Control of a Chaotic Relay System Using the OGY Method*

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The control method of Ott, Grebogi, and Yorke (OGY) is used to stabilize the unstable periodic orbits of a chaotic relay system. Small variations in the height of the relay output are used as control input. The influence of the control activation bound is studied in detail via the one-dimensional Poincaré map of the controlled system. The reduced sensitivity of the multi-step OGY method for higher period orbits can thus directly be verified.

Key words: Control of Chaos; OGY Control Method; Relay System; Stabilization of Unstable Periodic Orbits; Poincaré Map.

1. Introduction

Control of chaotic systems does in principle not differ from the control of general nonlinear systems. The system dynamics is changed by feedback control in order to make the system stable in the neighborhood of a desired operating point, or in order to improve the speed of the reaction. If the system has an unstable limit cycle which is suitable as an operating condition, stabilization of this orbit may be an attractive option, because in principle this can be done using arbitrarily small control action. A chaotic system contains an infinite number of unstable limit cycles. Thus the approach is particularly well suited for such systems. In addition, due to the mixing property, a change of the operating condition can be accomplished simply by waiting until the uncontrolled system comes sufficiently close to the desired orbit, and then capture it by switching on (linear) control. The described idea goes back to Ott, Grebogi, and Yorke [1] and has attracted considerable interest in the last years, now often cited as the OGY control method. A compilation of recent research concerning control of chaos, including other approaches, can be found in [2].

The central parameter in OGY control is the size of the orbit neighborhood in which the control is activated. We will call this parameter the *control activation bound*. It is obvious that there must be a trade-off between this bound and the time to capture the trajectory. Asymptotic results for this dependence are given in [1] for small control action. However, the case of large control action in general is very difficult to treat. Due to the nonlinearity of the system, the linear control law may fail to capture the trajectory. If the trajectory escapes but stays inside the chaotic attractor or its basin of attraction, then it eventually comes close enough later and may be captured then. But because of the change of dynamics due to feedback control, it is also possible that the trajectory leaves the attractor basin, so that the OGY control fails.

In this paper we investigate this problem for the case of a very simple chaotic system. It consists of an unstable harmonic oscillator in feedback with a relay with hysteresis. Some results on this system are presented in Section 2. The application of the OGY control method to this system is presented in Section 3. The Poincaré map of the system is only one-dimensional. Therefore the consequences of a variation of the control activation bound can be made transparent by investigating the Poincaré map of the controlled system. In this analysis, the notion of a *basin of immediate attraction* arises as a useful concept. This will be presented in Section 4.

For orbits of period $n > 1$ the OGY standard method of stabilization is known to be very sensitive. A modification using also the $(n-1)$ intermediate points on the Poincaré section for control was already suggested in [3]. For our relay system the superiority of this multi-step control method can be nicely illumi-

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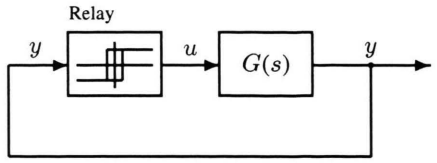

Fig. 1. The relay system.

nated by comparing the basins of immediate attraction of the controlled systems. This will be presented in Section 5.

2. The Relay System

The system under consideration consists of a relay and a linear subsystem, as illustrated in Figure 1. The relay toggles between the two states $\pm b$ and has a symmetrical input-output characteristic with hysteresis, as shown in Figure 2. For the switching threshold a and the switching height b we assume $a = 1$ and $b = 1$ without loss of generality. The linear subsystem with the transfer function

$$G(s) = \frac{1}{s^2 - 2\zeta s + 1} \tag{1}$$

is a harmonic oscillator with damping $-\zeta$. The relay system described above was first proposed by Cook [4] as an example of chaotic behavior. Although the state space of the continuous system part (1) is only two-dimensional, chaos may occur due to the non-unique input-output relationship introduced by the hysteresis. It should be noted that, as a consequence, the system equations are not time-invertible.

Introducing the state variables $x_1 = y$ and $x_2 = \dot{y}$, the state space equations of the relay system can be written as

$$\dot{x}_1 = x_2, \tag{2}$$

$$\dot{x}_2 = 2\zeta x_2 - x_1 + u, \tag{3}$$

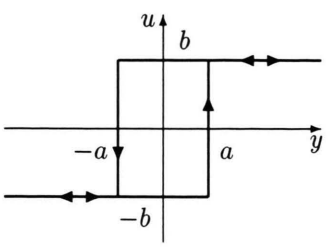

Fig. 2. The input-output characteristic of the relay.

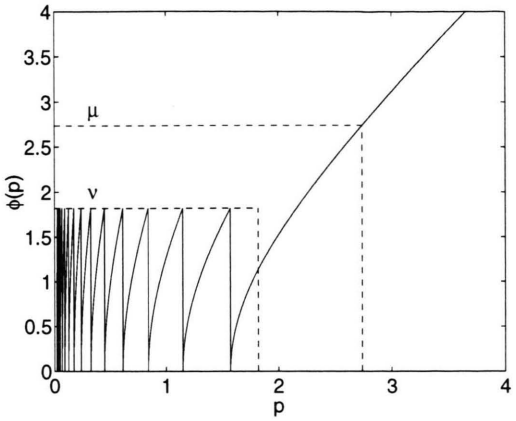

Fig. 3. Poincaré map of the relay system for $\zeta = 0.05$.

where u denotes the output signal of the relay. The input-output characteristic of the relay given in Fig. 2 can be represented by

$$u = \begin{cases} +1 & \text{if} \begin{cases} +1 < y & \text{or} \\ -1 \le y \le +1 & \text{and} \quad u_- = +1 \end{cases} \\ -1 & \text{if} \begin{cases} -1 \le y \le +1 & \text{and} \quad u_- = -1 \\ \text{or} \quad y < -1, \end{cases} \end{cases} \tag{4}$$

where u^- denotes the last switching state. We note that the phase space is a subspace of $\mathbb{R}^2 \times \{-1, 1\}$, namely it consists of two halfplanes, overlapping in the hysteresis zone. The relay output $u = x_3$ is a third (discrete) state variable which only takes on the values $x_3 = \pm 1$.

Obviously the system has the two equilibrium points $(x_1, x_2) = (1, 0)$ and $(x_1, x_2) = (-1, 0)$. Both are stable for $\zeta < 0$ and unstable for $\zeta > 0$. The dynamics of this relay system is studied in detail in [5]. For $\zeta < 0$ the system is stable, i. e. all trajectories asymptotically approach one of the stable equilibrium points after having crossed the switching lines for a finite number of times. For $\zeta > \zeta_0$ with $\zeta_0 \approx 0.06735$, the system is completely unstable. For $0 < \zeta < \zeta_0$ however, trajectories starting sufficiently close to one of the unstable equilibrium points evolve away from the equilibrium points but cannot leave a bounded region in state space. For this range of the parameter ζ the relay system exhibits chaotic behavior. Figure 3 shows the Poincaré map for such a case. It is obtained in the following way. For each starting point $(x_1, x_2) = (1, p)$, $p > 0$, on the right switching line the next switching point $(x_1, x_2) = (-1, -q)$, $q > 0$,

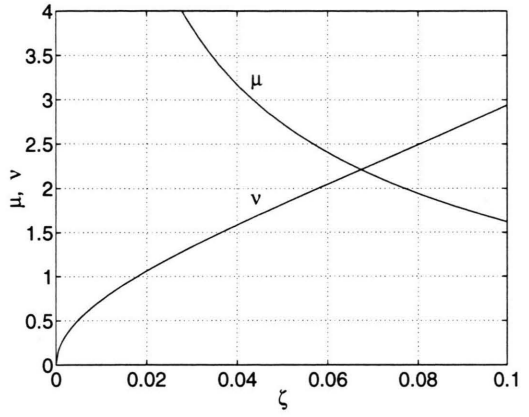

Fig. 4. Attractor parameters μ and ν as a function of system parameter ζ.

on the left switching line is calculated. For a sample trajectory we refer to Figure 5a. The solution of the system equations (2) and (3) is

$$x_1(t) = 1 + \frac{p}{\gamma} e^{\zeta t} \sin(\gamma t), \quad (5)$$

$$x_2(\tau) = \frac{p}{\gamma} e^{\zeta t} \left[\zeta \sin(\gamma t) + \gamma \cos(\gamma t)\right], \quad (6)$$

where $\gamma = \sqrt{1-\zeta^2}$. Setting $x_1(\tau) = -1$ and $x_2(\tau) = -q$, where τ is the switching time, and eliminating τ from these equations, we get the desired Poincaré map $q = \phi(p)$. Due to the symmetry of the system, all subsequent switching points are generated by repeated application of $\phi(p)$. From Fig. 3 it can be derived that the chaotic attractor is the interval $A_\nu = [0, \nu]$. Its basin of attraction is given by $A_\mu = [0, \mu)$, which is limited to the right by an unstable fixed point, see [5] for details. The dependence of ν and μ on ζ, which can be expressed analytically [5], is shown in Figure 4. We note that for $\zeta = \zeta_0$ we have $A_\mu = A_\nu$. The chaotic attractor has embedded densely within it an infinite number of unstable periodic orbits. Those of period 1 are the fixed points $\phi(p) = p$ in Figure 3. The orbits of higher period are the fixed points of the iterated map $\phi^{(k)}(p)$.

3. Application of OGY Control Method

We now describe the application of the OGY control method to the relay system discussed in the previous section. The method is based on the control of the discrete time system associated with the induced dynamic on a Poincaré section. In the case of our relay system, natural Poincaré sections are the two switching lines. Due to symmetry, the value of $|x_2|$ at the switching point can be chosen as the discrete state variable p_i.

For smooth continuous time dynamical systems, the induced Poincaré map is smooth and time invertible. To this sort of chaotic systems the OGY method usually has been applied. For our relay system, the Poincaré map is only one-dimensional and thus not invertible, and in addition it is discontinuous. However, for OGY's control method to be successful, the map only needs to be smooth in a neighbourhood of the fixed point, and this indeed is the case for our relay system. Thus the OGY method also works in this application.

For the stabilization of an unstable periodic orbit, every parameter of the system can be used as long as the discrete linearized system is controllable by this parameter. We use the height of the relay output $b = \bar{b} + \delta b$, where δb is a small correction to the standard value $\bar{b} = 1$. We note that this is equivalent to a variation in the system gain, which was chosen unity in (1).

The control parameter δb is adjusted at each switching point. Thus the Poincaré map now also depends on the variable height b of the relay output, that is

$$p_{i+1} = \phi(p_i, b_i). \quad (7)$$

Let p^* be an unstable fixed point of the Poincaré map (7) for the nominal value $\bar{b} = 1$ of the relay output, that is

$$p^* = \phi(p^*, \bar{b}).$$

This fixed point corresponds to one of the unstable periodic orbits of the system. For values of p_i close to p^* and values of b_i close to $\bar{b}$ the map (7) can be approximated by the linear map

$$\delta p_{i+1} = A\, \delta p_i + B\, \delta b_i, \quad (8)$$

where $\delta p_i = p_i - p^*$ and $\delta b_i = b_i - \bar{b}$ are the deviations from the nominal values, and the (one-dimensional) system matrices are given by

$$A = \left.\frac{\partial \phi}{\partial p}\right|_{(p^*, \bar{b})}, \quad B = \left.\frac{\partial \phi}{\partial b}\right|_{(p^*, \bar{b})}. \quad (9)$$

Now a linear state feedback

$$\delta b_i = -K\, \delta p_i \quad (10)$$

is applied to the discrete time system (8). Substituting the control law (10) into (8) yields

$$\delta p_{i+1} = (A - B\,K)\,\delta p_i. \tag{11}$$

From (11) it can be seen that the closed loop system is stable as long as

$$|\,\mathrm{eig}(A - B\,K)\,| < 1. \tag{12}$$

In [3] the pole placement technique is proposed for determining K. The unstable system poles are shifted to the origin, while the stable ones are left unchanged. For our case the discrete system has only one unstable pole. It is shifted to the origin if $\mathrm{eig}(A - BK) = 0$ or, because A and B are scalars,

$$K = \frac{A}{B}. \tag{13}$$

In this case (known to control engineers as deadbeat control) we have $\delta p_{i+1} = 0$, thus the error vanishes in the next step.

The calculation of the system matrices A and B can be done by numerical differentiation using the solution of (2) and (3) with the relay output set to $u = b$ and initial conditions $x_1(0) = 1$ and $x_2(0) = p$. The solution is given by

$$x_1 = b + e^{\zeta t}\,[c_1 \cos(\gamma t) + c_2 \sin(\gamma t)]\,, \tag{14}$$

$$x_2 = \zeta\, e^{\zeta t}\,[c_1 \cos(\gamma t) + c_2 \sin(\gamma t)] - \gamma\, e^{\zeta t}\,[c_1 \sin(\gamma t) - c_2 \cos(\gamma t)]\,, \tag{15}$$

where $c_1 = 1 - b$, $c_2 = (p - \zeta(1 - b))/\gamma$ and $\gamma = \sqrt{1 - \zeta^2}$. Then A is obtained from (14) and (15) by taking two solutions with fixed $b = \bar{b} = 1$ and two values of p close to p^*, e.g. $p = p_1 = p^* - \Delta p/2$ and $p = p_2 = p^* + \Delta p/2$, with Δp sufficiently small. The switching time τ can be obtained from (14) via interpolation. By inserting τ into (15) one obtains the corresponding next switching point $q = |x_2(\tau)|$. If we denote these next switching points for both trajectories by q_1 and q_2 and set $\Delta q = q_2 - q_1$, then A is obtained by

$$A \approx \frac{\Delta q}{\Delta p} = \frac{q_2 - q_1}{p_2 - p_1}\,.$$

Similarly B can be obtained. The central idea of OGY's method is to activate the control in (10) only if the trajectory is sufficiently close to the fixed point.

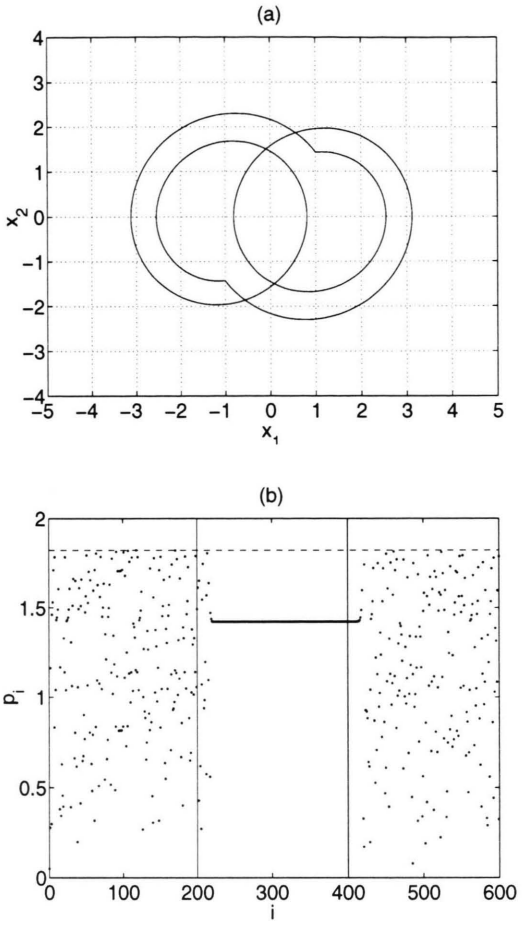

Fig. 5. The unstable period 1 orbit corresponding to the fixed point $p^* = 1.4226$. (b) Stabilization of this orbit via OGY control. Control is turned on at $i = 200$ and turned off at $i = 400$. The control activation bound is set to $\delta p_{\max} = 0.05$.

Because in the original paper [1] the emphasis was on *small control action*, the control activation bound was expressed in terms of the control action itself, i.e. $|\,\delta b_i\,| < \delta b_{\max}$. An alternative, which we will prefer here, is to base it on the distance to the fixed point, thus expressing the control activation bound in terms of the discrete state, i.e. $|\,\delta p_i\,| < \delta p_{\max}$. Note that for the relay system discussed in this paper the discrete time system is one-dimensional and thus $\delta b_{\max} = |\,K\,|\,\delta p_{\max}$. Thus in this case the two approaches differ only by scaling.

As an example, we now stabilize the (symmetric) period 1 orbit of our relay system, which corresponds to the fixed point $p^* = 1.4226$ of the Poincaré map in Figure 3. The trajectory of the orbit is shown in

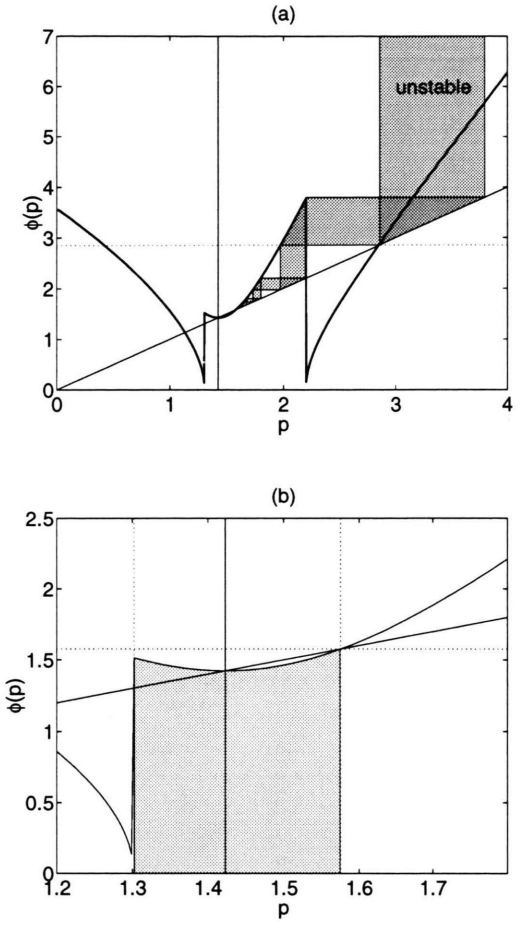

Fig. 6. (a) Poincaré map of the controlled relay system with $\delta p_{\max} = \infty$. The shading indicates points that escape to infinity. (b) Magnification close to the fixed point. The basin of immediate attraction A^* is shaded.

Figure 5a. Here and in all further figures we have chosen $\zeta = 0.05$.

Figure 5b shows the application of OGY control to the relay system for stabilizing this orbit. After switching on OGY control at $i = 200$, the trajectory continues executing a chaotic orbit for some time, unchanged from the uncontrolled case, because it does not come close enough to the fixed point. Eventually p_i falls into the range $|\delta p_i| < \delta p_{\max} = 0.05$. Then it quickly is brought to the fixed point. After deactivating the control at $i = 400$, the trajectory evolves exponentially off the fixed point and exhibits a chaotic behavior again.

4. Poincaré Maps of Controlled System

Because the Poincaré map of our relay system is one-dimensional, the consequences of the choice of the control activation bound can be easily understood by looking at the Poincaré map with activated OGY control. The Poincaré map of the uncontrolled system for $\zeta = 0.05$ was given in Figure 3. This corresponds to the case $\delta p_{\max} = 0$. We recall that the chaotic attractor is the set $A_\nu = [0, \nu]$, $\nu = 1.8213$, while its basin of attraction is $A_\mu = [0, \mu)$, $\mu = 2.7352$. The other extreme case is $\delta p_{\max} = \infty$, which means that the control is always activated, see Figure 6.

When using OGY control one has to be a bit careful about the terming of attracting sets. The aim of OGY control is to turn the whole basin of attraction of the (uncontrolled) chaotic attractor into a basin of attraction for the stabilized orbit. When control is always activated, a certain neighborhood of the stabilized orbit can be identified where all trajectories remain in this neighborhood, and thus are directly attracted to p^*. We will call this set A^* the *basin of immediate attraction* of p^*. In Fig. 6b this set is shown shaded. However, closely outside this basin there are regions where the trajectories escape to infinity. One such family of regions is marked in Figure 6a. It has to be emphasized, however, that these trajectories only escape to infinity if control is not deactivated for large distances from the desired orbit. Only with this deactivation, which is in fact the key idea in OGY's method, the whole chaotic attractor in A_ν, or more precisely, its whole basin of attraction A_μ, can become the basin of attraction of the stabilized orbit. We thus have to consider the following sets:

- A_μ Basin of the chaotic attractor,
- A_ν The chaotic attractor,
- $\bar{A}_\delta$ Largest allowed control activation neighborhood,
- A_δ Control activation neighborhood, i. e. $A_\delta = (p^* - \delta p_{\max}, p^* + \delta p_{\max})$,
- A^* Basin of immediate attraction of p^*.

If $A_\delta \subset A^*$, the situation is simple and save. Control is only activated if the trajectory is inside the basin of immediate attraction and thus cannot escape again. For the case shown in Fig. 6b it is obvious that the basin of immediate attraction is limited to the right by the unstable fixed point located at $p_r = 1.5752$, and to the left by the discontinuity located at $p_l = 1.3013$. The argument above implies that control should only be activated if $p \in A^* = (1.3013, 1.5752)$. Since

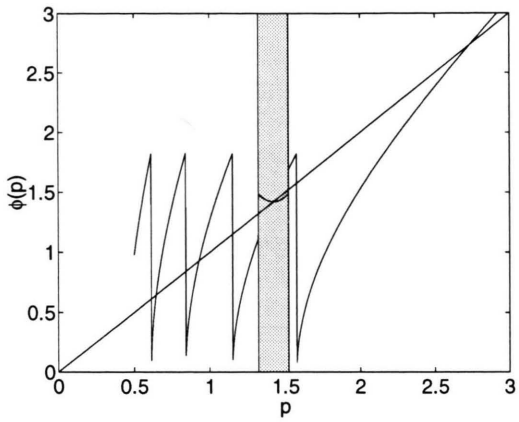

Fig. 7. Poincaré map of the OGY controlled system with $\delta p_{\max} = 0.1$, thus $A_\delta \subset A^*$. Shading marks A_δ.

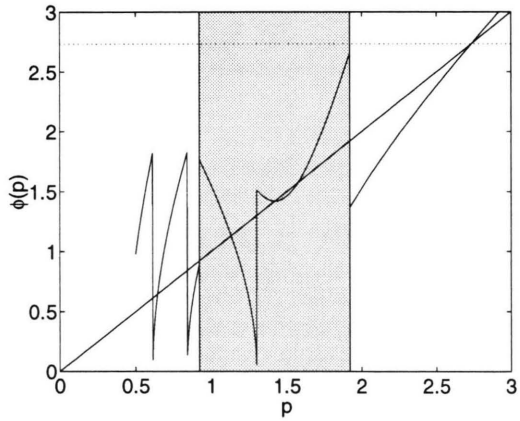

Fig. 8. Poincaré map of the OGY controlled system with $\delta p_{\max} = 0.5$, thus $A^* \subset A_\delta \subset \bar{A}_\delta$. Shading marks A_δ.

$p_r - p^* > p^* - p_l$ the value of $\delta p_{\max}$ should be at most $\delta p_{\max} = p^* - p_l = 0.1213$. Figure 7 shows the Poincaré map for such a case with $\delta p_{\max} = 0.1$. All trajectories starting in the range $|p - p^*| < \delta p_{\max}$ are inside the basin of immediate attraction A^* and are therefore attracted to the fixed point directly. All trajectories starting outside (but within the basin of attraction A_μ of the chaotic attractor) will execute a chaotic orbit, unchanged from the uncontrolled case, until they eventually meet the region $A_\delta = (p^* - 0.1, p^* + 0.1)$, whereupon they are captured forever.

If $\delta p_{\max}$ is increased, such that $A_\delta \supset A^*$, the situation becomes more complicated. The trajectories starting in $A_\delta \setminus A^*$ typically will not be captured directly, but escape. Now there are two cases. In the first one, shown in Fig. 8, OGY control may still be successful. Some trajectories escape into the attractor basin A_μ. Due to the mixing property of the uncontrolled system they will eventually reenter A_δ. If they even enter A^*, they are captured. Otherwise they may escape into the attractor basin A_μ again and the argument can be repeated. Whether they are finally captured to the fixed point p^* must be left to a more detailed investigation.

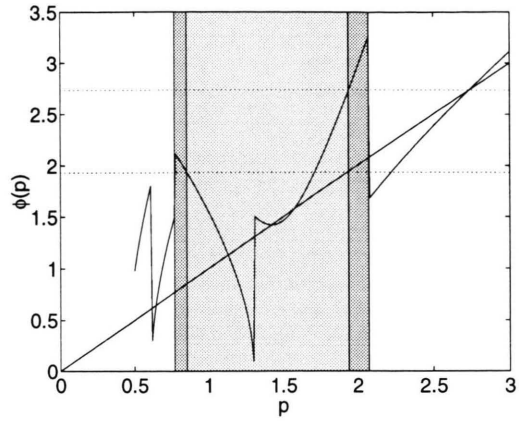

Fig. 9. Poincaré map of the OGY controlled system with $\delta p_{\max} = 0.65$, thus $A^* \subset \bar{A}_\delta \subset A_\delta$. Light shading marks $\bar{A}_\delta$, dark shading marks escaping trajectories.

The critical case is shown in Figure 9. Now some trajectories starting from $A_\delta \setminus A^*$ even escape from the attractor basin A_μ and thus escape to infinity, see the dark-shaded areas in Figure 9. Then OGY control failed. This motivates the introduction of the set $\bar{A}_\delta$ as the largest A_δ where escaping from A_μ does not occur.

As a consequence of the discussion above it follows that for successful OGY control it is recommended to choose $\delta p_{\max}$ such that A_δ is contained in the basin of immediate attraction A^*. If $\delta p_{\max}$ is chosen larger, stability of the control becomes very difficult to establish.

In principle the control gain K may be arbitrarily varied within the stabilizing range $K \in ((A-1)/B), (A+1)/B)$, which can be derived from (12). Figure 10 shows the Poincaré map close to the fixed point for $K = K_0 = A/B$ and the edges of the stabilizing range $K = K_{1,2} = (A \pm 1)/B$. No restriction of control activation is assumed. It is interesting to note that the size of the basin of immediate attraction is nearly independent of the choice of K. The basin, however, becomes rather unsymmetrical

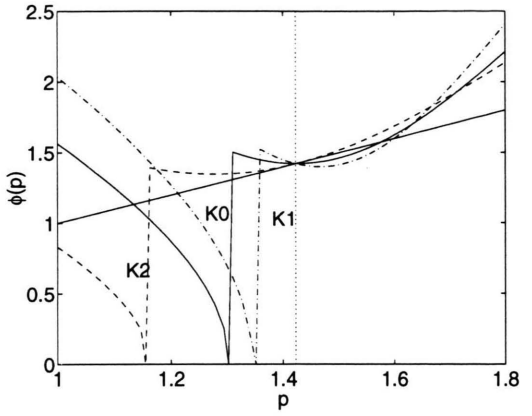

Fig. 10. Poincaré map of the controlled system using a control gain K, varying over the stabilizing range. $K_0 = A/B$, $K_1 = (A+1)/B$, $K_2 = (A-1)/B$.

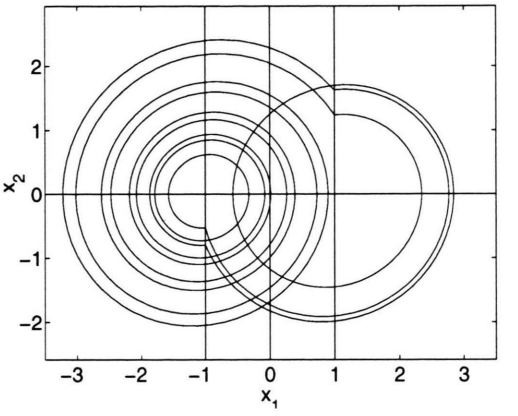

Fig. 11. Period 4 orbit of relay system.

if the stability boundaries are approached. If the control activation is based on the symmetric condition $|p - p^*| < \delta p_{\max}$, the deadbeat value K_0 obtains a certain justification by the fact that it approximately maximizes $\delta p_{\max}$ subject to $A_\delta \subset A^*$.

5. One-step Versus Multi-step Control for Higher Period Orbits

The straight forward control method for a period T orbit is to do just the same as for an orbit of period 1, but using the Tth iterate of the Poincaré map instead. However, the eigenvalues of the iterated Poincaré map are much larger, rendering the control problem much more sensitive. In [3] a straight forward modification to circumvent this problem is proposed. We consider a given orbit of period T,

$$p^*_{(i+1)} = \phi(p^*_i, \bar{b}) \text{ with } p^*_{(i+T)} = p^*_i.$$

Linearization of (7) along the orbit yields

$$\delta p_{i+1} = A_i\, \delta p_i + B_i\, \delta b_i \quad (i = 1, 2, \ldots, T), \tag{16}$$

where $\delta p_i = p_i - p^*_i$ and $\delta b_i = b_i - \bar{b}$ and

$$A_i = \left.\frac{\partial \phi}{\partial p}\right|_{(p^*_i, \bar{b})}, \quad B_i = \left.\frac{\partial \phi}{\partial b}\right|_{(p^*_i, \bar{b})}. \tag{17}$$

We now apply the OGY control method in the same way as for a period 1 orbit. Thus we introduce a state feedback

$$\delta b_i = -K_i\, \delta p_i \quad (i = 1, 2, \ldots, T) \tag{18}$$

to the discrete time systems (16), which is only activated if $|\delta p_i| < \delta p_{\max}$. For the gains K_i one can again choose

$$K_i = A_i / B_i \quad (i = 1, 2, \ldots, T).$$

We will now illustrate the superiority of the multi-step OGY control as described above, compared to the one-step method. We will stabilize a period 4 orbit of our relay system using the standard and the multi-step method. The sample orbit is shown in Figure 11.

First we look at the one-step method. The eigenvalue of A is much larger for a period 4 orbit than for the period 1 orbit in Section 3. The 4. iterate of the Poincaré map at $p^*_1 = 1.6311$ leads to the discrete linearized system

$$\delta p_{i+1} \;=\; 852.4\, \delta p_i - 588.7\, \delta b_i. \tag{19}$$

Then the range of stabilizing gains is

$$K \in \left(\frac{A+1}{B}, \frac{A-1}{B}\right) = (1.4448, 1.4482).$$

Note that the relative variation of the stabilizing gains is only $2/A = 2.3 \cdot 10^{-3}$, leading to an extreme sensitivity with respect to the choice of the correct gain. The basin of immediate attraction for the deadbeat control is shown shaded in Figure 12. The width of the basin is only 0.0022, which is two orders of magnitude less than the basin for the orbit of period 1 discussed in Section 4. The multistep-method additionally uses the three intermediate points visible in Fig. 11, namely $p^*_2 = 0.52994$, $p^*_3 = 1.2412$, and $p^*_4 = 0.79944$. The central part of the Poincaré map

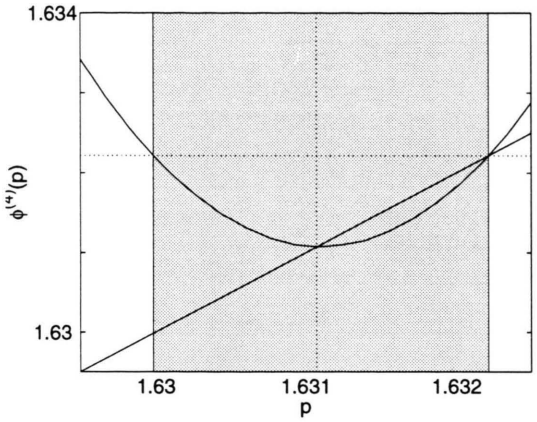

Fig. 12. Poincaré map for the controlled system close to the fixed point, using one-step control ($\delta p_{\max} = \infty$).

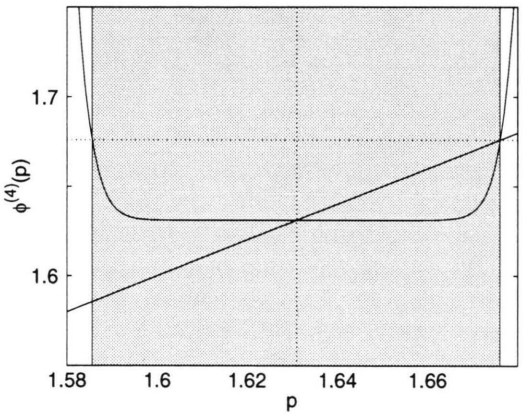

Fig. 13. Poincaré map for the controlled system close to the fixed point, using the multistep method. Note that the scale differs from Figure 12.

with permanent control activation for this method is shown in Figure 13. It turns out that the basin of immediate attraction A^* in this case is roughly by a factor of 40 larger than for the one-step control. Note that the basin is very flat, because the map actually includes four successive control actions. Thus for our special system, the general claim of superiority of the multi-step method in [3] can be directly confirmed.

5. Conclusions

We investigated some aspects of the OGY method of controlling chaos for a relay system using the one-dimensional Poincaré map. This allowed a direct analysis of the different cases that arise, if the control activation bound for the stabilization of an unstable periodic orbit is gradually increased. The basin of immediate attraction of the stabilized orbit could be explicitly derived for the considered system. The superiority of the multi-step OGY method for higher periodic orbits can clearly be demonstrated by comparing the respective basins of immediate attraction.

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