Log-Compound-Poisson Distribution Model of Intermittency in Turbulence

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The log-compound-Poisson distribution for the breakdown coefficients of turbulent energy dissipation is proposed, and the scaling exponents for the velocity difference moments in fully developed turbulence are obtained, which agree well with experimental values up to measurable orders. The underlying physics of this model is directly related to the burst phenomenon in turbulence, and a detailed discussion is given in the last section.

1. Intermittency of Turbulence

In Kolmogorov's 1941 theory [1], all statistically averaged quantities of fully developed turbulence at scale l depend only on the mean dissipation rate $\langle\varepsilon\rangle$ and l, where $\langle\ \rangle$ denotes the ensemble average, but the fluctuation of energy dissipation is disregarded. This theory gives the famous $-5/3$ power law of the energy spectrum and the linear scaling law of the scaling exponents for the velocity difference moments. Nevertheless, extensive experimental [2] and numerical [3] studies provide evidence that the deviation from Kolmogorov's similarity law becomes increasingly significant for the higher order velocity difference moments. This suggests that the fluctuation of energy dissipation should contribute considerably to the small scale statistics of turbulence. This problem of the internal intermittency of turbulence has generated a large literature and there are many models, such as: log-normal distribution model [4], log-stable distribution model [5], log-gamma distribution model [6], She-Levegue model [7], etc. [8].

In this paper we propose a new possible model, from which the derived scaling exponents are in good agreement with experimental values up to measurable orders.

Consider one-dimensional averaged turbulent energy dissipation density, as is the case in the experimental measurements of dissipation fluctuation. Let ε_r denote the energy dissipation density averaged over a scale r. We take three arbitrary scales: $r<s<l$ in the inertial range, and introduce corresponding breakdown coefficients:

$$e_{r,l} = \varepsilon_r/\varepsilon_l\,, \tag{1}$$

$$e_{r,l} = e_{r,s}\, e_{s,l}\,. \tag{2}$$

By making use of the fact that the energy dissipation density is non-negative, we obtain

$$0 \le e_{r,l} \le l/r\,. \tag{3}$$

Novikov [9] has proposed the assumption of scale similarity, which states that the probability distribution for $e_{r,l}$ depends only on the scale ratio, and $e_{r,s}$ and $e_{s,l}$ are statistically independent. From the scale similarity assumption and (3), the p-th order moment of the breakdown coefficient can be expressed as

$$\langle e_{r,l}^p\rangle = (l/r)^{\mu(p)}\,. \tag{4}$$

We now introduce the ratio of energy dissipation

$$q_{r,l} = \frac{r\varepsilon_r}{l\varepsilon_l} = e_{r,l}\,(r/l) \le 1\,, \tag{5}$$

which represents the ratio of the energy dissipated in scale r to that in scale l. The p-th order moment of $q_{r,l}$ has the form

$$\langle q_{r,l}^p\rangle = (l/r)^{\mu(p)-p}\,. \tag{6}$$

From (2), we have

$$q_{r,l} = q_{r,s}\, q_{s,l}\,. \tag{7}$$

By making use of the random variable

$$z_{r,l} = -\ln q_{r,l}\,, \tag{8}$$

where $0 \le z_{r,l} < \infty$, we get from (7)

$$z_{r,l} = z_{r,s} + z_{s,l}\,. \tag{9}$$

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For arbitrary ratio l/r and arbitrary integer n, let

$$k = (l/r)^{1/n}, \text{ and } s_i = rk^i, \quad i = 0, 1, 2, \ldots, n \,, \tag{10}$$

We can generalize (9) in the form

$$z_{r,l} = z_{s_0,s_1} + z_{s_1,s_2} + \ldots + z_{s_{n-1},s_n} \,. \tag{11}$$

On the basis of the scale similarity assumption, the random variables on the right hand side of (11) are independent of each other and have the same distribution function. Thus, for arbitrary l/r and arbitrary integer n, the random variable $z_{r,l}$ can be expressed as the sum of n independent random variables with a common distribution. Therefore, the distribution of $z_{r,l}$ is infinitely divisible [10]. This important feature seems to have first been noticed by Saito [6], then by Novikov [8], and other authors [11].

2. Compound-Poisson Distribution

The characteristic function of $z_{r,l}$ has the form

$$\begin{aligned}\Phi(\zeta, l/r) &= \langle \exp(i\zeta z_{r,l})\rangle = \langle \exp(-i\zeta \ln q_{r,l})\rangle \\ &= \int_0^1 q_{r,l}^{-i\zeta}\, p(q_{r,l}, r/l)\mathrm{d}q_{r,l} = \left(\frac{l}{r}\right)^{i\zeta + \mu(-i\zeta)} .\end{aligned} \tag{12}$$

The characteristic function Φ is infinitely divisible if for each n the condition

$$\Phi(\zeta, l/r) = \Phi^n(\zeta, l/r)^{1/n}) \tag{13}$$

in satisfied [10]. It is natural to suppose that the distribution function of $z_{r,l}$ is of the compound Poisson type, the characteristic function of which has the form

$$\Phi(\zeta, l/r) = \exp\left[c \ln \frac{l}{r}(\varphi - 1)\right], \tag{14}$$

where φ is a characteristic function of the arbitrary probability density function f (called the basic probability density in the following passages). Obviously, the condition (13) is satisfied by (14), and (14) is infinitely divisible. Although an infinitely divisible distribution need not be of the compound Poisson type, it can be proved the every infinitely divisible distribution is the limit of a sequence of compound Poisson distributions.

As $r \to l$, $z_{r,l} \to 0$. Therefore, the probability density for $z_{r,l}$ tends to the delta function $\delta\,(z_{r,l})$, as $r \to l$. Obviously, the characteristic function (14) satisfies this condition, since

$$\Phi(\zeta, l/r) \to 1 \quad \text{as} \quad r \to l\,. \tag{15}$$

If the basic probability density takes the form of a gamma density

$$f(x) = \frac{1}{\Gamma(t)} \beta^t x^{t-1} e^{-\beta x}\,, \tag{16}$$

and the corresponding characteristic function [10] is

$$\varphi(\zeta) = 1/(1 - i\zeta/\beta)^t\,, \tag{17}$$

then from (14) we obtain

$$\Phi(\zeta, l/r) = \left(\frac{l}{r}\right)^{c[1/(1-i\zeta/\beta)^t - 1]} . \tag{18}$$

The probability density for $z_{r,l}$ can be calculated by the Fourier inversion formula

$$p(z, l/r) = \frac{1}{2\pi}\int e^{-i\zeta z}\, \Phi(\zeta, l/r)\, \mathrm{d}\zeta\,. \tag{19}$$

Substituting (18) into (19), and using exponential expansion, we can obtain

$$p(z, l/r) = \exp[-c \ln(l/r) - \beta z] \cdot \sum_{n=0}^{\infty} \frac{c^n (\ln(l/r))^n}{n!\, \Gamma(nt)} \beta^{nt} z^{nt-1}\,. \tag{20}$$

If the basic probability density takes the form of a delta function

$$f(x) = \delta(x - \alpha)\,, \tag{21}$$

then the corresponding characteristic function can be written as

$$\varphi(\zeta) = \exp(i\alpha\zeta)\,, \tag{22}$$

and from (14) we get

$$\Phi(\zeta, l/r) = \left(\frac{l}{r}\right)^{c[\exp(i\alpha\zeta) - 1]} . \tag{23}$$

3. Scaling Exponents

In fully developed turbulence, the p-th order velocity structure function follows a power law in the inertial range

$$S_p(r) = \langle [u(x+r) - u(x)]^p \rangle \propto r^{\xi(p)}\,, \tag{24}$$

where u is the velocity component parallel to r, r the separation. The p-th order moment of locally averaged energy dissipation density over a scale r has a similar relation

$$\langle \varepsilon_r^p \rangle \propto r^{\tau(p)}\,. \tag{25}$$

The Kolmogorov refined similarity hypothesis [4] states that

$$\xi(p) = p/3 + \tau(p/3) . \qquad (26)$$

Let l equal the integral scale L, ε_L be a constant, and the breakdown coefficient $e_{r,L}$ be proportional to ε_r. From (4) we obtain

$$\langle \varepsilon_r^p \rangle \propto \langle e_{r,L}^p \rangle \propto r^{-\mu(p)} ,$$

and $\tau(p) = -\mu(p)$. (27)

If the basic probability density takes the form of a gamma density (16), from (12) and (18) we get

$$\mu(p) = p + c\,[1/(1+p/\beta)^t - 1] . \qquad (28)$$

The scaling exponents for the velocity difference moments are

$$\xi(p) = c\,\{1 - 1/[1+p/(3\beta)]^t\} . \qquad (29)$$

The parameter c is determined by the exact relation $\xi(3)=1$, and (29) can be rewritten as

$$\xi(p) = \{1 - 1/[1+p/(3\beta)]^t\}/[1 - 1/(1+1/\beta)^t] . \qquad (30)$$

The parameter β can be determined by $\xi(6)$. $\xi(6)=2-\mu$, where μ is the universal exponent defined by the spectrum of dissipation fluctuations. If we choose $\mu=2/9$, in agreement with most experiments [12], then $\xi(6)=1.7778$.

The prediction of the formula (30) for $t=1, 2, 3$, and comparison with experiments [13] and the She-Leveque model [7] are listed in Table 1. Evidently the prediction of formula (30) is consistent with the experiments [13] and the She-Levegue model [7] up to measurable orders $p \le 10$. The suitable choice of the parameter t may be determined by comparison of the measured probability density of $z_{r,l}$ and the theoretical formula (20).

Table 1. Scaling exponents.

p	Experiment [13]	SL model [7]	Formula (30) $t=1$	$t=2$	$t=3$
1	0.37	0.363	0.364	0.363	0.363
2	0.70	0.695	0.696	0.695	0.695
3	1.00	1.000	1.000	1.000	1.000
4	1.28	1.279	1.280	1.280	1.281
5	1.54	1.538	1.539	1.539	1.539
6	1.78	1.778	1.778	1.778	1.778
7	2.00	2.001	2.000	1.999	1.998
8	2.23	2.210	2.207	2.204	2.203
9		2.407	2.400	2.395	2.392
10		2.593	2.581	2.572	2.568
12		2.938	2.909	2.892	2.883
14		3.254	3.200	3.172	3.157
16		3.548	3.460	3.418	3.396
18		3.824	3.693	3.636	3.606
20		4.088	3.903	3.830	3.791
∞		∞	8.001	6.248	5.664

The interesting prediction of (30) is the saturation of the exponents for the velocity difference moments as $p \to \infty$

$$\xi(\infty) = 1/[1 - 1/(1+1/\beta)^t] . \qquad (31)$$

If the basic probability density takes the form of the delta function (21), then from (23) and (12) we get

$$\mu(p) = p + c\,[\exp(-\alpha p) - 1] . \qquad (32)$$

From (26), the scaling exponents for the velocity difference moments are

$$\xi(p) = c\,[1 - \exp(-\alpha p/3)] . \qquad (33)$$

Setting $\mu = 1 - \exp(-\alpha)$ and using $\xi(3)=1$, (33) can be rewritten as

$$\xi(p) = \frac{1}{\mu}[1 - (1-\mu)^{p/3}] , \qquad (34)$$

where $\mu=2/9$. This is just Chen-Cao's formula [14], which corresponds to the log-Poisson distribution model.

4. Physical Background

It is conjectured that in three-dimensional space the curl of velocity may become infinite in some small sets of the domain occupied by turbulence. In fact Leray [15] in 1932 proposed the possible appearance of singularities as an explanation for turbulence. In 1949, Batchelor and Townsend [16] observed a fairly definite alternation between periods of quiescence, during which the magnitude of the derivative of the velocity is small, and periods of activity during which the derivative fluctuates in an apparently random fashion and the vorticity tends to concentration in isolated regions. The burst phenomenon was first observed in wall turbulence [17]. At the wall a horseshoe-shaped vortex is beginning to be formed, and this vortex is deformed by the flow into a more and more enlongated U-shaped loop in streamwise direction. Due to self-induction, the top of the loop drifts away from the wall thereby coming into regions of ever-increasing velocity. Consequently the vorticity increases because of stretching processes. The local inflexional instability and breakdown of the top of the vortex produces a turbulence burst. The pressure waves associated with the turbulence burst are propagating through the whole boundary layer.

It is natural to suppose that the burst phenomena also occur in other turbulent flows such as in grid-generated turbulence, in turbulent wake-flows, etc. In the fully turbulent region the burst phenomena are obscured by the general background turbulence, and consequently not directly noticeable. However, by filtering out this background turbulence using a narrow bandwidth wave analyzer, Rao et al. [18] found that the "bursts" clearly show up in the filtered traces. The burst region is associated with high vorticity and large dissipation, while the dissipations in the unburst regions are very small and can be neglected. By a direct simulation of the Navier-Stokes equations Siggia [19] showed that the dissipation is concentrated in a tiny region of the space. We now have a general vivid picture of turbulence: here and there the vortices are formed sometimes, and due to stretching and inflexional instability they ultimately breakdown (bursts). The bursts are locally limited in space and time, but their influences are not local. The pressure waves and velocity waves associated with bursts are propagated through the whole region. The bursts are random variables with various intensities. Therefore, the pressure waves and velocity waves have various intensities and various frequencies and wave numbers, the superposition effect of which gives very complicated flow fluctuations. The bursts of the vortices are the cause of internal intermittency in turbulence.

Assuming that the turbulence is homogeneous, we consider the random occurance of bursts in a certain region of scale r (the location is not relevant to the question). We denote the random number of bursts in this region at time τ, assuming a zero count at $\tau=0$, by $N(\tau)$:

$$N(\tau)=\sum_{i=1}^{\infty} H(\tau-T_i), \quad t\geq 0, \tag{35}$$

where $H(\tau)$ is the Heaviside step function, and τ_i the time of the i-th burst. In order to obtain the probability distribution of $N(\tau)$: $P_n(\tau)=\text{Prob}\{N(\tau)=n\}$, the following basic assumptions are proposed [20]:

1. The probability that a burst will occur during the time interval $(\tau, \tau+\Delta)$ is $\lambda\Delta+O(\Delta)$, where λ is a constant and $O(\Delta)$ tends to zero faster than Δ;
2. The probability that more than one burst will occur in $(\tau, \tau+\Delta)$ is $O(\Delta)$. Therefore the probability of no change in $(\tau, \tau+\Delta)$ is $1-\lambda\Delta-O(\Delta)$;
3. The random number of bursts that occur in some time interval is independent of the random number of bursts that occur in any non-overlapping interval.

The probability of the occurrence of n bursts during the interval $(0, \tau+\Delta)$ can be realized in three mutually exclusive ways: (i) all n bursts will occur in $(0, \tau)$ and non in $(\tau, \tau+\Delta)$ with probability $P_n(\tau)[1-\lambda\Delta-O(\Delta)]$; (ii) exactly $n-1$ bursts will occur in $(0, \tau)$ and one burst in $(\tau, \tau+\Delta)$ with probability $P_{n-1}(\tau)[\lambda\Delta+O(\Delta)]$; (iii) exactly $n-k$ bursts will occur in $(0, \tau)$ and k bursts in $(\tau, \tau+\Delta)$, where $2\leq k\leq n$, with probability $O(\Delta)$. Taking all these possibilities together, we have for $n\geq 1$

$$P_n(\tau+\Delta)=P_n(\tau)[1-\lambda\Delta]+P_{n-1}(\tau)\lambda\Delta+O(\Delta), \tag{36}$$

and for $n=0$

$$P_0(\tau+\Delta)=P_0(\tau)[1-\lambda\Delta]+O(\Delta). \tag{37}$$

For $\Delta\to O$ we obtain from (36) the differential equations

$$\frac{d}{d\tau}P_n(\tau)=-\lambda P_n(\tau)+\lambda P_{n-1}(\tau), \quad n\geq 1 \tag{38}$$

and $$\frac{d}{d\tau}P_0(\tau)=-\lambda P_0(\tau). \tag{39}$$

The initial conditions are

$$P_0(O)=0, \quad P_n(O)=0. \tag{40}$$

From (37)–(40) it is easy to get the solutions

$$P_0(O)(t)=e^{-\lambda t}, \quad P_n(t)=e^{-\lambda t}(\lambda t)^n/n!. \tag{41}$$

This is the Poisson distribution, and $\{N(\tau), \tau\geq 0\}$ is called the Poisson counting process.

If we denote the energy dissipation associated with the i-th burst of the vortex in the region of scale r by D_i, assuming that D_i are independent and have identical distributions, and $\{N(\tau)\}$ is independent of $\{D_i\}$, then the total energy dissipation in the time intervall $(0, \tau)$ is

$$E(\tau)=\sum_{i=1}^{N(\tau)} D_i. \tag{42}$$

$\{E(\tau), \tau\geq 0\}$ is a compound-Poisson process. Here we neglect the energy dissipation in the time interval when there is no burst of the vortex. If T is a large enough time interval, then the energy dissipation in unit time is approximately

$$r\varepsilon_r=L\varepsilon_L q_{r,L}=E(T)/T, \tag{43}$$

where L is the integral scale. From (43), we obtain

$$z_{r,L}=-\ln q_{r,L}=-\ln E(T)+\ln(TL\varepsilon_L). \tag{44}$$

The second term on the right hand side of (44) is a constant. Therefore, the random variable $z_{r,L}$ has a log-compound Poisson distribution. The basic probability density function f (mentioned in Sect. 2) is just the probability density of the energy dissipation associated with a burst of the vortex. If $f(x)=\delta(x-a)$, then the intensity of every burst of the vortex is a constant. This is a simplified situation.

Recently, some numerical and experimental works [21–23] have revealed that transverse velocity increments are more intermittent than longitudinal ones. Chen et al. [24, 25] proposed that transverse velocity increments bear the same relation to locally averaged enstrophy (squared vorticity) as longitudinal velocity increments bear in the refined similarity hypothesis to locally averaged dissipation. Our paper only concerns the energy dissipation cascade and the scaling of longitudinal velocity increments. The present work may be generalized from dissipation to enstrophy to fit into the above new finding.

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