Exact Solutions to Double and Triple Sinh-Gordon Equations

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In this paper, two transformations are introduced to solve double sinh-Gordon equation and triple sinh-Gordon equation, respectively. It is shown that different transformations are required in order to obtain more kinds of solutions to different types of sinh-Gordon equations. – PACS: 03.65.Ge

Key words: Transformation; Sinh-Gordon.

1. Introduction

The sinh-Gordon (ShG for short) equation [1 – 8]

$$u_{xt} = \alpha \sinh u, \tag{1}$$

the double sinh-Gordon (DShG for short) equation [8 – 11]

$$u_{xt} = \alpha \sinh u + \beta \sinh 2u, \tag{2}$$

and the triple sinh-Gordon (TShG for short) equation [8, 11]

$$u_{xt} = \alpha \sinh u + \beta \sinh 2u + \gamma \sinh 3u, \tag{3}$$

are widely applied in physics and engineering, for example in integrable quantum field theories [1], non-commutative field theories [2], fluid dynamics [3], kink dynamics [8, 9, 11] and so on. Due to the wide applications of the sinh-Gordon type equations, many achievements have been obtained [3 – 8, 12]. For instance, the ShG equation is known to be completely integrable [4] because it possesses similarity reductions to the third Painlevé equation [5].

Due to the special form of the sinh-Gordon type equations, it is much more difficult to solve them directly. There is need for some transformations. In this paper, based on the transformations to be introduced, we will show systematic results about the solutions of the DShG equation (2) and the TShG equation (3).

2. The Intermediate Transformation and Solutions to DShG Equation

Due to the complexity of nonlinear evolution equations, it is usually very difficult to solve them directly.

One uses some kinds of intermediate transformations to simplify the original nonlinear evolution equations. For example, in References [13 – 16], the intermediate transformation is a generalized elliptic equation [17]

$$y'^2 = a_0 + a_1 y + a_2 y^2 + a_3 y^3 + a_4 y^4, \tag{4}$$

with

$$u = u(y), \tag{5}$$

where the prime denotes the derivative of y with respect to its argument.

In reference [12], Sirendaoreji introduced a new transformation

$$u' = F(u), \tag{6}$$

where $F(u)$ is a suitable function of sine, cosine, hyperbolic sine, hyperbolic cosine and so on. It has been shown that this transformation is very powerful in solving the sinh-Gordon equation (1), the sine-Gordon equation, the double sine-Gordon equation and some other special types of nonlinear evolution equations. Usually, for different nonlinear evolution equations, the exact forms of the intermediate transformations are different. Next we shall introduce two transformations $u' = F(u)$ of a new form and apply them to solve the DShG equation (2) and the TShG equation (3), respectively.

First of all, we suppose the solution of the DShG equation (2) takes the form

$$u = u(\xi), \ \xi = k(x - ct), \tag{7}$$

where k and c are wave number and wave speed, respectively.

With the Ansatz (7), the DShG equation (2) can be rewritten as

$$u'' = \alpha_1 \sinh u + \beta_1 \sinh 2u \tag{8}$$

with

$$\alpha_1 = -\frac{\alpha}{k^2 c}, \quad \beta_1 = -\frac{\beta}{k^2 c}. \tag{9}$$

We then suppose that Eq.(8) satisfies the transformation (6) with the following new form

$$u' = \frac{du}{d\xi} = a + b\cosh u, \tag{10}$$

where a and b are real constants to be determined.

From the intermediate transformation (10), we have

$$u'' = ab \sinh u + \frac{b^2}{2} \sinh 2u. \tag{11}$$

Combining (8) with (11) results in

$$\alpha_1 = ab, \quad \beta_1 = \frac{b^2}{2}, \tag{12}$$

from which

$$a = \pm \frac{\alpha_1}{\sqrt{2\beta_1}}, \quad b = \pm \sqrt{2\beta_1}, \tag{13}$$

with constraints

$$\beta_1 > 0. \tag{14}$$

In order to derive the exact solutions to the DShG equation (2), the intermediate transformation (10) must be solved. It can be integrated directly. Here we have four cases under consideration.

Case 1. If $a = b$, i.e. $\alpha = 2\beta$, we have

$$u_1 = 2\tanh^{-1}[\pm \sqrt{2\beta_1}(\xi - \xi_0)], \tag{15}$$

where ξ_0 is an integration constant.

Case 2. If $a = -b$, i.e. $\alpha = -2\beta$, then we have

$$u_2 = 2\coth^{-1}[\mp \sqrt{2\beta_1}(\xi - \xi_0)], \tag{16}$$

where ξ_0 is an integration constant.

Case 3. If $b^2 > a^2$, i.e. $4\beta^2 > \alpha^2$, then we have

$$u_3 = \cosh^{-1}\left[-\frac{\pm\alpha_1 \sin\sqrt{\frac{4\beta_1^2 - \alpha_1^2}{2\beta_1}}(\xi - \xi_0) \mp 2\beta_1}{\pm 2\beta_1 \sin\sqrt{\frac{4\beta_1^2 - \alpha_1^2}{2\beta_1}}(\xi - \xi_0) \mp \alpha_1}\right], \tag{17}$$

where ξ_0 is an integration constant.

Case 4. If $b^2 < a^2$, i.e. $4\beta^2 < \alpha^2$, then we have

$$u_4 = 2\tanh^{-1} \tag{18}$$

$$\left\{\frac{(\pm\alpha_1 \pm 2\beta_1)\left[\exp\sqrt{\frac{\alpha_1^2 - 4\beta_1^2}{2\beta_1}}(\xi - \xi_0) - 1\right]}{\sqrt{\alpha_1^2 + 4\beta_1^2} - \sqrt{\alpha_1^2 - 4\beta_1^2}\exp\sqrt{\frac{\alpha_1^2 - 4\beta_1^2}{2\beta_1}}(\xi - \xi_0)}\right\},$$

where ξ_0 is an integration constant.

The solutions u_1 to u_4 for the DShG equation (2) have not been reported in the literature.

3. The Intermediate Transformation and Solutions to TShG Equation

With the Ansatz (7), the TShG equation (3) can be rewritten as

$$u'' = \alpha_1 \sinh u + \beta_1 \sinh 2u + \gamma_1 \sinh 3u \tag{19}$$

with

$$\alpha_1 = -\frac{\alpha}{k^2 c}, \quad \beta_1 = -\frac{\beta}{k^2 c}, \quad \gamma_1 = -\frac{\gamma}{k^2 c}. \tag{20}$$

We suppose now that Eq. (19) satisfies the transformation (6) with the following new form

$$u' = \frac{du}{d\xi} = a\cosh\frac{u}{2} + b\cosh\frac{3u}{2}, \tag{21}$$

where a and b are real constants to be determined.

From the intermediate transformation (21), we have

$$u'' = \frac{1}{4}(a^2 + 2ab)\sinh u + ab\sinh 2u + \frac{3b^2}{4}\sinh 3u. \tag{22}$$

Combining (19) with (22) results in

$$\alpha_1 = \frac{1}{4}(a^2 + 2ab), \ \beta_1 = ab, \ \gamma_1 = \frac{3b^2}{4}, \tag{23}$$

from which four kinds of results can be derived,

Case 1.

$$a = e_1 + e_2, \ b = e_1 - e_2, \tag{24}$$

Case 2.

$$a = e_1 - e_2, \ b = e_1 + e_2, \tag{25}$$

Case 3.

$$a = -e_1 + e_2, \ b = -e_1 - e_2, \tag{26}$$

Case 4.

$$a = -e_1 - e_2, \; b = -e_1 + e_2, \tag{27}$$

where

$$e_1 \equiv \sqrt{\alpha_1 + \frac{\gamma_1}{3}}, \; e_2 \equiv \sqrt{\alpha_1 + \frac{\gamma_1}{3} - \beta_1}, \tag{28}$$

with the constraints

$$\alpha_1 + \frac{\gamma_1}{3} \geq 0, \; \alpha_1 + \frac{\gamma_1}{3} \geq \beta_1, \quad \gamma_1 > 0. \tag{29}$$

For the intermediate transformation (21), two cases need to be considered. The first one is that $a = 3b$, i.e. $4\beta = 3\alpha + \gamma$. Then the intermediate transformation (21) can be rewritten as

$$\frac{\mathrm{d}u}{\mathrm{d}\xi} = 4b\cosh^3\frac{u}{2}, \tag{30}$$

from which we have

$$\frac{\sinh\frac{u}{2}}{\cosh^2\frac{u}{2}} + \tan^{-1}(\sinh\frac{u}{2}) = 4b(\xi - \xi_0), \tag{31}$$

where ξ_0 is an integration constant.

So there are four solutions to the TShG equation (3). The first one is

$$u_1 = 2\sinh^{-1}v_1, \tag{32}$$

where v_1 is defined by

$$\frac{v_1}{1+v_1^2} + \tan^{-1}v_1 = 4(e_1 - e_2)(\xi - \xi_0), \tag{33}$$

the second one

$$u_2 = 2\sinh^{-1}v_2, \tag{34}$$

where v_2 is defined by

$$\frac{v_2}{1+v_2^2} + \tan^{-1}v_2 = 4(e_1 + e_2)(\xi - \xi_0), \tag{35}$$

the third one is

$$u_3 = 2\sinh^{-1}v_3, \tag{36}$$

where v_3 is defined by

$$\frac{v_3}{1+v_3^2} + \tan^{-1}v_3 = -4(e_1 + e_2)(\xi - \xi_0), \tag{37}$$

and the last one is

$$u_4 = 2\sinh^{-1}v_4, \tag{38}$$

where v_4 is defined by

$$\frac{v_4}{1+v_4^2} + \tan^{-1}v_4 = 4(-e_1 + e_2)(\xi - \xi_0). \tag{39}$$

The second case is that $a \neq 3b$, i.e. $4\beta \neq 3\alpha + \gamma$. Then the intermediate transformation (21) can be rewritten as

$$\frac{\mathrm{d}u}{\cosh\frac{u}{2}} - \frac{1}{2}\frac{\mathrm{d}\sinh\frac{u}{2}}{f + \sinh^2\frac{u}{2}} = (a - 3b)\mathrm{d}\xi,$$
$$f \equiv \frac{a+b}{4b}, \tag{40}$$

from which we have three kinds of results. If $f = 0$, i.e. $e_1 = \alpha + \frac{\gamma}{3} = 0$, $e_2 = \pm\sqrt{-\beta_1}$, then we have

$$u_5 = 2\sinh^{-1}v_5, \tag{41}$$

where v_5 is defined by

$$\tan^{-1}v_5 + \frac{1}{2v_5} = \pm 3\sqrt{-\beta_1}(\xi - \xi_0), \tag{42}$$

with the constraint

$$\beta_1 < 0, \tag{43}$$

where ξ_0 is an integration constant.

If $f > 0$, there are three cases under consideration, the first is

$$u_6 = 2\sinh^{-1}v_6, \tag{44}$$

where v_6 is defined by

$$\tan^{-1}v_6 - \frac{1}{2\sqrt{f}}\tan^{-1}(\frac{v_6}{\sqrt{f}})$$
$$= \pm 2(e_1 + 2e_2)(\xi - \xi_0), \tag{45}$$

with

$$f = \frac{e_1}{2(e_1 + e_2)}. \tag{46}$$

The second one is

$$u_7 = 2\sinh^{-1}v_7, \tag{47}$$

where v_7 is defined by

$$\tan^{-1} v_7 - \frac{1}{2\sqrt{f}} \tan^{-1}\left(\frac{v_7}{\sqrt{f}}\right) \\ = 2(-e_1 + 2e_2)(\xi - \xi_0), \tag{48}$$

with

$$f = \frac{e_1}{2(e_1 - e_2)}, \tag{49}$$

with the constraint

$$\beta_1 > 0. \tag{50}$$

And the third one is

$$u_8 = 2\sinh^{-1} v_8, \tag{51}$$

where v_8 is defined by

$$\tan^{-1} v_8 - \frac{1}{2\sqrt{f}} \tan^{-1}\left(\frac{v_8}{\sqrt{f}}\right) \\ = 2(e_1 - 2e_2)(\xi - \xi_0), \tag{52}$$

with

$$f = \frac{e_1}{2(-e_1 + e_2)}, \tag{53}$$

with the constraint

$$\beta_1 < 0. \tag{54}$$

If $f < 0$, there are two cases under consideration: The first one is

$$u_9 = 2\sinh^{-1} v_9, \tag{55}$$

where v_9 is defined by

$$\tan^{-1} v_9 - \frac{1}{4\sqrt{-f}} ln \frac{v_9 - \sqrt{-f}}{v_9 + \sqrt{-f}} \\ = 2(-e_1 + 2e_2)(\xi - \xi_0), \tag{56}$$

with

$$f = \frac{e_1}{2(e_1 - e_2)}, \tag{57}$$

with the constraint

$$\beta_1 < 0. \tag{58}$$

The second one is

$$u_{10} = 2\sinh^{-1} v_{10}, \tag{59}$$

where v_{10} is defined by

$$\tan^{-1} v_{10} - \frac{1}{4\sqrt{-f}} ln \frac{v_{10} - \sqrt{-f}}{v_{10} + \sqrt{-f}} \\ = 2(e_1 - 2e_2)(\xi - \xi_0), \tag{60}$$

with

$$f = \frac{e_1}{2(-e_1 + e_2)}, \tag{61}$$

with the constraint

$$\beta_1 > 0. \tag{62}$$

The solutions u_1 to u_{10} derived here have not be reported in the literature. These solutions are all implicit, there are still many efforts required in further studies.

4. Conclusions

In this paper, two transformations have been introduced to solve the DShG equation (2) and the TShG equation (3). It is shown that different transformations are required in order to obtain solutions to different types of sinh-Gordon equations. The solutions obtained here have not been previously reported in the literature. Of course, there are still more efforts needed to explore, which kinds of transformations are more suitable to solve different types of sinh-Gordon equations, for not all solutions obtained here are explicit ones.

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