Special Relativity via Modified Bessel Functions

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Z. Naturforsch. **55 a**, 745–753 (2000); received May 11, 2000

The recursive formulas of modified Bessel functions give the relativistic expressions for energy and momentum. Modified Bessel functions are solutions to a continuous time, one-dimensional discrete jump process. The jump process is analyzed from two inertial frames with a relative constant velocity; the average distance of a particle along the chain corresponds to the distance between two observers in the two inertial frames. The recursion relations of modified Bessel functions are compared to the 'k calculus' which uses the radial Doppler effect to derive relativistic kinematics. The Doppler effect predicts that the frequency is a decreasing function of the velocity, and the Planck frequency, which increases with velocity, does not transform like the frequency of a clock. The Lorentz transformation can be interpreted as energy and momentum conservation relations through the addition formula for hyperbolic cosine and sine, respectively. The addition formula for the hyperbolic tangent gives the well-known relativistic formula for the addition of velocities. In the non-relativistic and ultra-relativistic limits the distributions of the particle's position are Gaussian and Poisson, respectively.

Key words: Special Relativity; Recursion Relations of Modified Bessel Functions; Lattice Jumps; Size and Mass of an Electron; Doppler Effect.

1. Introduction

At the beginning of the last century there was considerable interest in the origin of the mass of an electron [1]. Experiments that measured the charge to mass ratio definitely showed that the mass increased sharply with the speed of an electron [2]. It was even suggested that the entire mass of an electron is electromagnetic in origin [3]. An electron in motion produces a magnetic field about its line of flight. The magnetic field has an energy associated with it. Energy is required to set the electron in motion so that mass can be associated with an electron because of the fields that it creates. This mass is entirely electromagnetic. Experiments were performed to select the correct model of an electron; the contenders were the Lorentz model, which was indistinguishable in its predictions from Einstein's special theory of relativity, and the Abraham model [4], whose aim was to provide for an electrodynamic foundation for all of mechanics.

These classical models of an electron have all but been abandoned [5] because they appear to introduce more problems than they solve. The contradictions of an electron with a finite extension and relativistic causality are well-known [6]. It is the purpose of this article to point out, however, that the recursive formulas of modified Bessel functions give the correct special relativistic expressions for the energy and the momentum. Modified Bessel functions occur in a wide variety of problems in probability theory when the times at which the jumps of a random walk occur are randomized [7]. In other words, modified Bessel functions make their appearance when the times of the steps in a discrete time random walk are randomly distributed according to a Poisson process. In this way, a one-dimensional probabilistic model of special relativity presents itself in terms of random jumps along a linear lattice. The jumping electron accelerates and de-accelerates emitting radiation which is analyzed in an inertial frame moving relative to the lattice. The average displacement of the particle along the lattice, in a given time interval, coincides with the distance between two observers in two inertial frames moving relative to one another.

A discrete jump process permits the electric charge to have a finite extension in space. Consider an electron as a rigid object of finite dimension [6]. When a pulse of radiation strikes one side of the surface of the electron it is instantaneously set into motion.

This implies that the impulse had to be transmitted instantaneously across the diameter of the electron, and this contradicts the relativistic law of causality. However, by considering events separated by a distance of the order of the particle's Compton wavelength, the smallest time that a signal can be transmitted between neighboring points on the lattice is the time it takes light to cross the particle's Compton wavelength. In other words, by dispensing with all knowledge of the process between the lattice points, perhaps due in part to the limitations of our measuring apparatus, we admit that there can be only a finite rate of change. Consequently, there is nothing to prohibit an electron having a finite extension in space since signals transmitted over such distances would not be open to observation.

But, cannot the Compton wavelength be reduced still further to the classical electron radius? Once a universal length, r_0 say, is specified, it can then be combined with the other two fundamental constants, $\hbar$ and c, to produce a quantity which has dimensions of mass, $\hbar/r_0 c$. If the value e^2/mc^2 is assigned to r_0, a further constant must be introduced, namely the electric charge, e. The fact that the introduction of the classical electron radius requires an additional constant, led Heisenberg [8] to conclude that the specification of the charge is extraneous to the specification of a universal length, or elementary mass. Only after the nature of the universal length has been clarified can the question of electronic charge be addressed. Moreover, since the Compton wavelength is $\hbar c/e^2$ times greater than the classical electron radius, ample room is left for an electron of finite extension.

2. Bessel Functions and Random Walks

Consider an infinite chain of regularly spaced masses m_0. A particle will be able to jump from one mass point to another, and when it does it emits a signal of frequency ϖ. This frequency should characterize the particle in its rest frame. The only nonvanishing energy is the rest energy, $m_0 c^2$, and when it is divided by Planck's constant, we obtain the frequency $\varpi = m_0 c^2/\hbar$. From these three constants we can form a length, namely the Compton length $\Lambda = \hbar/m_0 c$, and it determines the spacing between the mass points.

Consider a frame $\tilde{k}$ which moves at a velocity v with respect to the frame k of the linear lattice. The relation between the coordinates $(r, \varpi t)$ in the frame k and the coordinates $(\tilde{r}, \varpi\tilde{t})$, in the frame $\tilde{k}$, is given in the most general form by the formulas:

$$r = \tilde{r}\cosh\theta + \varpi\tilde{t}\sinh\theta \qquad (1)$$
$$\text{and } \varpi t = \varpi\tilde{t}\cosh\theta + \tilde{r}\sinh\theta,$$

where the 'angle' θ can depend only on the relative velocity of the two frames. In particular, if we consider the motion of the origin of the $\tilde{k}$ frame ($\tilde{r} = 0$), with respect to the k frame, we obtain the relative velocity:

$$\beta = r/\varpi t = q/ct = \tanh\theta, \qquad (2)$$

where $\beta = v/c$, and $q = r\Lambda$ is distance from the origin of the k frame.

The particle's position r along the chain coincides with the distance of two observers in frames k and $\tilde{k}$. The advantage of introducing the frame $\tilde{k}$ is that it will allow us to determine the relationship between two events that occur in $\tilde{k}$ at one point, $\tilde{r} = 0$, in space, and registered by one clock using the proper time interval, $\tilde{t}$, and the time interval between the same events, t, as registered by two clocks in k, in which the two events occur at different points. If the particle starts at the origin and gets to r, at time $\tilde{t}$, then among the n jumps that were made, $\frac{1}{2}(n+r)$ had to have been positive, and $\frac{1}{2}(n-r)$ negative. In order that these values be integers, $n - r = 2j$ must be even. This is the number of reversals that has occurred. Given equal probabilities for a jump to the left and the right, the probability to be at position $r \geq 0$ just after the nth jump is [7]

$$\binom{n}{\frac{1}{2}(n+r)} 2^{-n} = \binom{r+2j}{r+j} 2^{-n}.$$

Given the probability that $n = r + 2j$ jumps have occurred up until time $\tilde{t}$ is Poisson, $(\varpi\tilde{t})^n e^{-\varpi\tilde{t}}/n!$, the probability to be at $r \geq 0$ at time $\tilde{t}$ is [7]

$$e^{-\varpi\tilde{t}} \sum_{j=0}^{\infty} \frac{\left(\frac{1}{2}\varpi\tilde{t}\right)^{r+2j}}{(r+2j)!} \binom{r+2j}{r+j} \qquad (3)$$
$$= e^{-\varpi\tilde{t}} I_r(\varpi\tilde{t}) = P_r(\tilde{t}).$$

Averaging is required since we do not know how many jumps it will take to reach r. It is this randomization of the time steps, which is accomplished by the Poisson process, that converts a discrete into a continuous time random walk, and has brought in the modified Bessel function of order r, $I_r(\varpi\tilde{t})$. The symmetry of modified Bessel function, $I_{-r}(\varpi\tilde{t}) = I_r(\varpi\tilde{t})$, for integer values of r, and the sum $\sum_{r=-\infty}^{\infty} I_r(\varpi\tilde{t}) = e^{\varpi\tilde{t}}$ guarantee that the probability density $P_r(\tilde{t})$ is normalized.

We will now prove that (3) is the solution to a one-dimensional random walk. For simplicity we assume that a step to the left or the right occurs with equal probability $\frac{1}{2}\varpi\, d\tilde{t}$ in time $d\tilde{t}$. The continuity, or master, equation

$$\mathrm{d}_{\tilde{t}} P_r(\tilde{t}) = \tfrac{1}{2}\varpi \left(P_{r+1}(\tilde{t}) - 2P_r(\tilde{t}) + P_{r+1}(\tilde{t})\right), \quad (4)$$

has the usual initial condition that the walker starts at the origin, $P_r(0) = \delta_{0,r}$. At the initial instant, the readings of the two clocks of k and $\tilde{k}$ coincide since the two observers are at the same point. Afterwards, the frame $\tilde{k}$ will move away from the source of radiation located in the frame k at a constant velocity.

The solution to (4) is most conveniently obtained by employing the method of generating functions [9]. The generating function

$$\mathcal{G}(z,\tilde{t}) = \sum_{r=-\infty}^{\infty} z^r P_r(\tilde{t})$$

satisfies the boundary conditions $\mathcal{G}(z,0) = 1$ and $\mathcal{G}(1,\tilde{t}) = 1$. Multiplying the master equation (4) by z^r and summing result in a first order differential equation whose solution is:

$$\mathcal{G}(z,\tilde{t}) = \exp\left\{-\varpi\tilde{t} + \tfrac{1}{2}\varpi\tilde{t}\left(z + z^{-1}\right)\right\}.$$

This expression for the generating function is comparable with that of a modified Bessel function,

$$e^{\frac{1}{2}\varpi\tilde{t}(z+z^{-1})} = \sum_{r=-\infty}^{\infty} z^r I_r(\varpi\tilde{t}), \quad (5)$$

which is sometimes used as the definition of $I_r(\varpi\tilde{t})$. Consequently (3) is the solution to the master equation (4).

Introducing (3) into the master equation (4) gives the well-known recursion relation [10]

$$\mathrm{d}_{\tilde{t}} I_r(\varpi\tilde{t}) = \tfrac{1}{2}\varpi \left(I_{r-1}(\varpi\tilde{t}) + I_{r+1}(\varpi\tilde{t})\right) \quad (6)$$

for modified Bessel functions. The recursion relation is easily verified from the generating function (5). Differentiating (5) with respect to $\tilde{t}$, and equating the coefficients of z^r gives (6). A second recursive formula can be obtained by differentiating the generating function (5) with respect to z. Equating coefficients of z^{r-1} equal to zero results in [10]

$$r I_r(\varpi\tilde{t}) = \tfrac{1}{2}\varpi\tilde{t}\left(I_{r-1}(\varpi\tilde{t}) - I_{r+1}(\varpi\tilde{t})\right). \quad (7)$$

Writing the dummy variable in the expression for the generating function as $z = e^{\theta}$, (5) becomes

$$\mathcal{G}(\theta,\tilde{t}) = \sum_{r=-\infty}^{\infty} e^{\theta r} I_r(\varpi\tilde{t}) = e^{\varpi\tilde{t}\cosh\theta}. \quad (8)$$

Multiplying both sides of (6) by $e^{\theta r}$ and summing over all r result in

$$\partial_{\tilde{t}} \ln\mathcal{G} = \varpi\tfrac{1}{2}\left(e^{\theta} + e^{-\theta}\right) = \varpi\cosh\theta = \omega, \quad (9)$$

which defines the frequency ω. Multiplying the second recursion relation (7) by $e^{\theta r}$ and summing give

$$r/\varpi\tilde{t} = q/c\tilde{t} = \tfrac{1}{2}\left(e^{\theta} - e^{-\theta}\right) = \sinh\theta. \quad (10)$$

This coincides with the first moment of the distribution

$$\partial_{\theta} \ln\mathcal{G} = \varpi\tilde{t}\,\sinh\theta = r, \quad (11)$$

which determines the average distance that the particle is from the origin. If (11) is evaluated at $\theta = 0$ ($z = 1$), as is usually done when θ has no physical meaning, the particle will, on the average, show no tendency to wander from the origin at the proper time $\tilde{t}$. This implies that θ can be a function only of the relative velocity of the two inertial frames.

Equation (11) sheds new light on the meaning of the Lorentz transformation as specifying the mean position of a particle executing a random walk. It coincides with first equation in (1) when the motion is considered in the k frame of the origin of the $\tilde{k}$ frame ($\tilde{r} = 0$). However, (9) does not coincide with the second equation in (1) under the same condition. Converting frequencies into periods of the motion, $\omega = 2\pi/t$ and $\varpi = 2\pi/\tilde{t}$, results in $\tilde{t} = t\cosh\theta$, and not $t = \tilde{t}\cosh\theta$ at $\tilde{r} = 0$, as given by (1). It will turn out that (9) is the correct relativistic expression for the energy, but it does not transform like the frequency of a clock.

3. Recursion Relations and the Doppler Effect

An elegant method of deriving relativistic kinematics is the so-called 'k calculus', which is based entirely on the radial Doppler effect in one-dimension [11]. The k calculus is completely equivalent to the Lorentz transformation, and enjoys the added advantage of dispensing with the necessity of having to introduce different sets of coordinate axes. It consists of sending, reflecting and receiving light signals between two observers in two inertial frames. An observer moving at a constant velocity relative to a source registers a frequency different from the frequency emitted by the source. The source of radiation is the radiating electron when it accelerates in making a jump from one lattice site to another. One of the observers, O, is placed at the source, in the frame k, and the other observer, $\tilde{O}$, is moving relative to the radiating source at a constant velocity v in frame $\tilde{k}$. This is entirely equivalent to an electron moving with an average velocity $-v$ in frame k with respect to a stationary frame $\tilde{k}$. If radiation is emitted periodically with period T, $\tilde{O}$ in $\tilde{k}$ will receive these signals in a different time interval, as measured by his own clock. If T is the period in which the signals are emitted, then kT will be the period in which they are received. These periods are measured by clocks at rest in frames k and $\tilde{k}$, respectively. Without knowing the specific form of k, we know that it can only depend upon the relatively velocity between the two frames. This is a consequence of the Doppler effect: the change in frequency depends only on the relative motion.

If O sends out signals in intervals T and $\tilde{O}$ receives them in intervals kT, then, by the equivalence of all inertial frames, O will receive signals sent out by $\tilde{O}$ in intervals kT when $\tilde{O}$ sends them out in intervals T. This has the important consequence that signals sent out by O in intervals T, received by $\tilde{O}$ in intervals kT and reflected by him in intervals T will be received back at O in intervals $k(kT)$. This is to say that the time interval on the return journey will again be increased by an amount k. Hence if O sends out a signal at time T to $\tilde{O}$, which is immediately reflected back to O he will receive it in time $(k^2-1)T$. The time that it takes a signal to propagate between these two observers is $\frac{1}{2}(k^2-1)T$. And because the velocity of light is the same in both directions, the distance between the two observers is

$$q = \tfrac{1}{2}(k^2-1)cT. \tag{12}$$

We must now determine the time at which the observer $\tilde{O}$ reflected the signal, as measured by the observer O's clock. Since the event occurs at a position other than where O's clock is located, this time interval cannot be measured by, but rather must be ascribed to, O's clock. The signal was sent out in time T and received back in time k^2T so that the moment it was reflected is their average:

$$t = \tfrac{1}{2}(k^2+1)T. \tag{13}$$

Since both the distance (12) and time separation (13) refer to a single frame, their ratio determines the relative velocity, $c\beta$, where

$$\beta = \frac{k^2-1}{k^2+1}, \tag{14}$$

of observer $\tilde{O}$ with respect to O. Rearranging (14) yields

$$k = \sqrt{\frac{1+\beta}{1-\beta}}. \tag{15}$$

This is as far as the k calculus goes in determining the form of k [11]. However, it is already apparent that an exponential factor is involved since a change in the sign of the relative velocity transforms k into $1/k$.

In order to find the functional form of k, we consider three inertial frames k, $\tilde{k}$ and $\hat{k}$. The factor $k(O,\tilde{O})$ will depend only on the relative velocity between the frames k and $\tilde{k}$, while $k(O,\hat{O})$ will depend only on the relative velocity between frames k and $\hat{k}$. If a light signal is sent from O to $\tilde{O}$ and immediately on to $\hat{O}$, it will require the same time as a light signal sent from O directly to $\hat{O}$ due to the constant speed of light. The equivalence of their time intervals

$$k(O,\hat{O})T = k(O,\tilde{O})\,k(\tilde{O},\hat{O})T$$

implies that k is exponential: $k = e^{\theta}$, where θ can depend only on the relative velocity of the two frames. In other words, the time magnification factor, k, of the radial Doppler effect is the exponential of an *imaginary* phase shift between neighboring lattice sites. Consequently, the relative velocity (14) between frames k and $\tilde{k}$ is (2). Upon solving (2) for θ gives

$$e^{\theta} = \sqrt{\frac{1+\beta}{1-\beta}}, \tag{16}$$

which is (15). Furthermore, (12) is now seen to coincide exactly with the average distance the particle moves along the chain, (11), remembering that $q = r\Lambda$ and $\tilde{t} = kT$ is the proper time. In other words, we can determine the average distance covered by the particle along the chain by an observer moving in an inertial frame with respect to a stationary source.

It is well-known that the addition formula for the hyperbolic tangent accounts for the relativistic addition law for velocities. If $\tilde{v}$ is the relative velocity between frames k and $\tilde{k}$, and $\hat{v}$ is the relative velocity between frames $\tilde{k}$ and $\hat{k}$, then the relativistic law of addition of velocities is $c\beta$ where

$$\beta = \tanh\left(\tilde{\theta} + \hat{\theta}\right) = \frac{\tanh\tilde{\theta} + \tanh\hat{\theta}}{1 + \tanh\tilde{\theta}\,\tanh\hat{\theta}} = \frac{\tilde{\beta} + \hat{\beta}}{1 + \tilde{\beta}\,\hat{\beta}}.$$

Although the *a priori* probabilities for a jump to the left and to the right are equal, the exponential factor on the left-hand side of (16) is related to the probability of an electron taking a jump to the right, while $e^{-\theta}$ is related to the probability that an electron will take a jump to the left. The jump consists in accelerating the electron, either to the right or to the left. An accelerating electron radiates energy, and it is the frequency of this radiation that gets Doppler shifted. If the radiation is emitted in the direction of the observer then the time it will take to reach him is $e^{-\theta}T$, with a corresponding increase in the frequency. Analogously, if the transition is in the opposite direction, the frequency will be shifted toward the red, requiring a longer time to arrive, $e^{\theta}T$.

If O sends light signals at the interval T, we have seen in (13) that the time it takes to reach $\tilde{O}$ is $\frac{1}{2}\left(e^{2\theta} + 1\right)T$, as registered by two clocks in k. The ratio of this time interval to the proper time interval, $\tilde{t} = e^{\theta}T$, in the frame $\tilde{k}$ is:

$$t = \tilde{t}\cosh\theta. \tag{17}$$

Transforming from time intervals to frequencies, (17) becomes:

$$\omega = \varpi\sqrt{1 - \beta^2}. \tag{18}$$

Consequently, (17) gives the correct transformation of the frequency of a clock, or time dilation due to viewing a moving clock. Furthermore, since the relative velocities in the two frames are equal and opposite, $q/\tilde{q} = t/\tilde{t}$, (17) is also the expression for the Fitzgerald–Lorentz contraction, $\tilde{q} = q\sqrt{1 - \beta^2}$.

The ratio of the distance (12), measured in k, to the proper time of $\tilde{k}$ is

$$q/\tilde{t} = c\sinh\theta, \tag{19}$$

which is precisely the recursion relation (10). This relation could also be obtained from (17) and $q = vt$ ($\tilde{q} = 0$), where the relative velocity is given by (2). Why then does (17) give the incorrect energy relation when Planck's energy-frequency relation is used? Instead of the coordinate 'two-vector' $(\tilde{q}, ic\tilde{t})$, consider momentum two-vector, $(\tilde{p}, i\tilde{E}/c)$. Since the $\tilde{k}$ frame is at rest, only $\tilde{E}$ does not vanish. According to the Lorentz transformation, we have momentum and energy in the k frame given by $p = (\tilde{E}/c)\sinh\theta$ and $E = \tilde{E}\cosh\theta$, respectively. With the proper frequency given by $\varpi = \tilde{E}/\hbar$, we find

$$\omega = \frac{\varpi}{\sqrt{1 - \beta^2}}, \tag{20}$$

and $p = m_0 c\beta/\sqrt{1 - \beta^2}$. Combining the two relations, we obtain $\omega = (c/\beta)\kappa$, where c/β will later be identified as the phase velocity [*cf.*, (21) below]. The correct expression for the momentum is arrived at independently of the expression for the frequency. According to special relativity, the ratio of the momentum to the total energy, $\hbar\omega$, is proportional to the velocity, $c^2 p/\hbar\omega = v$. The recursion relation (9), or equivalently (20), gives $p = (\omega/\varpi)m_0 v$, while (18) gives the inverse relation $p = (\varpi/\omega)m_0 v$. Whereas the Doppler effect predicts that the frequency decreases with the velocity, the frequency (20) does not transform like the frequency of a clock (18).

In order to get the correct velocity dependence on the frequency the Lorentz transformation has been used in conjunction with the wave associated with the motion [12]. In a stationary frame, the phase of the wave is $\varpi\tilde{t}$. Viewed from another inertial frame with a relative velocity u, the phase becomes $\varpi\tilde{t} = \varpi\{t\cosh\theta - (q/c)\sinh\theta\} = \omega\left(t - q/u\right)$, with a frequency (20) and a phase speed

$$u = c^2/v = c\coth\theta. \tag{21}$$

The frequency relation (20) is the *inverse* of (18). As de Broglie concluded, "the difference between the relativistic variations of the frequency of a clock and the frequency of a wave is fundamental" [12]. Moreover, the relative velocity of the traveling wave

(21) is not the particle velocity; rather, it is the inverse of (2).

The fact that special relativity actually predicts that the frequency will be *increased* by the motion, and not *decreased* by it, caught de Broglie's attention and made it a focus of his research [12]. That wave amplitudes should only depend upon space and time through the combination $(t - q/u)$ introduces a phase velocity u which make them unsuitable for the transmission of signals because if the particle velocity is less than the speed of light, the phase speed will certainly be greater than the speed of light. de Broglie assumed an equivalence between the phase of the 'regulator' clock, that is associated with the particle, and the phase of the wave phenomenon that is associated with it. The phase, $\varpi\tilde{t} = \omega t - \kappa q$, has frequency (20) and wave number

$$\kappa c = \frac{\varpi\beta}{\sqrt{1-\beta^2}} = \varpi \sinh\theta, \qquad (22)$$

which is identical to the second recursion relation (10) of the modified Bessel functions for a wave number given by $\kappa = r/c\tilde{t}$. Dividing (20) by (22) does, in fact, give the phase velocity (21). Since the phase velocity is ω/κ, both positive and negative traveling waves can be obtained by keeping the frequency positive and letting the wave number κ assume both positive and negative values. Squaring both (20) and (22) and subtracting the latter from the former give

$$\omega^2 - (\kappa c)^2 = \varpi^2. \qquad (23)$$

Interpreting κ as the density of waves and ω as the flux of waves,

$$\partial_t \kappa + \partial_r \omega = 0$$

represents the conservation of waves [13]. It is equivalent to the expression for the group, or particle, velocity (2) since

$$v = -\frac{\partial_t \theta}{\partial_r \theta} = c \tanh\theta.$$

Rather, had we considered the dispersion equation for the master equation (4) we would have obtained:

$$\omega = -\varpi + \sqrt{\varpi^2 + (c\kappa)^2}, \qquad (24)$$

with the convention that the frequency be kept positive while the wave number can take on both negative, as well as positive, values. The group velocity

$$v = \frac{\mathrm{d}\omega}{\mathrm{d}\kappa} = \frac{c^2\kappa}{\sqrt{\varpi^2 + (c\kappa)^2}},$$

remains the same, but the product of the modified phase velocity and the group velocity

$$u \cdot v = \left(1 - \frac{\varpi}{\sqrt{\varpi^2 + (c\kappa)^2}}\right) c^2$$

shows that both the phase and group velocities are less than the speed of light. Such a wave can be used for signal transmission. Whereas the dispersion relation (23) corresponds to the Klein-Gordon equation,

$$c^2 \partial_q^2 \psi - \partial_t^2 \psi = \varpi^2 \psi,$$

for the wave amplitude ψ, (24) is equivalent to the telegrapher's equation,

$$-\partial_t^2 \psi + 2i\varpi \partial_t \psi = -c^2 \partial_q^2 \psi,$$

which has an intermediate position between the non-relativistic Schrödinger equation and the relativistic Klein-Gordon equation. The former is obtained in the limit $\varpi \gg c\kappa$.

Consider the Lorentz transformation law for momentum and energy:

$$\begin{pmatrix} \tilde{p}/m_0 c \\ i\tilde{E}/m_0 c^2 \end{pmatrix} = \begin{pmatrix} \cosh\theta' & -i\sinh\theta' \\ i\sinh\theta' & \cosh\theta' \end{pmatrix} \cdot \begin{pmatrix} p/m_0 c \\ iE/m_0 c^2 \end{pmatrix}. \qquad (25)$$

Since $p = m_0 c \sinh\theta$ and $E = m_0 c^2 \cosh\theta$, the Lorentz transform (25) implies $\tilde{p} = m_0 c \sinh(\theta + \theta')$ and $\tilde{E} = m_0 c^2 \cosh(\theta + \theta')$. The fact that the determinant of the Lorentz transformation is equal to unity is the condition for energy conservation, *viz.*,

$$\frac{E'^2}{(m_0 c^2)} - \frac{p'^2}{(m_0 c)^2} = \frac{E^2}{(m_0 c^2)} - \frac{p^2}{(m_0 c)^2} = 1. \qquad (26)$$

To demonstrate that (25) does in fact imply the conservation of energy, we write its components out

$$\frac{\tilde{E}}{m_0c^2} = \frac{E'E}{(m_0c^2)^2} + \frac{p'p}{(m_0c)^2}$$

and

$$\frac{\tilde{p}}{m_0c} = \frac{E'}{m_0c^2}\frac{p}{m_0c} + \frac{E}{m_0c^2}\frac{p'}{m_0c}.$$

Squaring both expressions and subtracting the latter from the former gives (26) for both the primed and unprimed sets of terms.

To conclude this section, we consider the Compton effect in the more general case where the electron is in motion prior to its collision with the photon. If λ and λ' are the wavelengths of the photon before and after collision, and ϕ is the angle of deviation of the photon, then energy and momentum conservation yield the relation between the two wavelengths as:

$$\lambda' \cosh\theta - \lambda\, e^{-\theta} = 2\Lambda \sin^2(\phi/2), \qquad (27)$$

where θ depends on the velocity of the electron prior to collision. If the electron is at rest then (27) reduces to the ordinary Compton effect. However, for large initial velocities, (27) becomes:

$$\lambda' = 4\Lambda \sin^2(\phi/2)\sqrt{\frac{1-\beta}{1+\beta}}, \qquad (28)$$

where we have used (16), and the initial velocity of the electron is $c\beta$. Expression (28) has the form of the radial Doppler shift in the wavelength. The wavelength of the incoming photon has disappeared and the wavelength λ', represents the shift in wavelength of $4\Lambda\sin^2(\phi/2)$ due to the initial velocity of the electron.

4. Relativistic Limits via Integral Bessel Formula

The modified Bessel function can be represented by the complex integral [10, p. 181]

$$I_r(\varpi\tilde{t}) = \frac{1}{2\pi i}\int_{\infty-\pi i}^{\infty+\pi i} e^{\varpi\tilde{t}\cosh\theta - r\theta}\,\mathrm{d}\theta \qquad (29)$$

when $|\arg(\varpi\tilde{t})| \le \frac{1}{2}\pi$, where the equality sign holds for $r > 0$. The contour is made up of three sides of a rectangle with vertices at $\infty-\pi i$, $-\pi i$, πi and $\infty+\pi i$. We will consider real $\tilde{t}$. The function

$$S(\theta) = r\theta - \varpi\tilde{t}\cosh\theta \qquad (30)$$

has a maximum at $\theta = \sinh^{-1}(r/\varpi\tilde{t})$, which is none other than (10). Transforming from proper time to the time in the frame at rest, $\tilde{t} = t\cosh\theta - (q/c)\sinh\theta$, gives the stationary condition as $\beta = \tanh\theta$, which is the same condition since $\tanh^{-1}\beta = \sinh^{-1}\left(\beta/\sqrt{1-\beta^2}\right)$. Introducing this stationary point into (30) results in

$$S(r,\tilde{t}) = r\sinh^{-1}(q/c\tilde{t}) - \varpi\tilde{t}\sqrt{1+(r/\varpi\tilde{t})^2}. \qquad (31)$$

This is precisely the expression that appears when (29) is evaluated by the method of steepest descent [14]

$$I_r(\varpi t) \sim \frac{e^{-S(r,\tilde{t})}}{\sqrt{2\pi\varpi\tilde{t}\sqrt{1+(r/\varpi\tilde{t})^2}}}. \qquad (32)$$

The function (31) plays a role analogous to a classical action for a path. The derivative of (31) with respect to q gives

$$\Lambda\partial_q S = \sinh^{-1}(q/c\tilde{t}) = \theta. \qquad (33)$$

Ordinarily, we would identify (33) with the wave number, but from the condition of the extremum of (30) we are prevented from doing so. However, the derivative of (31) with respect to time is still the negative of the frequency

$$\partial_{\tilde{t}} S = -\varpi\sqrt{1+(q/c\tilde{t})^2} = -\omega, \qquad (34)$$

which is seen to be (9) when (33) is introduced. Hence, the action (31) may be written as:

$$S(q,\tilde{t}) = \frac{q}{\Lambda}\sinh^{-1}\left(\frac{\beta}{\sqrt{1-\beta^2}}\right) - \frac{\varpi\tilde{t}}{\sqrt{1-\beta^2}}.$$

Introducing the asymptotic form of the modified Bessel function (32) into the expression for the probability density (3) gives:

$$P_r(\varpi\tilde{t}) = e^{-\varpi\tilde{t}} I_r(\varpi\tilde{t}) \qquad (35)$$

$$\sim \frac{e^{\varpi\tilde{t}\left(\sqrt{1+(r/\varpi\tilde{t})^2}-1\right) - r\sinh^{-1}(r/\varpi\tilde{t})}}{\sqrt{2\pi\varpi\tilde{t}\sqrt{1+(r/\varpi\tilde{t})^2}}}.$$

It is well-known that in the limit $r \ll \varpi\tilde{t}$, (35) tends to the Gaussian probability density [9]

$$P_r(t) \sim \frac{e^{-r^2/2\varpi t}}{\sqrt{2\pi\varpi t}} \quad (36)$$

for the displacement of the particle. The action, $S(q,t) = m_0 q^2/2t\hbar = p^2 t/2m_0\hbar$, corresponds to that of a non-relativistic free particle. There is no longer any distinction between the time intervals in the two frames; Galilean invariance prevails. From the logarithm of the generating function, $\ln\mathcal{G} = \frac{1}{2}\varpi t\theta^2$, the average distance covered by the particle in time t is found to be $\partial_\theta \ln\mathcal{G} = \varpi t\theta$. It tends to zero as θ does; the particle, on the average, will be found at the origin in the non-relativistic limit where the drift tends to zero.

The frequency and angle are given by

$$\partial_t S = -\frac{1}{2\varpi}\left(\frac{r}{t}\right)^2 = -\omega \quad (37)$$

and

$$\partial_r S = \frac{1}{\varpi}\frac{r}{t} = \beta = \theta, \quad (38)$$

respectively. Solving (38), which is the first term in the series expansion for $\sinh\theta$ [*cf.*, (10)], for the ratio r/t and introducing it into (37) gives the first term in the power series expansion of $\cosh\theta$, *viz.*, $\omega/\varpi = \frac{1}{2}\theta^2$ [*cf.*, (9)]. The velocity dependence on the frequency, $\omega \sim \varpi\beta^2/2$, has nothing to do with the radial Doppler effect, which for small velocities would be $\omega \sim\varpi(1-\beta)$.

In the opposite limit $q \gg c$, the asymptotic form of the modified Bessel function (32) reduces to $I_r(\varpi\tilde{t}) \sim (\frac{1}{2}\varpi\tilde{t})^r/r!$, where Stirling's approximation $r! \sim \sqrt{2\pi r}e^{-r}r^r$ has been used. In comparison with the master equation (4), where steps to the left and to the right occur with equal probability, steps to the left have now a vanishing probability. The master equation is now reduced to:

$$d_{\tilde{t}} P_r(\tilde{t}) = \varpi'\left\{P_{r-1}(\tilde{t}) - P_r(\tilde{t})\right\}, \quad (39)$$

where $\varpi' = \frac{1}{2}\varpi$. Steps are now taken only to the right with a lattice spacing twice as great, but still at random times. Normalization of the asymptotic modified Bessel function leads to the Poisson distribution:

$$P_r(\tilde{t}) \sim \frac{(\varpi'\tilde{t})^r}{r!}e^{-\varpi'\tilde{t}} \quad (40)$$

in the ultra-relativistic limit $q \ll c\tilde{t}$, or, equivalently, $p \gg m_0 c$.

Relativistic trajectories look quite different from non-relativistic ones: the Brownian paths, corresponding to (36), get straightened out. The logarithm of the generating function of the Poisson distribution, $\ln\mathcal{G} = \varpi'\tilde{t}\left(e^\theta - 1\right)$, gives the average position of the particle as:

$$\partial_\theta \ln\mathcal{G} = \varpi'\tilde{t}\,e^\theta = r, \quad (41)$$

which, unlike (11), does not vanish even when $\theta = 0$. The average distance covered by the particle, or, equivalently, the distance that observer $\tilde{O}$ has moved away from O in the time interval $\frac{1}{2}k^2T$, is $q = \frac{1}{2}k^2cT$. This is the limiting expression for (12) for $k \gg 1$.

From the dimensionless action,

$$S(r,\tilde{t}) = -r\left\{1 + \ln\left(\frac{\varpi'\tilde{t}}{r}\right)\right\},$$

for the Poisson distribution (40), the expressions for frequency and angle are found to be

$$\partial_{\tilde{t}} S = -r/\tilde{t} = -\omega, \quad (42)$$

and

$$\partial_r S = \ln\left(\frac{r}{\varpi'\tilde{t}}\right) = \theta, \quad (43)$$

respectively. According to (11) and (22), $r/\tilde{t} = \kappa c$ so that (42) is the expression for the ultra-relativistic energy $\hbar\omega = pc$. Expression (43) can thus be written as $p = m_0 c\,e^\theta$. Introducing (42) into (43) and using the definition of the angle θ, (16), result in:

$$\omega = \varpi'\sqrt{\frac{1+\beta}{1-\beta}}, \quad (44)$$

which is the exact relativistic equation describing the radial Doppler effect.

The non-relativistic limit, therefore, corresponds to long wavelengths which are completely insensitive to the lattice spacing. Alternatively, in the ultra-relativistic limit, corresponding to extremely short wavelengths, the particle motion is discontinuous, and the lattice spacing, 2Λ, is the minimum wavelength of an electron. Nothing can be said about the motion of the electron in between the lattice spacing, and this gives the electron its finite extension. An impulse could not be transmitted instantaneously across the electron since we would have no information on the position of the electron in lengths smaller than Λ. The time required for light to cross the particle's Compton length is $2\pi/\varpi$. This is the smallest time interval possible; the Doppler effect (17) requires all other time intervals to be greater.

Acknowledgements

This work is supported, in part, by contributions from ex-40% and 60% M.U.R.S.T.

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