On the Variable-coefficient Burgers-Hlavaty Equation

Yi-Tian Gao and Bo Tian

Departments of Applied Physics and Mathematics,
Beijing University of Aeronautics and Astronautics,
Beijing 100083, China

Reprint requests to Prof. B. T.; E-mail: tianb@netscape.net

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We study Hlavaty's generalization of the Burgers equation containing certain coefficient functions. We obtain a new auto-Bäcklund transformation and a family of exact analytical solutions along with the constraints on those coefficients.

Key words: Nonlinear Evolution Equations; Variable Coefficients; Bäcklund Transformation; Exact Solutions; Symbolic Computation.

In this paper we consider Hlavaty's variable-coefficient Painlevé-admissible extension of the Burgers equation [1, 2]

$$L(x,t)\,u_t+\frac{u^2\,L_x(x,t)}{L(x,t)}+\frac{2\,u\,L_x^2(x,t)}{L^2(x,t)}-2\,u\,u_x-\frac{u\,L_{xx}(x,t)}{L(x,t)}-u_{xx}+S(x,t)=0, \quad (1)$$

where $L(x,t)$ and $S(x,t)$ are a couple of analytical functions. Recently, the prolongation structure with the related issues for (1) has been studied by Karasu [3].

We will investigate the interesting case of $L\neq 0$. Computerized symbolic computation will be used. Firstly we plan to use to Painlevé expansion [4, 5]

$$u(x,t)=\phi^{-J}(x,t)\sum_{j=0}^{\infty}u_j(x,t)\,\phi^j(x,t), \quad (2)$$

where J is a natural number and $\phi=0$ defines the singular manifold. Balancing powers of ϕ at the lowest orders requires that $J=1$. Next we truncate the expansion at the constant level terms, i.e.

$$u(x,t)=\phi^{-1}(x,t)\sum_{j=0}^{1}u_j(x,t)\,\phi^j(x,t), \quad (3)$$

aiming to obtain a certain Bäcklund transformation and analytical solutions of (1), disregarding the integrability issue. Recent work in this direction is seen, e.g. in Ref. [6, 7].

When substituting (3) into (1), we make the coefficients of like powers of ϕ to vanish, so as to get

ϕ^{-3}: $u_0=\phi_x$ with $\phi_x\neq 0$, (4)

ϕ^{-2}: $\Theta=-L^2\,\phi_t+2\,L\,u_1\,\phi_x+L_x\,\phi_x+L\,\phi_{xx}=0$, (5)

ϕ^{-1}: $\Theta_x\,L-2\,L_x\,\Theta\equiv 0$, (6)

ϕ^{0}: u_1 needs to satisfy the original equation, i.e.

$$L\,u_{1,t}+\frac{u_1^2\,L_x}{L}+\frac{2\,u_1\,L_x^2}{L^2}-2\,u_1\,u_{1,x}-\frac{u_1\,L_{xx}}{L}-u_{1,xx}+S=0. \quad (7)$$

The set of equations (3–5) and (7) constitutes an *auto*-Bäcklund transformation, since the set is solvable. Let us have some explicitly-solved sample solutions in the following analysis.

Into (5) we substitute a few trial expressions,

$$\phi(x,t)=\nu(x)\,t^2+\beta(x)\,t+\lambda(x), \quad (8)$$

$$u_1(x,t)=\delta(x)\,t^2+\psi(x)\,t+\sigma(x), \quad (9)$$

$$L(x,t)=\varepsilon(x)\,t^2+\mu(x)\,t+\omega(x)$$

(assumed for a variable coefficient), (10)

where $\sigma(x)$, $\nu(x)$, $\beta(x)$, $\delta(x)$, $\psi(x)$, $\varepsilon(x)$, $\mu(x)$, $\omega(x)$, and $\lambda(x)$ are all differentiable functions. Then, equating to zero the coefficients of like powers of t in (5) yields

t^6: $\delta(x)=0$, (11)

t^5: the choice of $\varepsilon(x)=0$ with the understanding that $\mu(x)\neq 0$, (12)

t^4: $\nu(x)=\text{constant}=\nu$, (13)

t^3: $-\nu\,\mu(x)+\psi(x)\,\beta'(x)=0$
$\to$ for simplicity we choose $\nu(x)=0$ and $\beta(x)=\text{constant}=\beta\neq 0$, (14)

t^2: $\psi(x)=\dfrac{\beta\,\mu(x)}{2\,\lambda'(x)}$, where $\lambda'(x)\neq 0$, (15)

t^1: $\sigma(x)=-\dfrac{\mu'(x)}{2\,\mu(x)}+\dfrac{\beta\,\omega(x)-\lambda''(x)}{2\,\lambda'(x)}$, (16)

t^0: $\omega(x)=\alpha\,\mu(x)$, (17)

where α is a constant. This way, (5) has been satisfied, so that those trail expressions are allowed. Correspondingly, (7) implies that

$$S(x,t)=\sum_n S_n(x)\,t^n, \quad (18)$$

where $S_n(x)$'s are also differentiable. We substitute (18) into (7) and again equate to zero the coefficients of like powers of t, yielding (21) at the end of this paper.

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Lastly, having considered (3) and (4) and put everything together, we obtain

$$u(x,t)=\frac{\beta t\,\mu(x)}{2\,\lambda'(x)}+\frac{\lambda'(x)}{\beta t+\lambda(x)}-\frac{\mu'(x)}{2\,\mu(x)}+\frac{\alpha\,\beta\,\mu(x)-\lambda''(x)}{2\,\lambda'(x)} \tag{19}$$

with the constraints on the coefficient functions as

$$L(x,t)=(\alpha+t)\,\mu(x), \tag{20}$$

$$\begin{aligned}S(x,t)=&-\frac{\alpha\,\beta\,\mu(x)^2}{2\,\lambda'(x)}-\frac{\beta\,t\,\mu(x)^2}{2\,\lambda'(x)}+\frac{\alpha^2\,\beta^2\,\mu(x)\,\mu'(x)}{4\,\lambda'(x)^2}+\frac{\alpha\,\beta^2\,t\,\mu(x)\,\mu'(x)}{2\,\lambda'(x)^2}+\frac{\beta^2\,t^2\,\mu(x)\,\mu'(x)}{4\,\lambda'(x)^2}\\&-\frac{\alpha\,\beta\,\mu'(x)^2}{2\,\mu(x)\,\lambda'(x)}-\frac{\beta\,t\,\mu'(x)^2}{2\,\mu(x)\,\lambda'(x)}-\frac{3\,\mu'(x)^3}{4\,\mu(x)^3}-\frac{\alpha^2\,\beta^2\,\mu(x)^2\,\lambda''(x)}{2\,\lambda'(x)^3}-\frac{\alpha\,\beta^2\,t\,\mu(x)^2\,\lambda''(x)}{\lambda'(x)^3}\\&-\frac{\beta^2\,t^2\,\mu(x)^2\,\lambda''(x)}{2\,\lambda'(x)^3}-\frac{\alpha\,\beta^2\mu'(x)^2\,\lambda''(x)}{2\,\lambda'(x)^2}-\frac{\beta\,t\,\mu'(x)\,\lambda''(x)}{2\,\lambda'(x)^2}+\frac{2\,\alpha\,\beta\,\mu(x)\,\lambda''(x)^2}{\lambda'(x)^3}\\&+\frac{2\,\beta\,t\,\mu(x)\,\lambda''(x)^2}{\lambda'(x)^3}-\frac{3\,\mu'(x)\,\lambda''(x)^2}{4\,\mu(x)\,\lambda'(x)^2}-\frac{3\,\lambda''(x)^3}{2\,\lambda'(x)^3}+\frac{\alpha\,\beta\,\mu''(x)}{2\,\lambda'(x)}+\frac{\beta\,t\,\mu''(x)}{2\,\lambda'(x)}\\&+\frac{3\,\mu'(x)\,\mu''(x)}{2\,\mu(x)^2}-\frac{\alpha\,\beta\,\mu(x)\,\lambda^{(3)}(x)}{\lambda'(x)^2}-\frac{\beta\,t\,\mu(x)\,\lambda^{(3)}(x)}{\lambda'(x)^2}+\frac{\mu'(x)\,\lambda^{(3)}(x)}{2\,\mu(x)\,\lambda'(x)}\\&+\frac{2\,\lambda''(x)\,\lambda^{(3)}(x)}{\lambda'(x)^2}-\frac{\mu^{(3)}(x)}{2\,\mu(x)}-\frac{\lambda^{(4)}(x)}{2\,\lambda'(x)}\end{aligned} \tag{21}$$

where the differentiable functions $\mu(x)$ and $\lambda(x)$, and the constants α and β all remain arbitrary, except that $\lambda'(x)\neq 0$, $\mu(x)\neq 0$ and $\beta\neq 0$.

In conclusion, for (1) we have obtained the new family of exact analytical solutions (19–21), as well as a new *auto*-Bäcklund transformation as stated above.

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