Solutions Generated from the Symmetry Group of the (2 + 1)-Dimensional Sine-Gordon System

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Applying a symmetry group theorem on a two-straight-line soliton, some types of new localized multiply curved line excitations including the plateau-basin type ring solitons are obtained.

Key words: Finite Transformations; Localized Excitations.

1. Introduction

The symmetry study plays a very important role in almost all scientific fields, especially in the soliton theory because of the existence of infinitely many symmetries for integrable systems. The classical and the nonclassical Lie group approaches are two famous methods of deriving symmetries and conditional symmetries of differential equations. In 1974, Bluman and Cole [1] and later Olver and Rosenau [2] generalized the Lie approach to encompass symmetry transformations so that the invariants become a subset of the possible solutions of the PDE. Later, a new, say, direct method was developed by Clarkson and Kruskal [3, 4] in 1989 to obtain the similarity reductions of the PDE without using group theory.

On the other hand, Lax pairs have founded out to be very important in the study of integrable PDEs since Lax introduced them in 1968 [5]. Especially, it is an important method of determining the integrals of a PDE. It is also known that the Lax pairs can be used to find infinitesimal transformation invariances (symmetries) [6]. Very recently, we noticed that the Lax pair method can be directly used to obtain finite transformation invariances (groups) of integrable models [7]. In this short paper, the $(2+1)$-dimensional sine-Gordon system is discussed by using a new simple method to get both its Lie point symmetry group, the related (infinitesimal) symmetry algebra and then the exact solutions.

2. Results and Discussion

A $(2+1)$-dimensional master soliton system had been constructed by Konopelchenko and Rogers [8] in 1991 via a reinterpretation, and Loewner [9] generalized a class of infinitesimal Bäklund transformations originally in a gas dynamics context. A particular reduction leads to a symmetric integrable extension of the classical sine-Gordon equation, namely,

$$\left(\frac{\phi_x}{\sin\theta}\right)_x - \left(\frac{\phi_y}{\sin\theta}\right)_y + \frac{\phi_y\theta_x - \phi_x\theta_y}{\sin^2\theta} = 0, \quad \text{(1a)}$$

$$\left(\frac{\phi'_x}{\sin\theta}\right)_x - \left(\frac{\phi'_y}{\sin\theta}\right)_y + \frac{\phi'_y\theta_x - \phi'_x\theta_y}{\sin^2\theta} = 0, \quad \text{(1b)}$$

where $\theta_t = \phi + \phi'$. After that the model has been widely studied by many authors [10 – 15].

In [16], the similarity reductions of (1) have been given by one of the present authors. In [17], an equivalent group analysis for a gauge equivalent form of the system had also been given.

The $(2+1)$-dimensional sine-Gordon (2DsG) system (1) is generated as the compatibility condition of the particular Loewner Konopelchenko Rogers (LKR) triad [18]

$$\left[I\partial_x + \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\partial_y\right]\boldsymbol{\Phi} = 0,$$

$$\left[I\partial_t\partial_y + \frac{1}{2}\begin{pmatrix} 0 & \theta_t \\ -\theta_t & 0 \end{pmatrix}\partial_y - \frac{1}{2\sin\theta}\begin{pmatrix} \phi_y\cos\theta - \phi_x & -\phi'_y\sin\theta \\ \phi_y\sin\theta & \phi'_y + \phi'_x\cos\theta \end{pmatrix}\right]\boldsymbol{\Phi} = 0, \quad (2)$$

$$\left[I\partial_t\partial_x + \frac{1}{2}\begin{pmatrix} 0 & \theta_t \\ -\theta_t & 0 \end{pmatrix}\partial_x - \frac{1}{2\sin\theta}\begin{pmatrix} \phi_x\cos\theta - \phi_y & -\phi'_x\sin\theta \\ \phi_x\sin\theta & \phi'_x + \phi'_y\cos\theta \end{pmatrix}\right]\boldsymbol{\Phi} = 0.$$

It is also known that the $(2+1)$-dimensional sine-Gordon system (1) is equivalent to the following compact version [15]

$$u_{\xi\eta t}+u_\eta v_{\xi t}+u_\xi v_{\eta t}=0,\quad v_{\xi\eta}=u_\xi u_\eta, \tag{3}$$

where

$$\xi=\frac{1}{2}(y-x),\ \eta=\frac{1}{2}(y+x),\ u=\frac{1}{2}\theta$$

and v is determined by

$$v_{\xi t}=\frac{\phi'_\eta-\phi_\eta-\theta_{\eta t}\cos\theta}{2\sin\theta},\ v_{\eta t}=\frac{\phi_\xi-\phi'_\xi-\theta_{\xi t}\cos\theta}{2\sin\theta}.$$

The corresponding Lax pair of the representation (3) reads

$$\begin{pmatrix}\partial_\xi & u_\xi\\ -u_\eta & \partial_\eta\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}=0,$$
$$\begin{pmatrix}\partial_\eta\partial_t+v_{\eta t} & u_\eta\partial_t\\ -u_\xi\partial_t & \partial_\xi\partial_t+v_{\xi t}\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}=0. \tag{4}$$

The component form of (4) reads

$$\Phi_{1\xi}+u_\xi\Phi_2=0, \tag{5a}$$
$$-u_\eta\Phi_1+\Phi_{2\eta}=0, \tag{5b}$$
$$\Phi_{1\eta t}+v_{\eta t}\Phi_1+u_\eta\Phi_{2t}=0, \tag{5c}$$
$$-u_\xi\Phi_{2t}+\Phi_{2\xi t}+v_{\xi t}\Phi_2=0. \tag{5d}$$

Now let

$$\Phi=g\Psi, \tag{6}$$

where

$$\Phi=\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix},\ \Psi=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix},$$
$$\text{and } g=\begin{pmatrix}g_{11}\ g_{12}\\ g_{21}\ g_{22}\end{pmatrix}$$

is a matrix function of $\xi,\ \eta,\ t$ while Ψ is a function of the new variables $\xi_1\equiv\xi_1(\xi,\ \eta,\ t)$, $\eta_1\equiv\eta_1(\xi,\ \eta,\ t)$ and $\tau\equiv\tau(\xi,\ \eta,\ t)$. Substituting (6) into the first equation of (5) we have

$$\begin{aligned}&g_{11}(\Psi_{1\xi_1}\xi_\xi+\Psi_{1\eta_1}\eta_\xi+\Psi_{1\tau}\tau_\xi)\\&\quad+g_{12}(\Psi_{2\xi_1}\xi_\xi+\Psi_{2\eta_1}\eta_\xi+\Psi_{2\tau}\tau_\xi)\\&\quad+u_\xi g_{21}\Psi_1+u_\xi g_{22}\Psi_2=0.\end{aligned} \tag{7}$$

Requiring that Ψ satisfies the same Lax equations (5) but with new variables, *i.e.*,

$$\Psi_{1\xi_1}+u_{1\xi_1}(\xi_1,\eta_1,\tau)\Psi_2=0, \tag{8a}$$
$$-u_{1\eta_1}(\xi_1,\eta_1,\tau)\Psi_1+\Psi_{2\eta_1}=0, \tag{8b}$$
$$\Psi_{1\eta_1\tau}+v_{1\eta_1\tau}\Psi_1+u_{1\eta_1}\Psi_{2\tau}=0, \tag{8c}$$
$$-u_{1\xi_1}\Psi_{2\tau}+\Psi_{2\xi_1\tau}+v_{1\xi_1\tau}\Psi_2=0, \tag{8d}$$

the substitution of (8) into (7) and the comparison of the different derivatives of Ψ yield

$$\tau_\xi=0,\quad \eta_{1\xi}=0,\quad g_{12}=0, \tag{9}$$

and

$$(u_\xi g_{22}-g_{11}u_{1\xi_1}\xi_{1\xi})\Psi_2+u_\xi g_{21}\Psi_1=0,$$

from which we can get

$$u_\xi=\frac{g_{11}u_{1\xi_1}\xi_{1\xi}}{g_{22}},\quad g_{21}=0. \tag{10}$$

In the same way the substitution of (6) into the second equation of (5) leads to

$$\begin{aligned}&g_{21}(\Psi_{1\xi_1}\xi_\eta+\Psi_{1\eta_1}\eta_\eta+\Psi_{1\tau}\tau_\eta)\\&\quad+g_{22}(\Psi_{2\xi_1}\xi_\eta+\Psi_{2\eta_1}\eta_\eta+\Psi_{2\tau}\tau_\eta)\\&\quad-u_\eta g_{11}\Psi_1-u_\eta g_{12}\Psi_2=0.\end{aligned} \tag{11}$$

Applying (8) and (9) to (11) and comparing the different derivative of Ψ, we have

$$\tau_\eta=0,\quad \xi_{1\eta}=0, \tag{12}$$

and

$$u_\eta=\frac{g_{22}u_{1\eta_1}\eta_{1\eta}}{g_{11}}. \tag{13}$$

Finally, substituting (8) – (12) into the other two equations of (5), we get

$$\xi_{1t}=0,\quad \eta_{1t}=0, \tag{14}$$

$$u_\eta=\frac{g_{11}u_{1\eta_1}\eta_{1\eta}}{g_{22}},\quad u_\xi=\frac{g_{22}u_{1\xi_1}\xi_{1\xi}}{g_{11}}, \tag{15}$$

$$v_{\xi t}=\tau_t\xi_{1\xi}v_{1\xi_1 t},\quad v_{\eta t}=\tau_t\eta_{1\eta}v_{1\eta_1 t}. \tag{16}$$

From (9), (12) and (14) one can find that

$$\xi_1=\xi_1(\xi),\ \eta_1=\eta_1(\eta),\ \tau=\tau(t),\ g_{12}=g_{21}=0, \tag{17}$$

and from (10), (13), (15) and (16) one can obtain

$$\begin{aligned} u_1 &= u(\xi_1,\eta_1,\tau)+m(t),\\ v_1 &= v(\xi_1,\eta_1,\tau)+n(t)+k(\xi)+l(\eta). \end{aligned} \tag{18}$$

where $\xi_1(\xi)$, $\eta_1(\eta)$, $\tau(t)$, $m(t)$, $k(\xi)$, $l(\eta)$ and $n(t)$ are arbitrary functions of the indicated variables while $g_{11}=g_{22}$ are arbitrary constants.

In summary, the following theorem is assured:

Theorem 1. *If* $\{u=u(\xi,\ \eta,\ t),\ v=v(\xi,\ \eta,\ t)\}$ *is a solution of the sine-Gordon system* (3) *then* $\{u_1\equiv u_1(\xi,\ \eta,\ t),\ v_1\equiv v_1(\xi,\ \eta,\ t)\}$ *is expressed by* (17) *and* (18).

Theorem 1 can also be easily verified by the direct substitution of (18) with (17) into (3).

Furthermore, by restricting the arbitrary functions of the theorem as

$$\begin{aligned} &\xi_1=\xi+\varepsilon g(\xi),\ \eta_1=\eta+\varepsilon h(\eta),\ \tau=t+\varepsilon f(t),\\ &m(t)=\varepsilon p(t),\ n(t)=\varepsilon q(t),\\ &k(\xi)=\varepsilon r(\xi),\ l(\eta)=\varepsilon s(\eta), \end{aligned}$$

we can reobtain the general Lie point symmetries which are linear combinations of the following generators,

$$\begin{aligned} &\sigma_1(g)=g(\xi)\begin{pmatrix}u\\v\end{pmatrix}_\xi,\ \sigma_2(h)=h(\eta)\begin{pmatrix}u\\v\end{pmatrix}_\eta,\\ &\sigma_3(f)=f(t)\begin{pmatrix}u\\v\end{pmatrix}_t, \end{aligned} \tag{19}$$

$$\begin{aligned} &\sigma_4(p)=\begin{pmatrix}p(t)\\0\end{pmatrix},\ \sigma_5(q)=\begin{pmatrix}0\\q(t)\end{pmatrix},\\ &\sigma_6(r)=\begin{pmatrix}0\\r(\xi)\end{pmatrix},\ \sigma_7(s)=\begin{pmatrix}0\\s(\eta)\end{pmatrix}, \end{aligned} \tag{20}$$

and the Lie algebra constituted by $\sigma_1(g)$, $\sigma_2(h)$, $\sigma_3(f)$, $\sigma_4(p)$, and $\sigma_5(q)$ reads

$$[\sigma_1(g_1),\ \sigma_1(g_2)]=\sigma_1(g_1g_{2\xi}-g_2g_{1\xi}), \tag{21}$$

$$[\sigma_2(h_1),\ \sigma_2(h_2)]=\sigma_2(h_1h_{2\eta}-h_2h_{1\eta}), \tag{22}$$

$$[\sigma_3(f_1),\ \sigma_3(f_2)]=\sigma_3(f_1f_{2t}-f_2f_{1t}), \tag{23}$$

$$[\sigma_3(f),\ \sigma_4(p)]=\sigma_4(fp_t), \tag{24}$$

$$[\sigma_3(f),\ \sigma_5(f_2)]=\sigma_5(fq_t), \tag{25}$$

$$[\sigma_1(g_1),\ \sigma_6(r)]=\sigma_6(g_1r_\xi), \tag{26}$$

$$[\sigma_2(h_1),\ \sigma_7(s)]=\sigma_7(h_1q_\eta), \tag{27}$$

while other commutators are all identically zero.

Applying the above symmetry group Theorem 1 on some trivial known exact solutions, one can obtain many kinds of interesting new exact localized excitations. Since the discovery of the two-dimensional sine-Gordon system, the exact solutions of the model have been investigated by many authors. A Bäcklund transformation was constructed in [10] and certain coherent solitonic solutions thereby derived. Solitonic solutions of an important reduction of the $(2+1)$-dimensional sine-Gordon system have been investigated by Nimmo [11]. Doubly periodic wave solutions have been constructed by Chow [19]. Localized solutions of the two-dimensional sine-Gordon system were constructed via a binary Darboux transformation by Schief *et al.* [10]. In [12], Nimmo and Schief constructed nonlinear superposition principles and an associated integrable discretization of the two-dimensional sine-Gordon system. Localized solutions of a model with nontrivial boundaries have been constructed by Dubrovsky and Konopelchenko [20] and Dubrovsky and Formusatik [21]. Radha and Lakshmanan [22] studied the Painlevé property for the two-dimensional sine-Gordon system and have constructed dromion solutions. In [15] many kinds of localized excitations have been given by the multi-linear variable separation approach.

In this paper, we only apply the symmetry group Theorem 1 on the following two-straight-line soliton solution ($\varphi_1\equiv k_1\xi+l_1\eta+\omega_1t$, $\varphi_2\equiv k_2\xi+l_2\eta+\omega_2t$)

$$u=-2\arctan\frac{(k_1+k_2)(l_1+l_2)(\mathrm{e}^{\varphi_1}+b\mathrm{e}^{\varphi_2})}{(k_1+k_2)(l_1+l_2)+b(k_2-k_1)(l_1-l_2)\mathrm{e}^{\varphi_1+\varphi_2}}, \tag{28}$$

$$v=\ln\left[\left(1+\frac{b(k_2-k_1)(l_1-l_2)\mathrm{e}^{\varphi_1+\varphi_2}}{(k_1+k_2)(l_1+l_2)}\right)^2+(\mathrm{e}^{\varphi_1}+b\mathrm{e}^{\varphi_2})^2\right]+\frac{k_1k_2(l_2\omega_2-l_1\omega_1)t\xi}{l_1k_2-l_2k_1}+\frac{l_1l_2(k_2\omega_2-k_1\omega_1)t\eta}{k_1l_2-k_2l_1}, \tag{29}$$

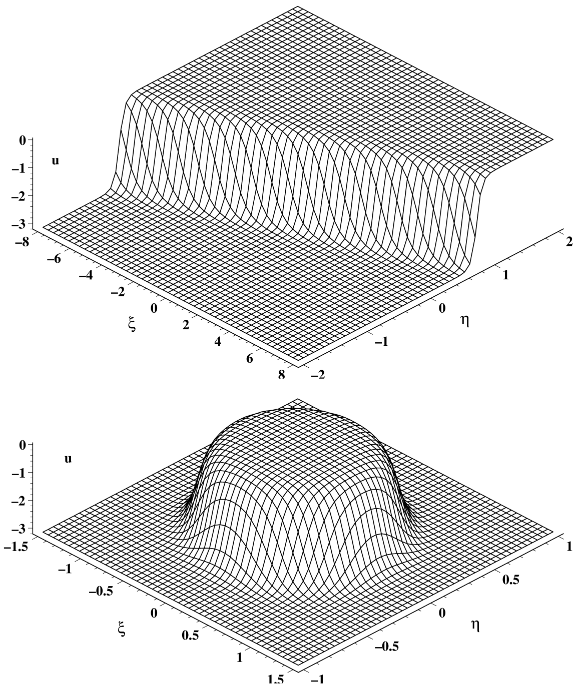

Fig. 1. A typical straight-line soliton solution of the $(2+1)$-dimensional sine-Gordon equation (*top*) and its special ring shape group deformation (*bottom*).

where l_1, k_1, l_2, k_2, ω_1, ω_2 and b are all arbitrary constants.

When we take $b=0$, (28) and (29) become a single straight-line soliton, say, for the field u. We have

$$u = -2\arctan \mathrm{e}^{k_1\xi + l_1\eta + \omega_1 t}. \tag{30}$$

Fig. 1a shows a typical structure of the single straight-line kink soliton (30) with

$$k_1 = 2,\ l_1 = -20,\ \omega_1 = 3$$

at time $t=0$. Applying the symmetry group Theorem 1 on (30) yields a curved-line soliton

$$u = -2\arctan e^{\xi_1(\xi)+\eta_1(\eta)+\tau(t)} + m(t) \tag{31}$$

with an arbitrary time-dependent background $m(t)$ and the time-dependent curve is determined by

$$\xi_1(\xi) + \eta_1(\eta) + \tau(t) = 0. \tag{32}$$

Furthermore, if the curve given by (32) is closed, then (30) becomes a so-called plateau type ring soliton solution and the density of the potential energy quantity

$$1 - \cos\theta \equiv -w \tag{33}$$

denotes the bowl type ring soliton structure [15].

Fig. 1b exhibits a typical plateau type ring soliton (31) with

$$\xi_1(\xi) = 8\xi^2, \quad \eta_1(\eta) = 20\eta^2,$$
$$\tau(t) = 6\cos(t), \quad m(t) = 0.$$

For a two-straight-line soliton solution (28), to avoid the singularity, the constant b should be selected appropriately such that

$$A_{12} \equiv b(k_2^2 - k_1^2)(l_1^2 - l_2^2) \geq 0.$$

$A_{12} = 0$ is related to the resonant soliton case ("Y" shape soliton or three soliton solution).

Applying Theorem 1 to the two-straight-line soliton solution (28), we have

$$u = -2\arctan \frac{(k_1+k_2)(l_1+l_2)(\mathrm{e}^{\Phi_1} + b\mathrm{e}^{\Phi_2})}{(k_1+k_2)(l_1+l_2) + b(k_2-k_1)(l_1-l_2)\mathrm{e}^{\Phi_1+\Phi_2}} + m(t), \tag{34}$$

with

$$\Phi_1 \equiv k_1\xi_1(\xi) + l_1\eta_1(\eta) + \omega_1\tau(t),$$
$$\Phi_2 \equiv k_2\xi_1(\xi) + l_2\eta_1(\eta) + \omega_2\tau(t). \tag{35}$$

Usually, (34) denotes a special two sets of time-dependent curved line soliton interaction solution.

Figure 2 displays two special structures of the two-straight-line soliton solution expressed by (28) with different parameter selections:

(a)

$$k_1 = 2,\ l_1 = -20,\ \omega_1 = 3,$$
$$k_2 = 1,\ l_2 = 2,\ \omega_2 = 4,\ b = -27, \tag{36}$$

and (b)

$$k_1 = 2,\ l_1 = 15,\ \omega_1 = 3,$$
$$k_2 = 1,\ l_2 = 2,\ \omega_2 = 4,\ b = -27. \tag{37}$$

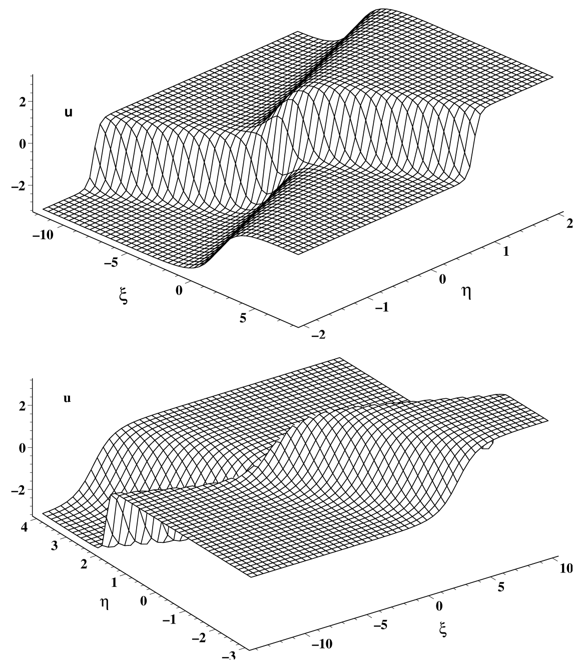

Fig. 2. Two special examples of two-straight-line soliton solutions expressed by (28) at time $t = 0$ with the parameter selections (36) (*top*) and (37) (*bottom*) respectively.

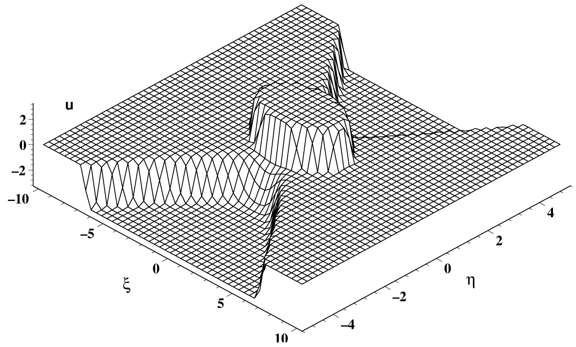

Fig. 3a. $t = -\pi$.

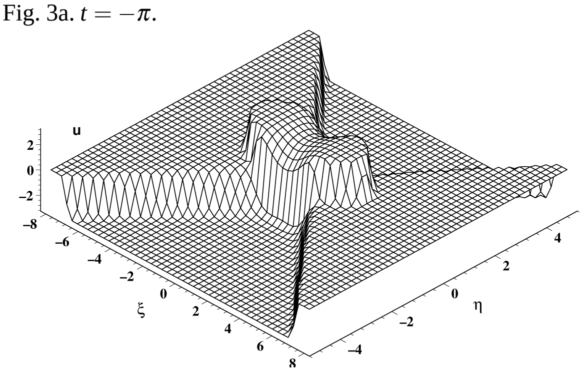

Fig. 3b. $t = -\frac{29}{30}\pi$.

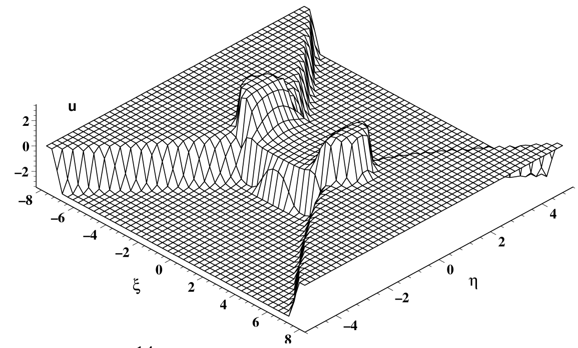

Fig. 3c. $t = -\frac{14}{15}\pi$.

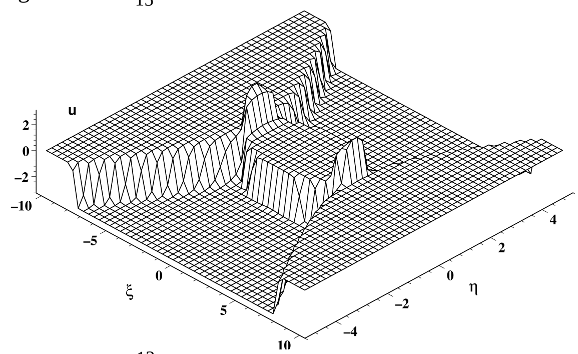

Fig. 3d. $t = -\frac{13}{15}\pi$.

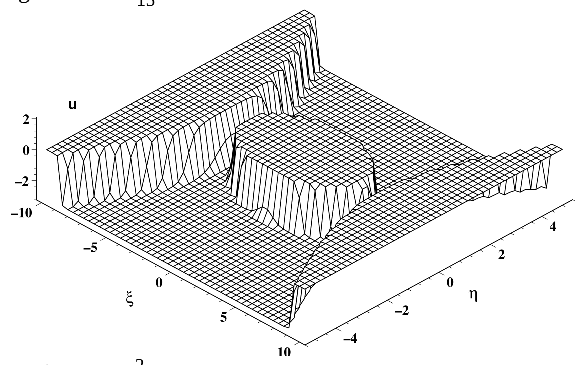

Fig. 3e. $t = -\frac{2}{3}\pi$.

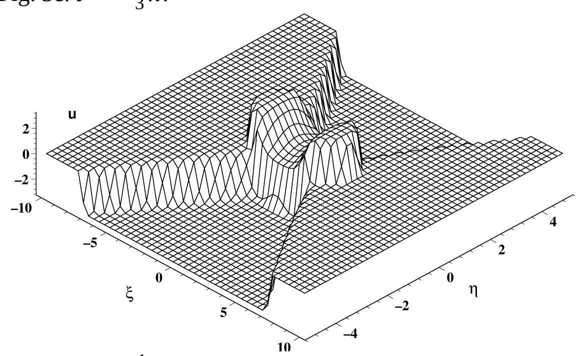

Fig. 3f. $t = -\frac{1}{20}\pi$.

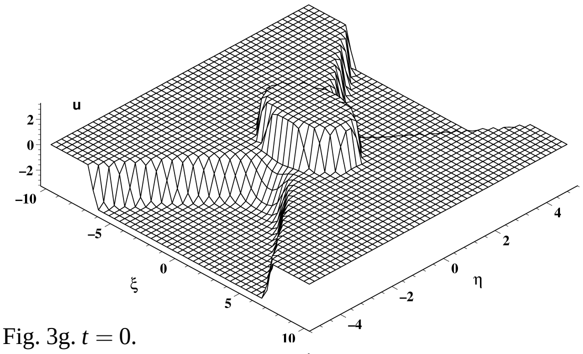

Fig. 3g. $t = 0$.

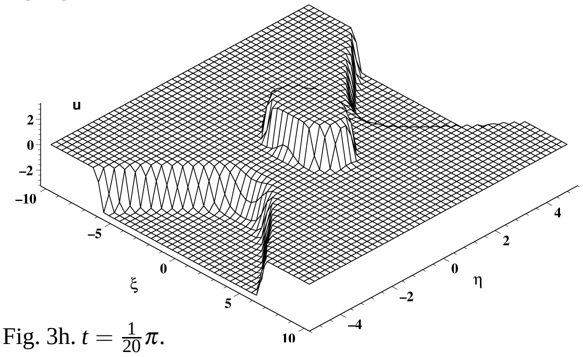

Fig. 3h. $t = \frac{1}{20}\pi$.

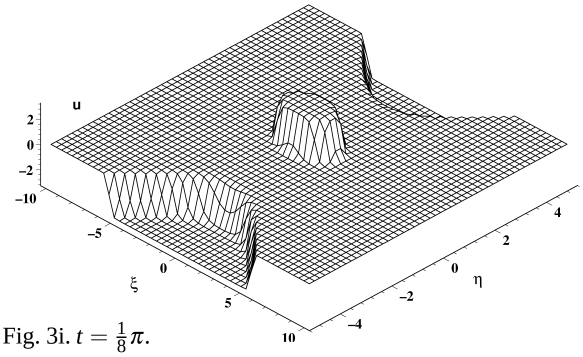

Fig. 3i. $t = \frac{1}{8}\pi$.

Fig. 3. The time evolution plot of the elliptic-hyperbolic pair curved line solitons of the $(2+1)$-dimensional sine-Gordon equation expressed by (34) with (35), (36) and (38) at the specified times.

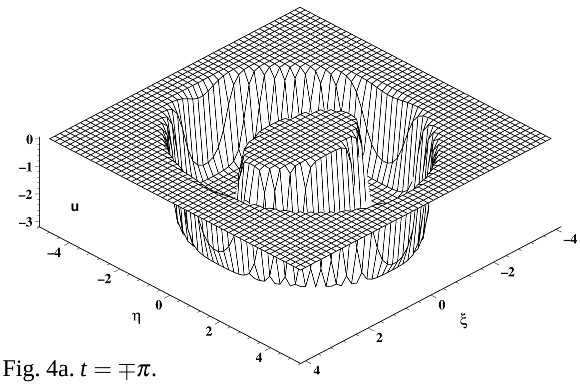

Fig. 4a. $t = \mp\pi$.

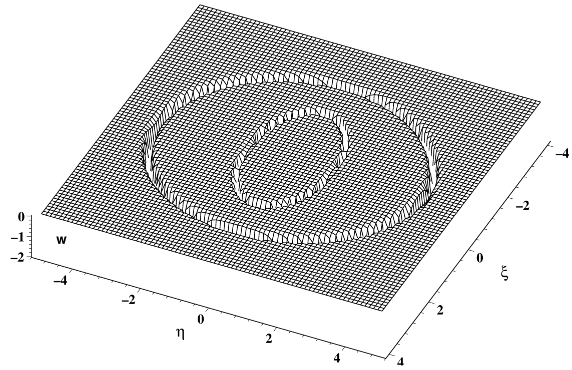

Fig. 4b. The parameters are the same as in (1) but for the quantity w instead of u.

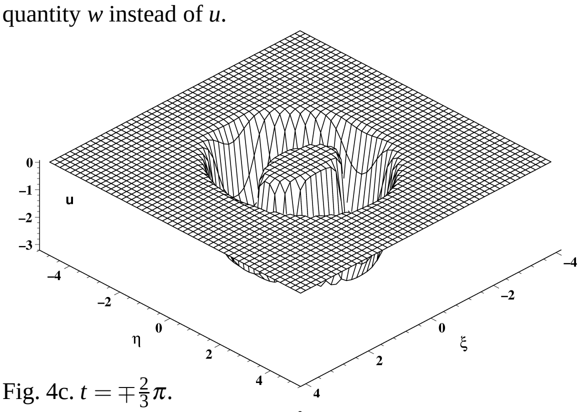

Fig. 4c. $t = \mp\frac{2}{3}\pi$.

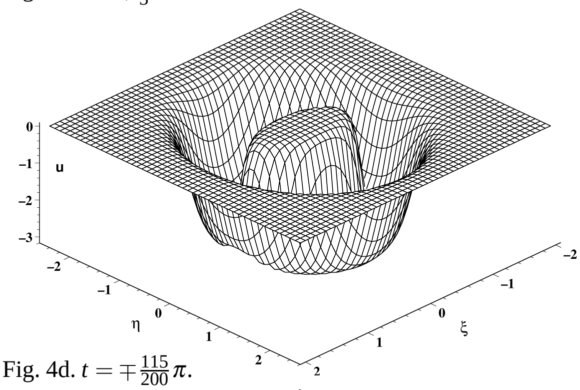

Fig. 4d. $t = \mp\frac{115}{200}\pi$.

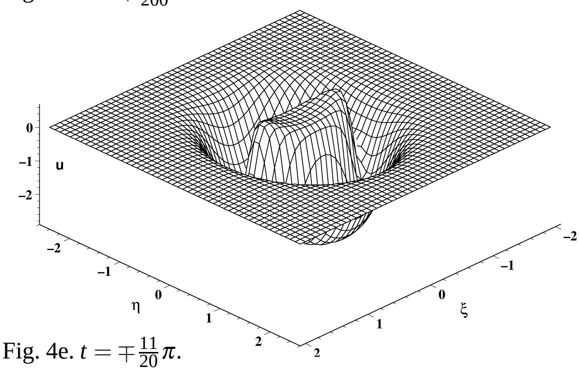

Fig. 4e. $t = \mp\frac{11}{20}\pi$.

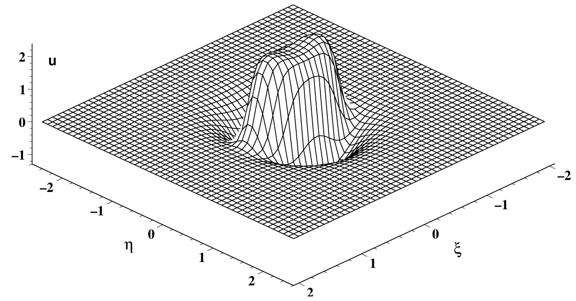

Fig. 4f. $t = \mp\frac{105}{200}\pi$.

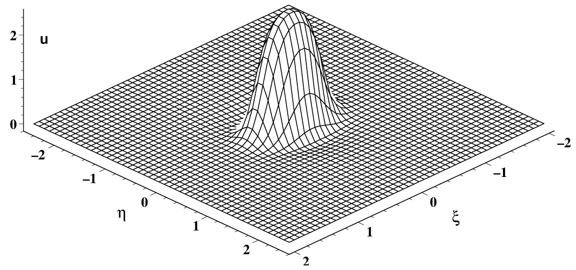

Fig. 4g. $t = \mp\frac{1}{2}$.

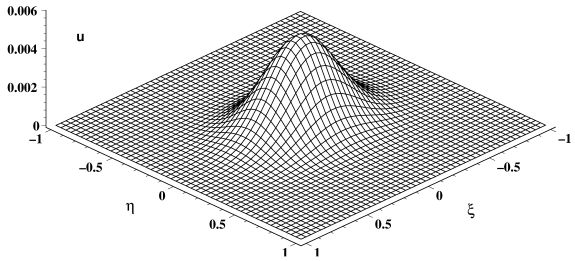

Fig. 4h. $t = \mp\frac{2}{5}\pi$.

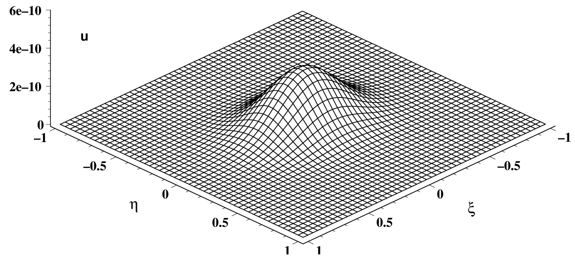

Fig. 4i. $t = 0$.

Fig. 4. The time evolution plot of the elliptic-elliptic pair ring solitons of the $(2+1)$-dimensional sine-Gordon equation expressed by (34) with (35), (37) and (39) at the specified times.

Applying the group Theorem 1 to Fig. 2a with

$$\xi_1(\xi) = \xi^2, \quad \eta_1(\eta) = -\eta^2 + 1, \\ \tau(t) = 10\sin(t), \quad m(t) = 0 \tag{38}$$

yields an elliptic-hyperbolic pair curved line soliton solution as shown in Figure 3.

Differently, the application of Theorem 1 to Fig. 2b and the special selections

$$\xi_1(\xi) = 4\xi^2, \quad \eta_1(\eta) = \eta^2, \\ \tau(t) = 8\cos(t), \quad m(t) = 0 \tag{39}$$

results in an elliptic-elliptic pair ring soliton solution as exhibited in Figure 4.

3. Conclusion

In summary, starting from the Lax expression of the $(2+1)$-dimensional sine-Gordon system, the symmetry group and then the Lie symmetries and the related algebra can be reobtained via a simple combination of a gauge transformation of the spectral function and the transformations of the space time variables.

Because many kinds of special solutions of the $(2+1)$-dimensional sine-Gordon system have been given by many authors, one can find many kinds of more general exact solutions by applying the group transformation theorem on the known ones.

Especially, applying the group transformation theorem on the multiple straight line soliton solutions, one can obtain various types of multiple curved line excitations. A special single plateau type ring soliton, an elliptic-hyperbolic pair curved line soliton solution and an elliptic-elliptic pair ring soliton solution are explicitly plotted in Figs. 1, 3 and 4.

In [15], one of the present authors has obtained a special type of variable separation solutions. It is straightforward to see that the variable separation solutions of [15] are group transformation invariant under the Theorem 1 (for $m(t) = 0$). On the other hand, the group deformed solutions obtained in this paper, say, the solutions expressed by (34) are beyond the variable separation solutions of [15].

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