Effect of Modulation on Thermal Convection Instability

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The linear stability problem for a fluid in a classic Benard configuration is considered. The applied temperature gradient is the sum of a steady component and a time-dependent periodic component. Only infinitesimal disturbances are considered. The time-dependent perturbation is expressed in Fourier series. The shift in critical Rayleigh number is calculated and the modulating effect of the oscillatory temperature gradient on the stability of the fluid layer is examined. Some comparison is made with known results.

1. Introduction

This paper concerns the stability of a fluid layer confined between two horizontal planes and heated from below as well as from above in a periodic manner with time. Chandrasekhar [1] has given a comprehensive review of this stability problem. Donnelly [2] has investigated experimentally the circular couette flow, i.e. the flow between two coaxially rotating cylinders (Taylor instability) when the inner cylinder has a velocity which varies periodically with time while the outer cylinder is at rest. He found that the onset of instability is delayed by the modulation of the angular speed of the inner cylinder with the degree of stabilization rising from zero at high frequency to a maximum at a frequency of 0.274 (ν/d^2), where d is the gap between the cylinders and ν is the kinematic viscosity.

Since the problem of Taylor stability and Benard stability are very similar, Venezian [3] has worked out the thermal analogue of Donnelly's experiment. One of the aims of Venezian's paper is to compare his solution with some experimental results, obtained by Donnelly. Venezian's theory does not give any such finite frequency as obtained by Donnelly, but he found that for the case of modulation only at the lower surface, the modulation would be stabilizing with maximum stabilization occuring as the frequency goes to zero. However in his explanation it was suggested by Venezian that linear stability theory ceases to be applicable when the frequency of modulation is sufficiently small.

Rosenblat and Herbert [4] have investigated the linear stability problem in the case of low modulation frequency and provided an asymptotic solution of the problem. They employed the periodicity criterion and amplitude criterion to calculate the critical Rayleigh number. The free-free boundary conditions are used in both the above analyses. Rosenblat and Tanaka [5] have used the Galerkin procedure to solve the linear problem by using the more realistic rigid wall boundary conditions. A similar problem has been considered earlier by Gershuni and Zhukhovitskii [6]. In their work, however, the temperature fluctuations obey a rectangular law instead of being sinusoidal as used by other researchers.

Gresho and Sani [7] have treated the linear stability problem with rigid boundaries and found that gravitational modulation can significantly affect the stability limits of the system. Finucane and Kelly [8] have carried out an analytical-experimental investigation to confirm the results of Rosenblat and Herbert. Besides investigating the linear stability, Roppo et al. [9] have also carried our the weakly non-linear analysis of the problem. A numerical solution of the linear Rayleigh-Benard convection was obtained by Weimin and Charles [10], and the results are compared with the analytic solution. Kelly and Hu [11] have investigated the onset of thermal convection in the presence of an oscillatory, non-planar shear flow on linear basis. Recently Aniss et al. [12] have worked out a linear problem of the convection parametric instability in the case of a Newtonian fluid confined in a Hele-Shaw cell and subjected to a vertical periodic motion. In their asymptotic analysis they have investigated the influence of the gravitational modulation on the instability threshold.

The object of the present study is to find the critical conditions for the onset of convection. Here a more realistic modulated temperature profile is being taken, which is similar to the variation of the atmospheric temperature near to the earth's surface during one complete day-night cycle. The temperature profile has been expressed by a Fourier series, and the results of this profile have been compared with the results of other profiles as well as with the Venezian results.

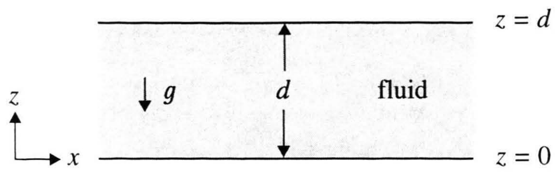

Fig. 1.

2. Formulation

Consider a fluid layer of a viscous, incompressible fluid, confined between two parallel, horizontal stress-free planes, a distance d apart. The system is of infinite extent in the horizontal direction. The configuration is shown in Figure 1.

The governing equations in the Boussinesq approximation are

$$\frac{\partial V}{\partial t} + V.\nabla V + \frac{1}{\rho_R}(p - p_H) = \nu \nabla^2 V + \alpha g (T - T_H) k, \quad (2.1)$$

$$\nabla . V = 0, \quad (2.2)$$

and
$$\frac{\partial T}{\partial t} + V.\nabla T = \kappa \nabla^2 T, \quad (2.3)$$

where ρ_R, ν, κ, and α are the fluid properties; i.e. the reference density, the kinematic viscosity, the thermometric conductivity and the coefficient of volume expansion, respectively. k is the vertical unit vector, $V = (u, v, w)$ the fluid velocity, g the acceleration due to gravity, and p_H and T_H are the hydrostatic pressure and temperature, respectively, and are determined by

$$\frac{\partial p_H}{\partial z} = -\rho_H g \quad (2.4)$$

and
$$\frac{\partial T_H}{\partial t} = \frac{\kappa \partial^2 T_H}{\partial z}, \quad (2.5)$$

where ρ_H is the hydrostatic density of the fluid. The precise form of T_H clearly depends on the nature of the applied heating. To write the boundary conditions, we consider a temperature profile as shown in Fig. 2 and given by

$$T(t) = \begin{cases} \dfrac{2\omega t}{\pi} & 0 \le t \le \dfrac{\pi}{2\omega} \\ 1 & \dfrac{\pi}{2\omega} \le t \le \dfrac{5\pi}{6\omega} \\ \dfrac{12}{7}\left(1 - \dfrac{\omega t}{2\pi}\right) & \dfrac{5\pi}{6\omega} \le t \le \dfrac{2\pi}{\omega}, \end{cases}$$

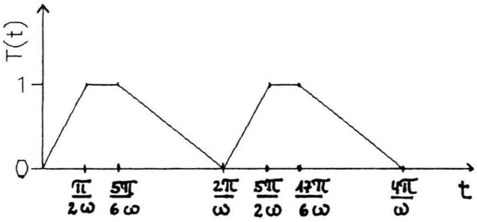

Fig. 2.

where ω is the modulating frequency and $2\pi/\omega$ the period of oscillation. The temperature profile shown in Fig. 2 is similar to the variation of the atmospheric temperature near to the earth's surface during one complete day-night cycle.

The Fourier series of the above function is given by

$$T(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos m\omega t + \sum_{m=1}^{\infty} b_m \sin m\omega t, \quad (2.6)$$

where

$$a_0 = \frac{14}{12}, \quad (2.6a)$$

$$a_m = \frac{2}{m^2\pi^2}\left[\frac{-10}{7} + \cos\frac{m\pi}{2} + \frac{3}{7}\cos\frac{5m\pi}{6}\right], \quad (2.6b)$$

and
$$b_m = \frac{2}{m^2\pi^2}\left[\sin\frac{m\pi}{2} + \frac{3}{7}\sin\frac{5m\pi}{6}\right]. \quad (2.6c)$$

By shifting the origin we write

$$T(t) = \sum_{m=1}^{\infty} a_m \cos m\omega t + \sum_{m=1}^{\infty} b_m \sin m\omega t, \quad (2.7)$$

where a_m and b_m are as given above.

Now we write the externally imposed wall temperatures as follows:

i) When the temperature of the lower boundary as well as of the upper boundary is modulated, we have

$$T(t) = T_R + \beta d \left[1 + \frac{\varepsilon}{2} \left\{ \sum_{m=1}^{\infty} a_m \cos m\omega t \right.\right. \quad (2.8a)$$

$$\left.\left. + \sum_{m=1}^{\infty} b_m \sin m\omega t \right\} \right] \quad \text{at } z = 0,$$

$$= T_R + \beta d \frac{\varepsilon}{2} \left[\sum_{m=1}^{\infty} a_m \cos (m\omega t + \phi) \right. \quad (2.8b)$$

$$\left. + \sum_{m=1}^{\infty} b_m \sin (m\omega t + \phi) \right] \quad \text{at } z = d,$$

ii) When the upper boundary is held at a fixed constant temperature, then

$$T(t) = T_R + \beta d \left[1 + \varepsilon \left\{ \sum_{m=1}^{\infty} a_m \cos m\omega t + \sum_{m=1}^{\infty} b_m \sin m\omega t \right\} \right] \quad \text{at } z = 0, \quad (2.9a)$$

$$= T_R \quad \text{at } z = d, \quad (2.9b)$$

Here ε represents a small amplitude, β is the thermal gradient, ϕ the phase angle and T_R the reference temperature. For both the cases (i) and (ii), the differential equation (2.5) can be solved. We write

$$T_H(z, t) = T_S(z) + \varepsilon T_1(z, t), \quad (2.10)$$

where $T_S(z)$ is the steady temperature field and εT_1 the oscillating part. Then the solution is

$$T_S(z) = T_R + \Delta T \frac{(z - d)}{d} \quad (2.11a)$$

and

$$T_1(z, t) = \mathrm{Re} \left[\sum_{m=1}^{\infty} a_m \left\{ a(\lambda_m) e^{\lambda_m z/d} + a(-\lambda_m) e^{-\lambda_m z/d} \right\} e^{im\omega t} \right]$$
$$- \mathrm{Im} \left[\sum_{m=1}^{\infty} b_m \left\{ a(\lambda_m) e^{\lambda_m z/d} + a(-\lambda_m) e^{-\lambda_m z/d} \right\} e^{im\omega t} \right], \quad (2.11b)$$

where

$$a(\lambda_m) = \frac{\Delta T}{2} \frac{e^{-i\phi} - e^{-\lambda_m}}{e^{\lambda_m} - e^{-\lambda_m}} \quad \text{for case (i)} \quad (2.12)$$

and

$$a(\lambda_m) = -\Delta T \frac{e^{-\lambda_m}}{e^{\lambda_m} - e^{-\lambda_m}} \quad \text{for case (ii)}. \quad (2.13)$$

Also $\lambda_m^2 = - im\omega\, d^2/\kappa$. (2.14)

Here the basic solution of (2.1)–(2.3), (2.8) and (2.9) is

$$V = (u, v, w) = 0, \quad T = T_H(z, t)$$
$$\text{and} \quad p = p_H(z, t). \quad (2.15)$$

Our aim is to examine the behaviour of infinitesimal disturbances to the basic solution (2.15). With this in view, we substitute

$$V = (u, v, w), \quad T = T_H + \theta, \quad p = p_H + p \quad (2.16)$$

into (2.1)–(2.3). If we scale length, time, temperature, velocity and pressure by the units d, d^2/κ, ΔT, κ/d, and $\kappa \nu \rho_R/d^2$, respectively, then the governing equations in linear form are

$$\frac{1}{P} \frac{\partial V}{\partial t} + \nabla p = \nabla^2 V + R\theta k, \quad (2.17)$$

$$\nabla . V = 0, \quad (2.18)$$

$$\frac{\partial \theta}{\partial t} + w = \frac{\partial T_0}{\partial z} = \nabla^2 \theta, \quad (2.19)$$

where $P = \nu/\kappa$ is the Prandtl number and $R = \alpha g \Delta T d^3/\nu\kappa$ is the Rayleigh number. In the above equations, the variables are in their non-dimensional form.

The temperature gradient $\frac{\partial T_0}{\partial z}$, obtained from the dimensionless form of (2.10) is

$$\frac{\partial T_0}{\partial z} = -1 + \varepsilon f, \quad (2.20)$$

where

$$f = \mathrm{Re} \left[\sum_{m=1}^{\infty} a_m\, g(\lambda_m)\, e^{-im\omega^* t} \right] - \mathrm{Im} \left[\sum_{m=1}^{\infty} b_m\, g(\lambda_m)\, e^{-im\omega^* t} \right] \quad (2.21)$$

$$g(\lambda_m) = A(\lambda_m)\, e^{\lambda_m z} + A(-\lambda_m)\, e^{-\lambda_m z}, \quad (2.22)$$

$$A(\lambda_m) = \lambda_m\, a(\lambda_m) \quad (2.23)$$

and $\lambda_m^2 = -\, im\omega^*$ and $\omega^* = \omega d^2/\kappa$ (2.24)
(non-dimensional frequency)

Henceforth the asterisk will be dropped and ω will be considered as the non-dimensional frequency. For convenience, the entire problem has been expressed in terms of w and θ.

From (2.17) we write

$$\left[\frac{1}{P}\frac{\partial}{\partial t} - \nabla^2\right]\nabla^2 w = R\nabla_1^2\,\theta, \tag{2.25}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \text{ and } \nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Now from (2.25) and (2.19) we write

$$\left[\frac{1}{P}\frac{\partial}{\partial t} - \nabla^2\right]\left[\frac{\partial}{\partial t} - \nabla^2\right]\nabla^2 w = -\,R\frac{\partial T_0}{\partial z}\nabla_1^2\, w. \tag{2.26}$$

Free-free boundary conditions are being applied in this problem, therefore at $z = 0$ and 1, we have

$$w = \frac{\partial^2 w}{\partial z^2} = 0 \tag{2.27}$$

and also $\theta = 0$ (externally fixed temperature).

Then with the help of (2.25) we write

$$w = \frac{\partial^2 w}{\partial z^2} = \frac{\partial^4 w}{\partial z^4} = 0 \quad \text{at } z = 0 \text{ and } 1. \tag{2.28a}$$

From (2.26) it follows that $\dfrac{\partial^6 w}{\partial z^6} = 0$ for $z = 0$ and 1.

Now by differentiating (2.26) we conclude, successively, that all the even derivatives of w vanish for $z = 0$ and 1. Thus

$$\frac{\partial^{(2m)} w}{\partial z^{(2m)}} = 0 \quad (m = 1,2,3,------) \quad \text{at } z = 0 \text{ and } 1. \tag{2.28b}$$

Equation (2.26) can be solved using the boundary conditions (2.28). For solution we Fourier-analyze the disturbances and write

$$w = w(z,t)\exp\left[i(a_x x + a_y y)\right] \text{ etc.,} \tag{2.29}$$

so that

$$\nabla_1^2 w = -\,a^2 w, \tag{2.30}$$

where $a = (a_x^2 + a_y^2)^{1/2}$ is the horizontal wave-number.

For the sake conciseness of notation, the exponential factor will be left out.

3. Solution

We seek the eigenfunctions w and eigenvalues R of (2.26) and (2.28) for a temperature profile that departs from the linear profile $\dfrac{\partial T_0}{\partial z} = -1$ by quantities of order ε. For this we write the expansions

$$w = w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + --------, \tag{3.1a}$$

$$R = R_0 + \varepsilon R_1 + \varepsilon^2 R_2 + --------, \tag{3.1b}$$

This type of expansion was first used in connection with convection problems by Malkus and Veronis [13] to consider the effects of finite-amplitude convection. A similar expansion has been used among others by Schlüter, Lortz, and Busse [14], Ingersoll [15], Roppo, Davis, and Rosenblat [9] and by Jenkins and Proctor [16]. Substituting (3.1) into (2.26) and separating the terms of different powers of ε, we get the system as

$$Lw_0 = 0, \tag{3.2}$$

$$Lw_1 = R_1\nabla_1^2 w_0 - R_0 f\nabla_1^2 w_0, \tag{3.3}$$

$$Lw_2 = R_1\nabla_1^2 w_1 + R_2\nabla_1^2 w_0 - R_1 f\nabla_1^2 w_0 - R_0 f\nabla_1^2 w_1, \tag{3.4}$$

where

$$L \equiv \left[\frac{1}{P}\frac{\partial}{\partial t} - \nabla^2\right]\left[\frac{\partial}{\partial t} - \nabla^2\right]\nabla^2 - R_0\,\nabla_1^2. \tag{3.5}$$

The function w_0 is the solution of the classical Benard problem ($\varepsilon = 0$). The marginally stable solutions for that problem are

$$w_0^{(n)} = \sin n\pi z,$$

which correspond to the eigenvalues

$$R_0^{(n)} = \frac{(n^2\pi^2 + a^2)^3}{a^2}.$$

For a fixed value of a, the least eigenvalue is

$$R_0 = \frac{(\pi^2 + a^2)^3}{a^2} \tag{3.6}$$

corresponding to

$$w_0 = \sin \pi z. \tag{3.7}$$

Here (3.6) and (3.7) will serve as the starting solution. The equation for w_1 reduces to

$$Lw_1 = R_1 a^2 \sin \pi z + R_0 a^2 f \sin \pi z. \tag{3.8}$$

The solubility condition of (3.8) requires that the coefficient R_1 is zero. In fact all odd coefficients R_1, R_3, R_5 – – – – – – are zero.

To find the solution of (3.8) we expand the right hand side of it in a Fourier series and thus obtain an expression for w_1 by inverting the operator L term by term. For this we write

$$g(\lambda) \sin k\pi z = \sum_{n=1}^{\infty} g_{nk}(\lambda) \sin n\pi z, \tag{3.9}$$

where

$$g_{nk}(\lambda) = 2 \int_0^1 g(\lambda) \sin k\pi z \sin n\pi z \, dz,$$

$$g_{nk}(\lambda_m) = \frac{-4nk\pi^2\lambda_m^2 [1 + (-1)^{n+k+1} e^{-i\phi}]}{2[\lambda_m^2 + (n-k)^2\pi^2][\lambda_m^2 + (n+k)^2\pi^2]} \quad \text{for case (i)} \tag{3.10}$$

and $$g_{nk}(\lambda_m) = \frac{-4nk\pi^2\lambda_m^2}{[\lambda_m^2 + (n-k)^2\pi^2][\lambda_m^2 + (n+k)^2\pi^2]} \quad \text{for case (ii).} \tag{3.11}$$

From (3.8) we write

$$Lw_1 = R_0 a^2 \operatorname{Re}\left[\sum_{m=1}^{\infty} a_m g(\lambda_m) e^{-im\omega t} \sin \pi z\right] - R_0 a^2 \operatorname{Im}\left[\sum_{m=1}^{\infty} b_m g(\lambda_m) e^{-im\omega t} \sin \pi z\right].$$

Using (3.9) in the above equation, we write

$$Lw_1 = R_0 a^2 \operatorname{Re}\left[\sum_{m,n=1}^{\infty} A_n(\lambda_m) e^{-im\omega t} \sin \pi z\right] - R_0 a^2 \operatorname{Im}\left[\sum_{n=1}^{\infty} B_n(\lambda_m) e^{-im\omega t} \sin n\pi z\right]. \tag{3.12}$$

where

$$A_n(\lambda_m) = a_m g_{nl}(\lambda_m), \; B_n(\lambda_n) = b_m g_{nl}(\lambda_m)$$

and $$D_n(\lambda_m) = g_{nl}(\lambda_m). \tag{3.13}$$

Also we have

$$L \sin n\pi z \, e^{-im\omega t} = L_{mn}(\omega) \sin n\pi z \, e^{-im\omega t}, \tag{3.14}$$

where

$$L_{mn}(\omega) = \frac{m^2\omega^2}{P} q_n + im\omega\left[1 + \frac{1}{P}\right] q_n^2 - q_n^3 + q_1^3 \tag{3.15}$$

with $$q_n = n^2\pi^2 + a^2. \tag{3.15a}$$

Using (3.14) and (3.12) gives

$$w_1 = R_0 a^2 \operatorname{Re}\left[\sum_{m,n=1}^{\infty} \frac{A_n(\lambda_m) e^{-im\omega t}}{L_{mn}(\omega)} \sin n\pi z\right] - R_0 a^2 \operatorname{Im}\left[\sum_{n=1}^{\infty} \frac{B_n(\lambda_m) e^{-im\omega t}}{L_{mn}(\omega)} \sin n\pi z\right]. \tag{3.16}$$

The equation for w_2 reduces to

$$Lw_2 = -R_2 a^2 w_0 + R_0 a^2 f w_1. \tag{3.17}$$

The solubility condition of (3.17) requires that the right hand side of it should be orthogonal to $\sin \pi z$, therefore

$$R_2 = 2R_0 \int_0^1 \overline{f w_1} \sin \pi z \, dz, \tag{3.18}$$

where the bar denotes a time average. From (3.8)

$$f \sin \pi z = \frac{1}{a^2 R_0} L w_1,$$

so that

$$\overline{f w_1} \sin \pi z = \frac{1}{a^2 R_0} \overline{w_1 L w_1}$$

or $$\overline{f w_1} \sin \pi z = \frac{a^2 R_0}{2} \operatorname{Re}\left[\sum_{m,n=1}^{\infty} \frac{A_n(\lambda_m)}{L_{mn}(\omega)} \sin n\pi z \cdot \sum_{m,n=1}^{\infty} \tilde{A}_n(\lambda_m) \sin n\pi z\right] + \frac{a^2 R_0}{2} \operatorname{Re}\left[\sum_{m,n=1}^{\infty} \frac{B_n(\lambda_m)}{L_{mn}(\omega)} \sin n\pi z \cdot \sum_{m,n=1}^{\infty} \tilde{B}_n(\lambda_m) \sin n\pi z\right],$$

where ~ denotes the conjugate complex. Then

$$R_2 = \frac{a^2 R_0^2}{2} \operatorname{Re} \sum_{m,n=1}^{\infty} \frac{|A_n(\lambda_m)|^2}{L_{mn}(\omega)} + \frac{a^2 R_0^2}{2} \operatorname{Re} \cdot \sum_{m,n=1}^{\infty} \frac{|B_n(\lambda_m)|^2}{L_{mn}(\omega)}$$

or $$R_2 = \frac{a^2 R_0^2}{2} \sum_{m,n=1}^{\infty} |D_n(\lambda_m)|^2 \, C_{mn} (a_m^2 + b_m^2), \tag{3.19}$$

where

$$C_{mn} = \frac{L_{mn}(\omega) + \tilde{L}_{mn}(\omega)}{2\,|L_{mn}(\omega)|^2}. \tag{3.20}$$

Equation (3.17) can be solved for w_2 and the procedure continued to evaluate further corrections to w and R. However we shall stop at this step.

In general R is a function of the horizontal wave number a and the amplitude of the perturbation ε, therefore we write

$$R(a, \varepsilon) = R_0(a) + \varepsilon^2 R^2(a) + -------. \tag{3.21}$$

Let the least value of R be R_c at $a = a_c$. This critical value of a occurs when

$$\frac{\partial R}{\partial a} = 0. \tag{3.22}$$

Also $a_c = a_0 + \varepsilon a_1 + \varepsilon a_2 + ------$. (3.23)

$$C_{mn} = \frac{(m^2\omega^2/P)(n^2+1/2)\pi^2 - (n^2+1/2)^3\pi^6 + (27/8)\pi^6}{[(m^2\omega^2/P)(n^2+1/2)\pi^2 - (n^2+1/2)^3\pi^6 + (27/8)\pi^6]^2 + m^2\omega^2(1+1/P)^2(n^2+1/2)^4\pi^8}. \tag{4.6}$$

From (3.21), (3.22), (3.23), and (3.6) we get

$$a_0 = \frac{\pi^2}{2} \tag{3.24}$$

and $R_c(\varepsilon) = R_{0c} + \varepsilon^2 R_{2c} + -------$. (3.25)

Thus to order ε^2, R_c is determined by (3.25).

4. Results

We have $R_0 = \dfrac{(\pi^2 + a^2)^3}{a^2}$ and $a_0^2 = \dfrac{\pi^2}{2}$, therefore $R_0(a_0) = R_{0c} = 657.51$.

The values of R_{2c} have been calculated in the following three cases:

a) when the plate temperatures are modulated in phase, i.e. $\phi = 0$,
b) when the plate temperatures are modulated out of phase, i.e. $\phi = \pi$,

and c) when only the bottom plate temperature is modulated, the upper plate is held at a fixed constant temperature.

Let $$d_{mn} = \frac{-4n\pi^2\lambda_m^2}{[\lambda_m^2 + (n-1)^2\pi^2][\lambda_m^2 + (n+1)^2\pi^2]}, \tag{4.1}$$

then from (3.13) and (3.10) we have

for case (a) (4.2)

$$D_n(\lambda_m) = d_{mn} \quad \text{if } n \text{ is even}, \quad = 0 \ \text{ if } n \text{ is odd},$$

for case (b) (4.3)

$$D_n(\lambda_m) = 0 \quad \text{if } n \text{ is even}, \quad = d_{mn} \ \text{ if } n \text{ is odd},$$

and from (2.11) for case (c)

$$D_n(\lambda_m) = d_{mn} \quad \text{for all values of } n. \tag{4.4}$$

Using (2.24), we can write

$$|d_{mn}|^2 = \frac{16 m^2 n^2 \pi^4 \omega^2}{[m^2\omega^2 + (n-1)^4\pi^4][m^2\omega^2 + (n+1)^4\pi^4]}. \tag{4.5}$$

Also, at $a^2 = a_0^2 = \dfrac{\pi^2}{2}$ we have

From (3.19) we get

$$R_{2c} = \frac{729}{64}\pi^{10} \sum_{m,n=1}^{\infty} |d_{mn}|^2 \, C_{mn}(a_m^2 + b_m^2). \tag{4.7}$$

In the limiting cases, we consider the case when ω is very small i.e. $\omega \to 0$.

For $n = 1$ $$|d_{m1}|^2 C_{m1} \to \frac{(a_m^2 + b_m^2)}{(27/8)\,P\,(1 + 1/P)^2\pi^6},$$

and for $n \neq 1$ we get

$$|d_{mn}|^2 C_{mn} \to \frac{-16 m^2 n^2 \omega^2 (a_m^2 + b_m^2)}{(n^2-1)^5\,[n^4 + (5/2)n^2 + 13/4]\,\pi^{10}}.$$

So $$R_{2c} = R_{\mathrm{Pr}} - \beta\omega^2, \tag{4.8}$$

where

$$R_{\mathrm{Pr}} = \frac{27\pi^4}{8P(1+1/P)^2} \sum_{m=1}^{\infty} (a_m^2 + b_m^2) \tag{4.9}$$

and

$$\beta = \frac{729}{4} \sum_{\substack{m=1 \\ n=2}}^{\infty} \frac{m^2 n^2 (a_m^2 + b_m^2)}{(n^2-1)^5\,[n^4 + (5/2)n^2 + 13/4]\,\pi^{10}}.$$

Then for case (a)

$$R_{2c} = -2.703573\times10^{-7}\omega^2, \tag{4.10}$$

for case (b)

$$R_{2c} = R_{Pr} - 1.235901\times10^{-9}\omega^2, \tag{4.11}$$

and for case (c)

$$R_{2c} = R_{Pr} - 2.715932\times10^{-7}\omega^2. \tag{4.12}$$

Now we consider the temperature profiles as shown in the Figs. 3 and 4. They are known as saw-tooth function and step function profiles, respectively.

The Fourier expansion of these profiles is

$$T(t) = \sum_{m=1}^{\infty} a_m \cos m\omega t\,, \tag{4.13}$$

where

$$a_m = \frac{4}{\pi^2 m^2}(1-\cos m\pi) \tag{4.14}$$

and

$$a_m = \frac{4}{\pi m}\sin(m\pi/2) \tag{4.15}$$

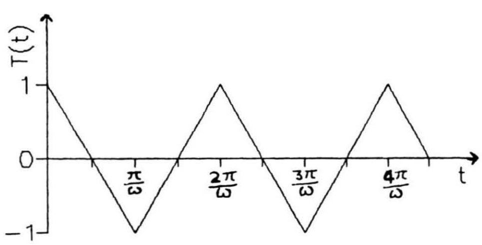

Fig. 3.

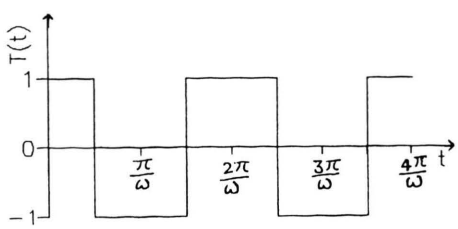

Fig. 4.

for the Figs. 3 and 4, respectively. Then (4.7) for the above profiles reduces to

$$R_{2c} = \frac{729}{64}\pi^{10}\sum_{m,n=1}^{\infty}|d_{mn}|^2\, C_{mn}\, a_m^2\,,$$

where a_m is as given above for the two profiles.

Here the values of R_{2c} for the three temperature profiles have been calculated as functions of ω for various values of P and are compared with Venezian's [3] results. In the Venezian results a sinusoidal function was taken for modulating the boundary temperatures.

5. Discussion

In the limiting case, when the frequency ω is very small, we find that for in-phase modulation the effect of modulation is to destabilize the system with convection occurring at an earlier point than in the unmodulated system. This agrees with the results of Krishnamurty [17]. The effect of modulation is more destabilizing for a step function and less destabilizing for a saw-tooth function than that calculated by Venezian. The least destabilizing effect was obtained for day-night profiles as shown in the Figs. 5, 6 and 7. Here the dependence on the Prandtl number can appear only at large values of ω.

When the modulation is out of phase, the effect is stabilization, decreasing with frequency (Figs. 8 and 9).

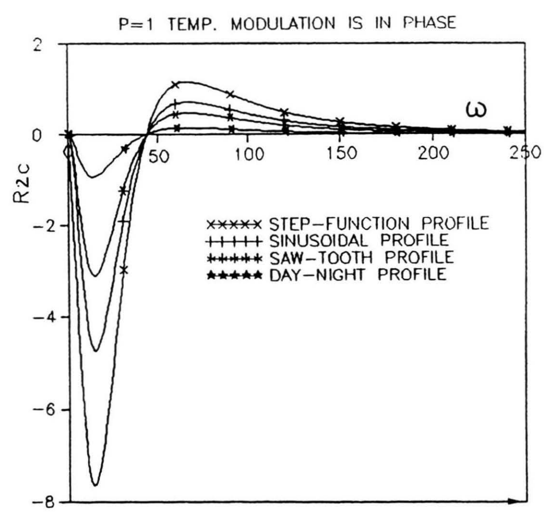

Fig. 5.

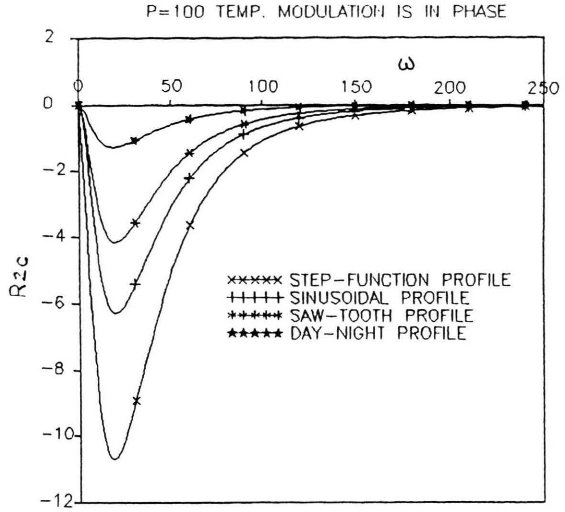

Fig. 6.

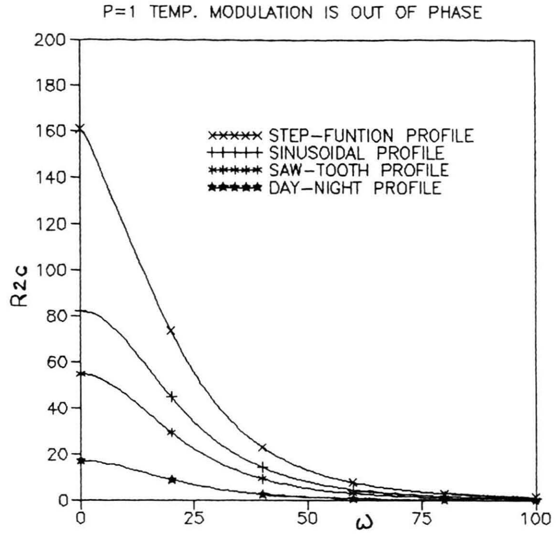

Fig. 8.

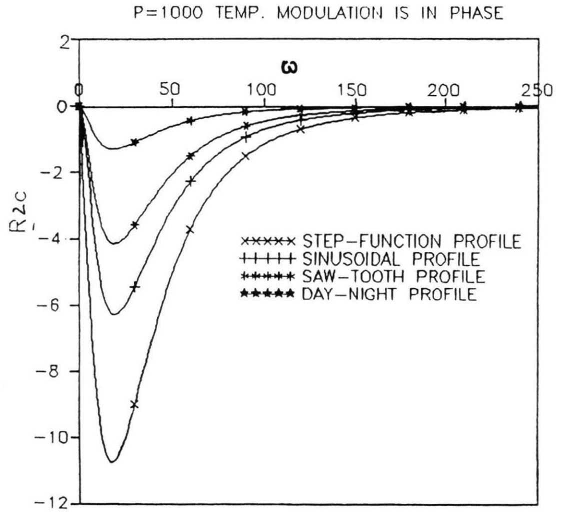

Fig. 7.

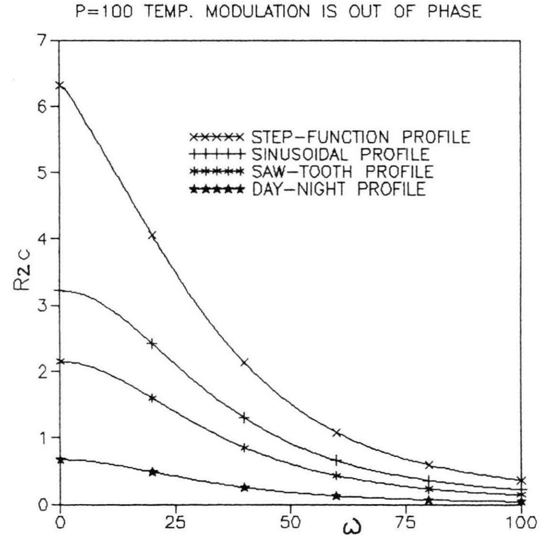

Fig. 9.

However, when the Prandtl number is very large, R_{Pr} is sufficiently small and so it is overtaken by the other terms in the sum (Fig. 10).

But when only the bottom plate temperature is modulated, there is no significant difference in this case (b) near $P = 1$. However for large P, R_{Pr} can become sufficiently small to be overtaken by the other terms in the sum, as shown in the Figs. 12 and 13.

In case of in-phase modulation the effect of modulation on the onset of convection is zero at zero frequency. But for out of phase modulation or when the upper plate is at a fixed constant temperature, the effect is of maximum stabilization at this frequency.

Also it is clear from (4.7) that due to the term $1/\omega^2$ in R_{2c} the effect of modulation disappears altogether as $\omega \to \infty$. This behaviour is in qualitative agreement with

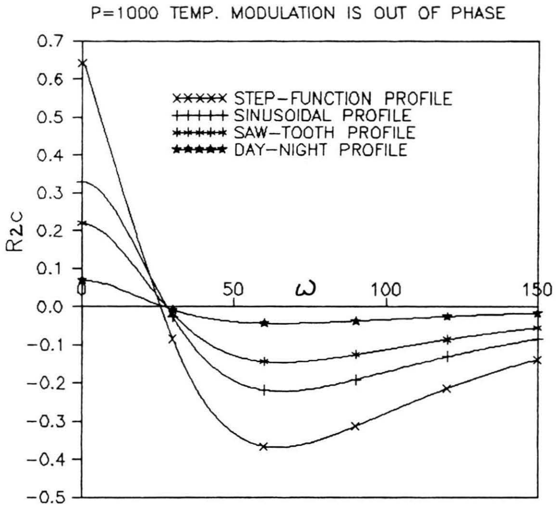

Fig. 10.

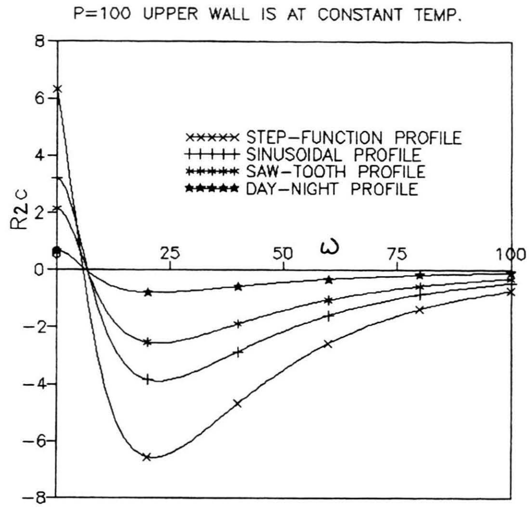

Fig. 12.

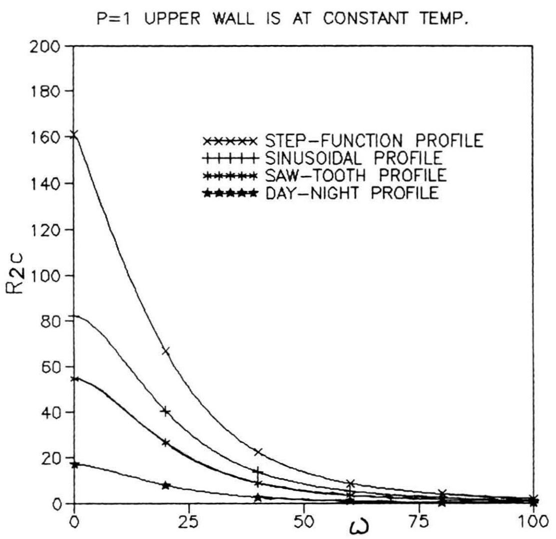

Fig. 11.

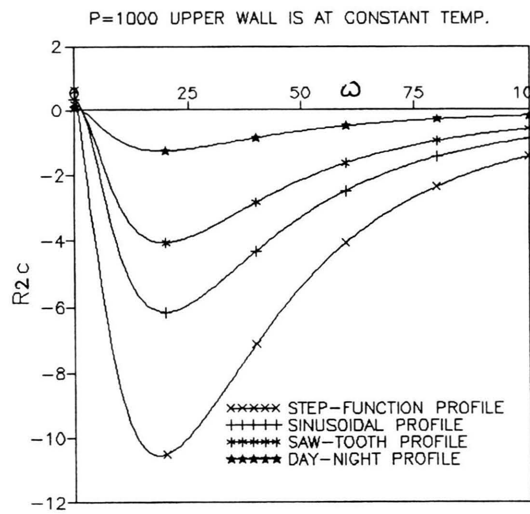

Fig. 13.

the results of Rosenblat and Tanaka [5] and with the experiment of [2] for the analogous rotating cylinder problem.

For intermediate values of ω, the effect of changing the frequency can be seen in the numerator of C_{mn}. We have $C_{m2} = 0$, when $\omega = \pi^2\,(78\mathrm{P})^{1/2}/2m$, so that in case (a) R_{2c} should be zero near $\omega = \pi^2\,(78\mathrm{P})^{1/2}/2$. The peak negative value of R_{2c} occurs near $\omega = 20$. Over the entire range of P, this value is about -11 for the step-function, -4 for the saw-tooth function, -1.5 for the day-night profile and -6.5 as calculated by Venezian.

When the frequency of modulation is small, the effect of modulation is felt throughout the fluid layer. If the modulation is in phase, the temperature profile consists of the steady straight-line section plus a time-dependent part that oscillates with time. It is because of this time-dependent part that the convection occurs at lower Rayleigh number than that predicted by the linear theory with steady temperature gradient, (Figures 5–7). Also when the temperature modulation is out of phase or the upper plate is at constant temperature, the convective wave propagates across the fluid layer thereby inhibiting the instability, and so convection occurs at a higher Rayleigh number that that predicted by the linear theory with steady temperature gradient. Different degrees of penetration of the convective waves across the fluid layer, corresponding to different temperature profiles are responsible for different graphs in a figure.

The above analysis is based on the assumption that the amplitude of the modulating temperature is small compared to the imposed steady temperature difference and the convective currents are weak, so that non-linear effects may be neglected. However, violation of these assumptions would alter the results signficantly at low modulating frequency.

Thus at low modulating frequency the amplitude of modulation should be small, and for convective currents to be weak, the frequency of modulation should be such that $\omega > \varepsilon$, as suggested by Venezian [3].

Acknowledgements

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