).
[5] J. Benitez, P. R. Martinez y Romero, H. N. Nunez-Yepez, and A. L. Salas-Brito, Phys. Rev. Lett. **64**, 1643 (1990).
[6] A. Boumali and H. Hassanabadi, Eur. Phys. J. Plus. **128**, 124 (2013).
[7] O. Panella and P. Roy, Phys. Rev. **A90**, 042111 (2014).
[8] Y.-L. Hou, Q. Wang, Z.-G. Yuan, Z.-W. Long, and J. Jing, Phys. Rev. **A91**, 032120 (2015).
[9] L. Menculini, O. Panella, and P. Roy, Phys. Rev. **D91**, 045032 (2015).
[10] Y.-L. Hou, Q. Wang, Z.-G. Yuan, Z.-W. Long, and J. Jing, Ann. Phys. **354**, 10 (2015).
[11] J. A. Franco-Villafane, E. Sadurni, S. Barkhofen, U. Kuhl, F. Mortessagne, et al., Phys. Rev. Lett. **111**, 170405 (2013).
[12] C. Quimbay and P. Strange, arXiv:1311.2021 (2013).
[13] C. Quimbay and P. Strange, arXiv:1312.5251 (2013).
[14] A. Kempf, J. Math. Phys. **35**, 4483 (1994).
[15] A. Kempf, G. Mangano, and R. B. Mann, Phys. Rev. D**52**, 1108 (1995).

[16] H. Hinrichsen and A. Kempf, J. Math. Phys. **37**, 2121 (1996).
[17] A. Kempf, J. Phys. A Math. Gen. **30**, 2093 (1997).
[18] C. Quesne and V. M. Tkachuk, Sigma **3**, 016 (2007).
[19] P. Gaete and E. Spallucci, EPL **105**, 21002 (2014).
[20] B. Vakili and M. A. Gorji, J. Stat. Mech. **10**, P10013 (2012).
[21] P. Pedram, EPL **101**, 30005 (2013).
[22] Kh. Nouicer, J. Phys. A Math. Gen. **38**, 10027 (2005).
[23] Kh. Nouicer, J. Phys. A Math. Theor. **40**, 2125 (2007).
[24] Kh. Nouicer, J. Phys. A Math. Gen. **39**, 5125 (2006).
[25] C. Quesne and V. M. Tkachuck, J. Phys. A Math. Gen. **38**, 1747 (2005).
[26] T. V. Fityo, I. O. Vakarchuk, and V. M. Tkachuk, J. Phys. A Math. Gen. **39**, 2143 (2006).
[27] C. Quesne and V. M. Tkachuk, J. Phys. A Math. Gen. **39**, 10909 (2006).
[28] M. Betrouche, M. Maamache, and J. R. Choi, Adv. High Energy Phys. **2013**, Article ID 383957 (2013).
[29] L. Menculini, O. Panella, and P. Roy, Phys. Rev. **D87**, 065017 (2013).
[30] H. Hassanabadi, Z. Molaee, and S. Zarrinkamar, Adv. High Energy Phys. **2014**, Article ID 459345 (2014).
[31] Hassanabadi, E. Maghsoodi, A. N. Ikot, and S. Zarrinkamar, Adv. High Energy Phys. **2013**, Article ID 923686 (2013).
[32] O. Bertolami and R. Queiroz, Phys. Lett. **A375**, 4116 (2011).
[33] C. Bastos, O. Bertolami, N. C. Dias, and J. N. Prata, Int. J. Mod. Phys. **A28**, 1350064 (2013).
[34] C. Bastos, O. Bertolami, N. C. Dias, and J. N. Prata, J. Math. Phys. **49**, 072101 (2008).
[35] C. Bastos, N. C. Dias, and J. N. Prata, Comm. Math. Phys. **299**, 709 (2010).
[36] O. Bertolami, J. G. Rosa, C. M. L. de Aragao, P. Castorina, and D. Zappala, Phys. Rev. **D72**, 025010 (2005).
[37] D. Nath and P. Roy, Ann. Phys. **351**, 13 (2014).
[38] L. M. Abreu, E. S. Santos, and J. D. M. Vianna, J. Phys. A Math. Theor. **43**, 495402 (2010).
[39] S. Flugge, Practical Qunatum Mechanics, Springer-Verlag, Berlin 1974.
[40] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Dover, New York 1965.
[41] T. K. Jana and P. Roy, Phys. Lett. A**373**, 1239 (2009).
[42] G. E. Andrews, R. Askey, and R. Roy, Special Functions, Cambridge University Press, Cambridge, 1999.
[43] A. Boumali and H. Hassanabadi, Can. J. Phys. **93**, 542 (2015).
[44] A. Boumali, Phys. Scr. **90**, 045702 (2015).
[45] A. Boumali, EJTP **12**, 121 (2015).