

# Avalanche Dynamics in Piles of Two Types of Sand

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The avalanche statistics of some piles of two types of sand are studied theoretically. Both symmetrical and asymmetrical piles do not preserve the behaviour of self-organized criticality. We suggest a distribution (we call it Fisher distribution) interpolating between fractal and Levy-Weibull distributions which describes these dynamics.

Some real systems usually drive themselves into a critical state. These systems are said to be in a state of self-organized criticality [1]. The theory of self-organized criticality aims to describe interactive, dissipative and self sustaining systems with many degrees of freedom. Hence it complements the chaos concept. These systems are characterized by power law statistics. Self-organized criticality has applications in many fields, e.g., sand piles [2, 3], earthquake dynamics [4, 5], dynamics of biological populations [6], interacting economic systems [7], etc.

The statistics of the systems which display self-organized criticality are characterized by power law distributions [1] of the form

$$P(x) \propto x^{-\alpha}, \quad (1)$$

where  $P(x)$  is the probability distribution function of the argument  $x$ . The exponent  $\alpha$  must be greater than unity to ensure normalizability. This reflects the fractal distribution of  $x$ , which means the absence of a characteristic scale. This concept of fractality has been widely observed in nature [8].

Recently, there are some arguments, e.g. [9], suggesting that for very large  $x$  the probability distribution functions obey the Levy-Weibull formula [10]

$$P(x) \propto e^{-x^\beta}, \quad 0 < \beta < 1. \quad (2)$$

Also this distribution has applications in many fields [9, 11].

Here, we are interested in the concept of self-organized criticality in sand piles of more than one type of sand. Consider a square lattice with open boundaries, integer variables  $Z(i, j)$  representing the height in the sand pile at the site  $(i, j)$ . Initially the height at any site is a random number of sand grains between 0 and 3. A sand grain is

randomly added to a site. Tumbling occurs if the height at a site exceeds 3 sand grains; then four of them redistribute to the four nearest neighbours. This process is repeated until all  $Z(i, j)$  becomes less than 4, hence the avalanche ends. Further grains are randomly added. For each avalanche we calculate its size (the number of tumblings),  $s$ , and the number of distinct sites,  $s_d$ . Dropping 6000 sand grains on a  $100 \times 100$  lattice, we found that the probability distributions of these quantities obey the power laws

$$P(s) \propto s^{-1.08}, \quad (3)$$

$$P(s_d) \propto s_d^{-1.02}. \quad (4)$$

Our result is very close to that obtained in [6].

Now we construct a pile of two types of sand. Consider a 2-dimensional lattice with open boundaries, two integer variables  $X(i, j)$  and  $Y(i, j)$  representing the number of sand grains of the X- and Y-type of sand, respectively. Assume that the grains of the Y-type are heavier than those of the X-type. Initially all  $X(i, j)$  are randomly taken between 0 and 3, and all  $Y(i, j)$  are randomly taken between 0 and 2. Tumbling occurs if

$X(i, j) = 4$ , the redistribution is spatially symmetric. (5)

$Y(i, j) = 3$ , it redistributes to the left, right and down sites. (6)

$X(i, j) = 3$  and  $Y(i, j) = 1$ , X-type redistributes to the left, right and down sites, while Y-type redistributes to the right site. (7)

$X(i, j) = 2$  and  $Y(i, j) = 2$ , X-type redistributes to the left and right sites, while Y-type redistributes to the up and down site. (8)

The rules (5) and (6) are identical to the single type of sand rules. Rules (7) and (8) represent the mixing between the two types of sand. Changing the assumed

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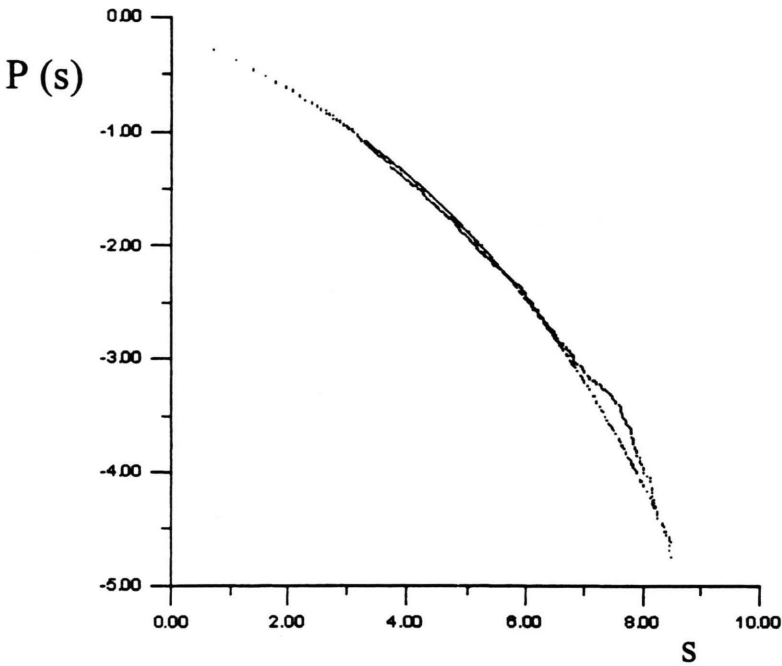


Fig. 1. A log-log plot for the distribution of the avalanche sizes,  $s$  with its fitting for the symmetric case

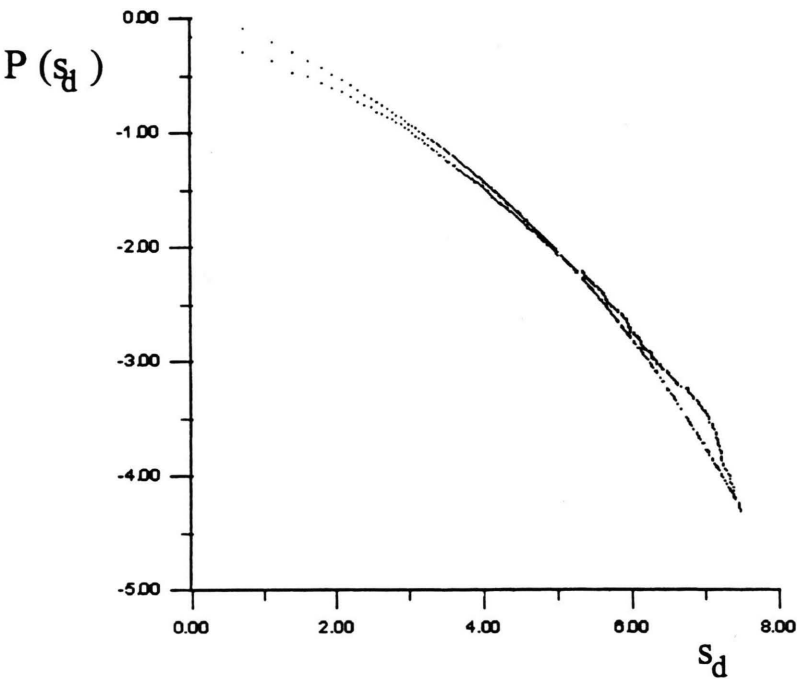


Fig. 2. A log-log plot for the distribution of the number of distinct sites,  $s_d$  with its fitting for the symmetric case

directions in rules (7) and (8) does not affect the results.

We studied a symmetrical pile of two types of sand by adding two sand grains (one of each type) randomly on a single site and letting the lattice relax. Repeating this

process 3000 times on a  $100 \times 100$  lattice, we found that the probability distributions obey power laws of the form

$$P(s) \propto s^{-0.57}, \tag{9}$$

$$P(s_d) \propto s_d^{-0.45}. \tag{10}$$

Table 1. The values of  $C$ ,  $\alpha$  and  $\beta$  of our distribution for the avalanche sizes and the number of distinct sites in the symmetric and asymmetric cases.

	Avalanche size			Number of distinct sites		
	$C$	$\alpha$	$\beta$	$C$	$\alpha$	$\beta$
Symmetric case	2.4	0.003	0.2	3.0	0.03	0.22
Asymmetric case	3.1	0.07	0.19	3.1	0.01	0.22

We got the same result when adding one X-type grain followed by one Y-type grain.

Asymmetry is introduced to the system by adding three X-type grains, followed by adding one Y-type grain, and so on. We added 6000 sand grains on a  $100 \times 100$  lattice and found that the probability distributions follow a similar statistics as

$$P(s) \propto s^{-0.57}, \quad (11)$$

$$P(s_d) \propto s_d^{-0.63}. \quad (12)$$

Then both symmetrical and asymmetrical piles of two types of sand preserve a power law behaviour with exponents larger than  $-1$ . Thus there must exist a cut-off scale other than the system size to ensure normalizability. This means that no fractal distribution is observed.

Consider the probability distribution

$$P(x) = C x^{-\alpha} e^{-x^\beta} \quad (13)$$

as an interpolating formula between fractal and Levy-Weibull distributions. This generalized distribution has previously appeared for the cluster statistics in the percolation theory [12], and we propose to call it Fisher [13] distribution.

The statistics of our system is fitted very well by (13). This is shown in Fig. 1 for the avalanche size and Fig. 2 for the number of distinct sites for the symmetric case. The values of  $C$ ,  $\alpha$  and  $\beta$  in the symmetrical and asymmetrical cases are shown in Table 1.

The presence of a non-negligible exponential term in the probability distribution of the quantities describing our system ensures the non-fractal behaviour. Therefore the system does not display self-organized criticality. This agrees with the results of some experiments on rice piles [14] and some types of rotating disks models [15].

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