A Theoretical Approach to Control Chebyshev Polynomials Chaos

Andrzej Posiewnik

Instytut Fizyki Toretycznej i Astrofizyki Uniwersytet Gdański, Wita Stwosza 57, Gdańsk, Poland

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We theoretically investigate the Güémez-Matias method for controlling chaotic behaviour. We obtain an algorithm to determine the value of the control parameter γ.
Numerical simulations confirme our theoretical approach.

In the last couple of years several methods have been developed to control chaotic dynamical systems. It is interesting that most of these techniques make use of the sensitive dependence of chaotic dynamical systems on the initial conditions. A review of some of the techniques, with emphasis on work which demonstrates that the sensitivity to tiny perturbations can make chaotic problems ideally suited for control, is given in [1].

The existing methods of chaos control can be divided into two categories: feedback and non-feedback methods.

Feedback methods [2, 3] stabilize orbits already existing in the system.

Nonfeedback methods [4, 5] apply a small driving force or small modulation to some system parameters or variables.

These methods modify the dynamical system in such a way that stable orbits or fixed points appear.

One of the most well-known feedback methods of controlling chaos is the Ott-Grebogi-Yorke (OGY) method [6]. This procedure takes advantage of the fact that unstable periodic states in chaotic systems typically have a stable direction (stable manifold) which can be put to use. This method has the advantage that it requires no a priori analytical model, but the serious drawback that it often takes a long time before reaching the neighbourhood of one of the points on the path leading to the target. A considerable improvement can be achieved by combining the targeting approach and the OGY method.

In this paper we analyse a nonfeedback method of chaos control invented by Güémez and Matias [7, 8].

In the G-M method, a periodic orbit is stabilized by performing rather nonspecific changes in the system variables. Periodic proportional perturbations in the form of pulses are applied to a chaotic system. The perturbation is directly applied to the system variables. No knowledge of the system behaviour is required.

In this paper we consider one-dimensional maps, although our results can easily be extended to more general systems.

Let the dynamics of our system be given by a differentiable map of the form

$$x_{n+1} = T(x_n)\,; \quad x_n \in R^1\,, \quad n = 1, 2, \ldots\,.$$

In the G-M method one performs changes in the system variables in the form of instantaneous pulses spaced in time, i.e.

$$x_{n+1} = T(x_n)\,(1 + \gamma\delta_{n,p})\,; \quad \gamma \in R^1\,,$$

where $\delta_{n,p}=1$ if n is a multiple of $p \in N$. (N is the set of natural numbers) and zero otherwise.

Thus every p time steps the variable x is modified by means of a proportional feedback of strength $\gamma \in R^1$. As G-M showed, using the method with suitable γ one can stabilize periodic orbits of a periodicity equal to a multiple of p.

Let us assume that for some value of the parameter γ there exists for the perturbed dynamical system an asymptotically stable orbit of period k. If we denote the initial point of the orbit by x^*, x^* should be a solution of the equation

$$(1+\gamma)\,T^k(x) = x\,, \tag{1}$$

and the condition of asymptotic stability reads

$$\left|(1+\gamma)\,\frac{\mathrm{d}}{\mathrm{d}x}\,T^k\,(x)\right|_{x=x^*} < 1\,. \tag{2}$$

Now let us apply the G-M method to a special class of dynamics given by the Chebyshev polynomials.

Reprint requests to A. Posiewnik;
e-mail: fizap@paula.univ.gda.pl

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Consider the family of functions

$$T_p(x) = \cos(p\theta)\,, \tag{3}$$

where p is a nonnegative integer, $x = \cos\theta$ and $0 \le \theta \le \pi$.

The function $T_p(x)$ is defined by (3) in the interval $-1 \le x \le 1$, which we also denote by I. $T_p(x)$ is a single-valued function and may be written

$$T_p(x) = \cos p\,(\arccos x)\,,$$

where

$$0 \le \arccos x \le \pi\,.$$

One can easily see that $T_p(x)$ is a polynomial of degree p. $T_p(x)$ is called the Chebyshev polynomial of degree p.

The first few Chebyshev polynomials are

$$T_0(x) = 1\,,\quad T_1(x) = x\,,\quad T_2(x) = 2x^2 - 1\,,$$
$$T_3(x) = 4x^3 - 3x\,,\quad T_4(x) = 8x^4 - 8x^2 + 1\,.$$

The Chebyshev polynomials $T_p(x)$ define mappings $x \to T_p(x)$ of I onto I for each $p = 0, 1, 2, \ldots$. If i, j are nonnegative integers, then $T_i(T_j(x)) = T_{i\cdot j}(x)$.

Definition

Let (X, B, μ) be a separable finite measure space, and let τ be a mapping of X onto itself that is measurable, i.e. such that $\mathrm{B} \in B$ implies $\tau^{-1}(\mathrm{B}) \in B$. τ is said to be *measure preserving* if

$$\mu(\tau^{-1}\mathrm{B}) = \mu(\mathrm{B})\,,\quad \mathrm{B} \in B\,,$$

and if τ is measure preserving, it is called *strongly mixing* if

$$\lim_{k\to\infty} \mu(\tau^{-k} A \cap \mathrm{B}) = \frac{\mu(A)\,\mu(\mathrm{B})}{\mu(X)}$$

for all $A, \mathrm{B} \in B$.

Every strongly mixing transformation is *ergodic*; i.e. if

$$\tau^{-1}(A) = A$$

for some $A \in B$, then either $\mu(A) = 0$ or $\mu(A) = \mu(X)$.

Theorem [9]

Each T_p with $p > 1$ is strongly mixing, hence ergodic.

Let us consider a dynamics given by the iterations $x_{n+1} = T_p(x_n)$, where T_p is the p-th Chebyshev polynomial, and the initial conditions are

$$-1 \le x_0 = \cos\theta_0 \le 1\,.$$

Then one can write

$$x_{n+1} = \cos(p\theta_n)\,,\quad \text{where } x_n = \cos\theta_n\,,$$

and the iterates of x are chaotic and given analytically by

$$x_n = \cos(p^n \theta_0)\,.$$

One can easily show that the conditions (1) and (2), applied to the dynamics given by the Chebyshev polynomial T_p, read

$$(1+\gamma)\cos(p^k\,\theta_0) = \cos\theta_0\,,$$
$$\cos(\theta_{n+1}) = \cos(p\,\theta_n)\,,$$
$$\left|\prod_{i=0}^{k-1} \frac{p\sin(p\,\theta_i)}{\sin(\theta_i)}\right| < \frac{1}{1+\gamma}\,.$$

We invented a simple Fortran program which for a given k (period) and p (number of the Chebyshev polynomial) finds the γ for which there exist asymptotically stable k-periodic orbits. Table 1 gives some results. We numerically checked the results obtained by the program, and very good agreement was obtained.

k	p	γ
2	2	−0.2757
2	4	−0.2889
3	2	−0.07658
2	4	−0.4439
5	2	0.07605
4	2	−0.12641

Table 1. p – number of the Chebyshev polynomial; k – period of stabilized orbit.

Remarks

1. The precision of a computational procedure significantly influences for instance the period of the orbit (obtained as a result of the control).

2. Frequently an application of a program A that is less precise than a program B will result in an orbit of a smaller period than would be obtained by applying program B. This is because a controlled orbit sometimes passes nearby a point x_0 and returns to its neighbouring point $x_0 + \delta$. If δ is a number that is smaller than the smallest recognizable number, then the less precise program A identifies the points x_0 and $x_0 + \delta$. Lesser precision results in loss of part of the information. In some extreme cases a non-periodical orbit is computed as periodical because of a lesser precision. No method exists that would

allow to predict the relationship between precision and the computed orbit's parameters. Therefore all computations associated with controlling chaos should be conducted with the maximum precision.

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