Phase Transitions in the Generalized Entropy Spectrum of Nonhyperbolic Dynamical Systems

R. Stoop

Institute of Mathematics, Federal Polytechnical University (ETH) of Zurich, CH-8001 Zurich, Switzerland

Z. Naturforsch. **46a**, 1117–1122 (1991); received September 25, 1991

In this contribution, the Generalized Thermodynamic Formalism is applied to a nonhyperbolic dynamical system. The associated generalized entropy function is discussed. The model considered is equipped with a generating partition of two symbols; in contrast to analogous hyperbolic models, the support of the generalized entropy function is two-dimensional. Furthermore, the phase-transition-like behavior caused by the nonhyperbolicity of the measure is shown to lead to the manifestation of a critical line in the surface described by the generalized entropy function. It is discussed how the different combinations of phase-transitions in the more specific entropy spectra $f(\alpha)$, $S_G(\varepsilon)$ and $\phi(\lambda)$ are determined by the form of the generalized entropy function.

Key words: Dissipative chaotic systems, Scaling behavior, Thermodynamical formalism.

1. Introduction

The scaling behavior of experimental dynamical systems has become a wide-ranging field of research since it was realized that for a description of such a system, various mutually independent characterizations are necessary. The most prominent among these characterizations are the fractal dimensions and the Lyapunov exponents, on the one hand, and the different concepts of entropy and complexity, on the other hand. For a convenient description of a system, also fluctuations of the first two quantities should be taken into account. This can be achieved by considering scaling functions of fractal dimensions and Lyapunov exponents [1–8].

Recently, the usefulness of a joint approach of Lyapunov exponents and fractal dimensions to the scaling behavior of a dissipative dynamical system has been made evident by different people [9, 10], including the present author [11–19]. Within this approach, the scaling properties due to the Lyapunov exponents are interpreted as scaling properties of the support, on which the measure is distributed according to the underlying symbolic dynamics, following its own inherent rules. From the entropy function of the joint scaling properties (called generalized entropy function) the well known entropy-like scaling functions of generalized Lyapunov exponents and fractal dimensions are obtained by restriction.

In earlier contributions, e.g. [12, 13, 15], the form of the generalized entropy function for three-scale Cantor sets was discussed. More specifically, for the generic case of restricted symbolic representations it was shown [17] that calculations based on the 'canonical' partition function (3) lead to severe numerical problems, which, however, in the case of finite grammatical rules, can be overcome by the use of appropriate zeta functions. In contrast hitherto, the present contribution deals with effects in the generalized entropy function which arise from a nonhyperbolicity of the system considered.

2. The Model

As the specific model for our investigation we consider a system which can be described by the combination of a hyperbolic support map with a nonhyperbolic measure map. As a very simple example of a hyperbolic support map we take the tent map

$$f\colon\ x \to x/l_1, \qquad \text{for} \quad x \in [0, l_1/(l_1+l_2)],$$
$$x \to (1-x)/l_2, \quad \text{for} \quad x \in [l_2/(l_1+l_2), 1], \tag{1}$$

where $l_1 = \frac{2}{9}$ and $l_2 = \frac{2}{7}$. The map of the measure, instead, is given by

$$g\colon\ x \to 4(1-x)\,x, \tag{2}$$

Reprint requests to Dr. R. Stoop/co Priv. Doz. Dr. J. Parisi, Physics Institute, University of Zurich, Schönberggasse 9, CH-8001 Zurich, Switzerland.

0932-0784 / 91 / 1200-1117 $ 01.30/0. – Please order a reprint rather than making your own copy.

the fully developed logistic map. Note that in this way, no restrictions are imposed on the associated binary grammar.

3. The Generalized Thermodynamic Formalism

The mathematical description of any model as an appropriate dynamical system usually starts with the definition of a suitable generating partition. For a given system, many diverse generating partitions are possible, in principle, depending on the point of view one is interested in. If one wants to investigate the dynamical behavior, however, a partition is necessary which is 'compatible' with the dynamical scaling behavior, i.e., a dynamical partition is required.

Using such a partition consisting of M (in our application $M=2$) symbols it is proceeded in analogy to statistical mechanics, and the partition function [1]

$$Z_G(q,\beta,n) = \sum_{j\in(1,\ldots,M)^n} l_j^\beta\, p_j^q \tag{3}$$

is defined. Here the size of the j'^{th} region R_j of the partition is denoted by l_j, whereas the probability of falling into this region is denoted by p_j ($p_j = \int_{R_j} \varrho(x)\,\mathrm{d}x$, where $\varrho(x)$ denotes the natural measure). To account for the nonisotropy of the attractor, they can be thought of as vectorlike. β, q are sometimes called "filtering exponents". Local scaling of l and p in n (where n denotes the 'level' of the partition) is expected. In this way, the length scale l and the probability p give rise to scaling exponents ε and α through

$$l_j = e^{-n\varepsilon_j}, \tag{4}$$

$$p_j = l_j^{\alpha_j}. \tag{5}$$

These exponents should again be considered as vectors. Using the above expression, from the partition function the generalized free energy F_G can be derived [9]:

$$F_G(q,\beta) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{j\in(1,\ldots,M)^n} e^{-n\varepsilon_j(\alpha_j q+\beta)}. \tag{6}$$

where log denotes the natural logarithm. Alternatively, the generation process of the system can be described for hyperbolic problems as a fixed-point or eigenvalue equation [20a] by means of the iterative version of the Frobenius-Perron equation [12, 22]

$$\lambda(q,\beta)\, Q_{n+1}(x,y) = \sum_{\varepsilon=0,1} \frac{Q_n(f_\varepsilon^{-1}(x), g_\varepsilon^{-1}(y))}{|f'(f_\varepsilon^{-1}(x))|^\beta\, |g'(g_\varepsilon^{-1}(y))|^q} \tag{7}$$

for $n\to\infty$, where ε labels the choice being made between the two inverses of the maps f, g. Note that, according to the above, the sum is performed for f and g over the same symbolic substrings. Starting from any smooth density $Q_0(x,y)$ in $(0,1)\times(0,1)$, a unique eigenvalue $\lambda(q,\beta)$ assures convergence towards a finite $Q(x,y)$. The dependence on x, y disappears for large n, for x, y in the invariant set.

The free energy arises herefrom as the largest eigenvalue of the associated Frobenius-Perron operator

$$[LQ](x,y) = \lambda(q,\beta)\, Q(x,y). \tag{8}$$

Therefore, in order to derive the generalized free energy or Gibbs potential, the relation

$$\lambda(q,\beta) = \exp\{F_G(q,\beta)\} \tag{9}$$

can be used. In a series of contributions, the properties of the eigenvalues and associated eigenfunctions were investigated [20b–22]. It was shown especially that generically hyperbolic and nonhyperbolic phases could be distinguished in the generalized free energy function. The separation between the two phases was characterized as the set of points where F_G was not real-analytic as a function of its parameters q and β. In the nonhyperbolic phase (in analogy to the nongeneralized case) a linear dependence on the weightening exponents was found. The latter property will prove especially useful for the numerical calculations in Section 4. In this paper, instead of the generalized free energy, we are more concerned with the investigation of the generalized éntropy function $S_G(\alpha,\varepsilon)$, which, for our model, will show characteristic, generic properties. The generalized entropy function $S_G(\alpha,\varepsilon)$ is introduced through the 'global' scaling assumption that the number of regions N which have scaling exponents between (α,ε) and $(\alpha+\mathrm{d}\alpha, \varepsilon+\mathrm{d}\varepsilon)$ scales as

$$N(\alpha,\varepsilon)\,\mathrm{d}\alpha\,\mathrm{d}\varepsilon \sim e^{S_G(\alpha,\varepsilon)}\,\mathrm{d}\alpha\,\mathrm{d}\varepsilon. \tag{10}$$

Writing the partition function formally as an integral, via a saddle-point approach, the relationship between the generalized free energy F_G and the generalized entropy S_G is found [9, 13]:

$$S_G(\alpha,\lambda) = F_G(q,\beta) + (\langle\alpha\rangle q + \beta)\langle\varepsilon\rangle. \tag{11}$$

The angular brackets indicate hat those values of α and ε leading to the maximum of the Z_G (as a function of given q and β) have been chosen. In the following, the brackets will be omitted. The free energy F_G or the generalized entropy S_G describe in this way the scaling behavior of the dynamical system equivalently. Note that the information-theoretical 'Renyi-entropies' evolve from F_G for $\beta = 0$.

From the generalized free energy and entropy, respectively, two additional free energies and entropies can be derived by restriction: For $q = 0$, we obtain the free energy first discussed by Oono and Takahashi [3] as the maximum value of $S_G(\alpha, \varepsilon)$ with respect to variation of α alone for given ε. The associated free energy and entropy are in this case denoted by $F_G(\beta)$ and $S_G(\varepsilon)$, respectively. Similar approaches to the dynamical scaling behavior have been put forward in [4–8]. If we consider the probabilities arising from the natural measure, the generalized Lyapunov exponents arise from $F_G(q = 1, \beta)$; we denote the associated scaling function by $\phi(\lambda)$.

Furthermore, the fractal dimensions (see, e.g., [1], and, in a slightly different form, [23–24]) can also be obtained from the zeros $\beta_0(q)$ of $F_G(q, \beta)$ for given q. The entropy-like function $f(\alpha)$ introduced in [1] (often called dimension spectrum) is then given by $S_G(\alpha_0, \varepsilon_0) = \varepsilon_0 f(\alpha_0)$, where ε_0 and α_0 lead to this zero of F_G for given q and appropriately chosen $\beta(q)$. From (11) it follows immediately that $-\beta_0(q) = \alpha_0 q - f(\alpha_0)$. Due to the relationship with the fractal Hausdorff-dimensions $D(q)$ $(-\beta_0 = (q-1) D(q))$, this point of view is often called the probabilistic approach. An analog Legendre transformation relation can also be derived for the 'dynamical' approach for $F_G(\beta)$: it also follows from (11) that $S_G(\varepsilon) = \varepsilon\beta + F_G(\beta)$. However, instead of $F_G(\beta)$, $-F_G(\beta)$ can be used; the latter convention leads to the identical Legendre transform situation if compared with the former approach.

4. Phase Transitions

a) Terminology

In order to associate the present problem with a more familiar physical situation, let us interprete (3) as the isobar-isotherm partition function of an elastic Ising chain with long-range, multispin interaction: ε_i then corresponds to the energy, $\alpha_i \varepsilon_i$ to the length (per spin) of a state; q and β are recognized to correspond to the pressure and to the inverse temperature, respectively.

b) Manifestation in the Generalized Free Energy or Gibbs Potential

For hyperbolic systems, the application of (7), is straightforward. To get rid of finite-n corrections, the difference between two neighbouring levels can be considered (e.g. [22, 25]): $\log\lambda(q, \beta) \simeq \log Z_n(q, \beta) - \log Z_{n+1}(q, \beta)$. Typically, exponential convergence is encountered. For nonhyperbolic systems, (7) still holds, but its derivation needs some more care [22]. The nonhyperbolicity of the measure or of the support (in our example we chose it to be the measure map) comes usually from the existence of a local maximum of these maps or from singularities in their invariant sets. This property leads to a phase-transition-like behavior of $F_G(q, \beta)$; around the points of nonanalytic behavior of $F_G(q, \beta)$ the convergence is then found to be much worse.

The appearance of points of nonanalytical behavior can generically be explained as follows [20b–22]. In addition to the eigenvalue obtained from smooth, singularity free eigenfunctions $Q(x, y)$, the use of singular initial functions leads to new eigenvalues $\lambda_{\text{nonhyp}}(q, \beta)$. The whole system chooses then its free energy from the requirement $F_G = \max(F_{G\,\text{hyp}}, F_{G\,\text{nonhyp}})$, where both contributions to the generalized free energy are smooth functions of their arguments. As the two individual functions intersect, a first-order phase-transition is created, for a subfamily of the family $\{(q, \beta)\}$. Due to the smoothness of the arguments, the points of such a nonanalytical behavior must be connected; they thus form what has been proposed should be called the 'critical line' [23]. Since in the case relevant to us the nonhyperbolic contribution stems only from the neighbourhood of the origin, the relation $F_{G\,\text{nonhyp}}(q, \beta) = -\beta\log c - q\log c'$ can be obtained, where c, c' denote the slopes of the map of the support and of the measure, respectively, at point 0. In a similar way, also the overall estimate $F_G(q, \beta) \geqq -\frac{1}{z}\beta\log c - \frac{1}{z'} q\log c'$, where z, z' are the respective order of the maxima, can be derived [22]. An example of a hyperbolic and a nonhyperbolic behavior is given in Figs. 1 a, b (for convenience we plot $-F_G(q, \beta)$ for different, fixed q-values versus β).

c) Manifestation in the Generalized Entropy Function

In earlier contributions [11–19], the form of the generalized entropy function for hyperbolic models and its relation to the more specific entropy functions $S_G(\varepsilon)$, $\Phi(\lambda)$ and $f(\alpha)$, were investigated. Furthermore, the connection to experimentally obtained scaling functions was discussed. Hyperbolic systems lead generally to strictly convex specific entropy functions,

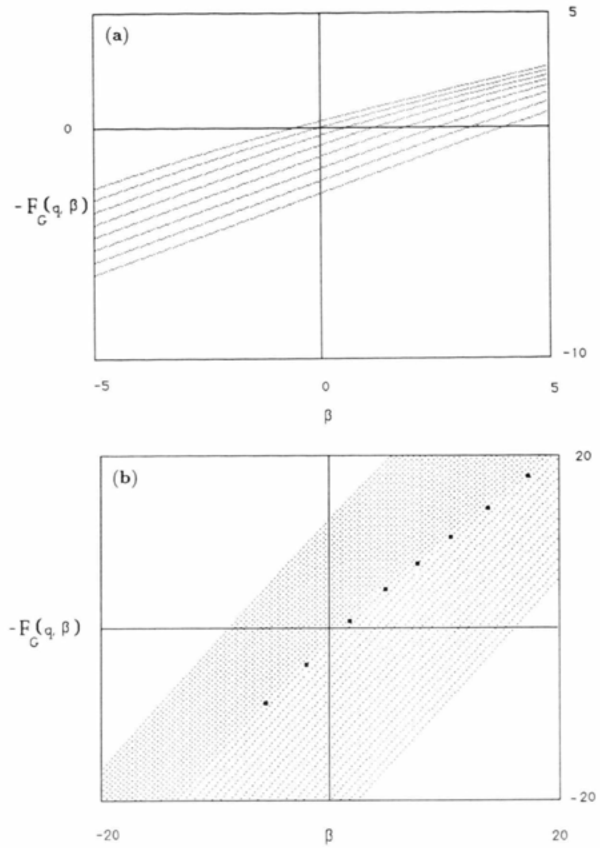

Fig. 1. a) Generalized free energy for a hyperbolic system (two coupled tent maps). The figure shows the lines obtained from fixed equidistant q-values, starting from $q = -5$ (bottom curve) and ending at $q = 2$ (top curve). No phase-transition-like effect is observed.
b) Generalized free energy of the nonhyperbolic system described in the text. The existence of a critical line is indicated by heavy dots. The q-values, held fixed for each line, range from -20 to 20. No care has been taken for logarithmic corrections, an approximation of level $n = 10$ has been used.

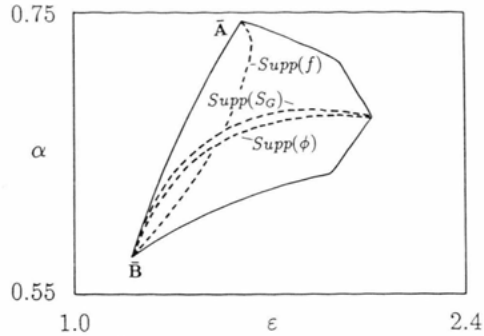

Fig. 2. Scaling behavior of a hyperbolic three-scale Cantor set with restriction. – The figure shows the α-ε-area on which the entropy $S_G(\alpha, \varepsilon)$. The lines are indicated along which the functions $S_G(\varepsilon)$, $\phi(\lambda)$, and $f(\alpha)$ are evaluated. These functions themselves are strictly convex in their arguments.

whereas in the more relevant nonhyperbolic case a linear segment is obtained (compare, e.g., [26–32]). This effect has even been observed in high dimensional experimental systems, where the dynamical and probabilistic scaling functions have been reconstructed from time series [11, 13, 14]. A reprint of an example of the support of the generalized entropy of a hyperbolic model is given in Figure 2. From the form of the support of the generalized entropy function it can immediately be concluded (note the characteristic location of the lines along which the functions $S_G(\varepsilon)$ and $f(\alpha)$ are evaluated) that different combinations of phase transitions in the different spectra could be distinguished from the place of the symbol sequence which triggers off the phase-transition effect. To give an example, if sequence $\bar{A} = AAAAA\ldots$ leads to such an effect, a phase-transition is obtained for the $f(\alpha)$-spectrum, whereas sequence $\bar{B} = BBBBB\ldots$ would lead to phase-transitions in $S_G(\varepsilon)$, $\phi(\lambda)$ and $f(\alpha)$, at the same time, and so on. It is therefore useful to discuss the form of the generalized entropy function for the nonhyperbolic model.

In our model, the interchange of the symbol positions within a given symbolic string is no longer allowed, because the probabilities do not commute, due to the nonlinearity of the measure map. Therefore, already a two-symbol system can lead to a nondegenerate entropy function (compare the discussion of the generalized entropy function in [12]). Using an approximation of finite level ($n = 10$) we obtain the form of the generalized entropy function as described in Figure 3a. Emanating from the top segment labeled by N, a 'sheet' touches the hyperbolic part along the critical line L (see Figure 3b). In the same figure, the support of the hyperbolic aspect of the system is shown by the dotted line S. From the form of the generalized entropy function it follows that the more specific scaling functions $S_G(\varepsilon)$, $\phi(\lambda)$ and $f(\alpha)$ inherit the phase-transition effect, this fact depending most prominently on the location of the line N with respect to the hyperbolic part. The partial entropy spectra $S_G(\varepsilon)$ and $f(\alpha)$ are shown in Figure 4. Note the linear parts on the r.h.s. of the scaling functions (full lines). Their locations in the specific scaling functions are again seen to depend directly on the position of N with respect to the hyperbolic part of the generalized entropy function.

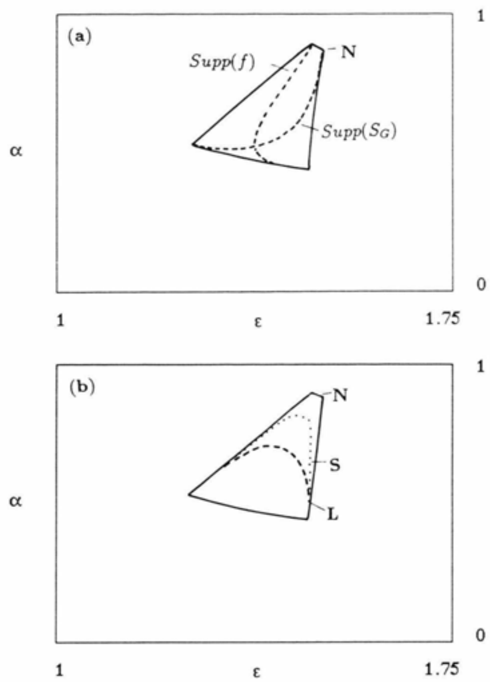

Fig. 3. Scaling behavior of the nonhyperbolic model, using a approximation of level $n=10$. a) Support of $S_G(\alpha, \varepsilon)$. The lines are indicated along which the functions $S_G(\varepsilon)$ and $f(\alpha)$ are evaluated. b) Location of the critical line (dashed curve). The hyperbolic part of the generalized entropy function of the nonhyperbolic model is indicated by the dotted line.

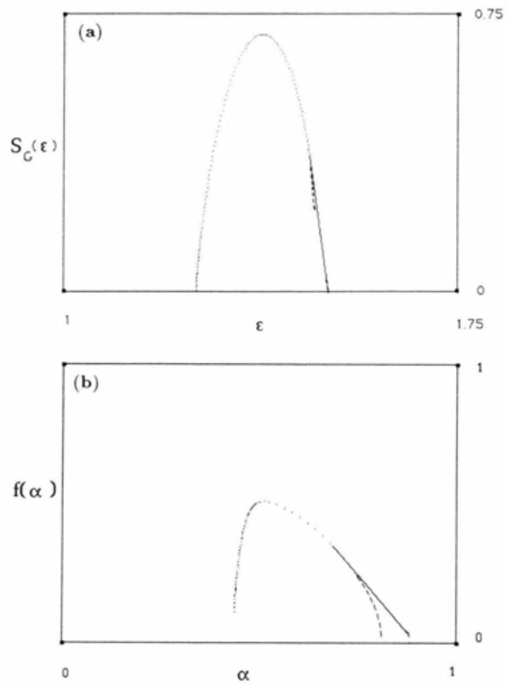

Fig. 4. a) Geometrical scaling function $S_G(\varepsilon)$ for the nonhyperbolic model. b) Probabilistic scaling function $f(\alpha)$ for the nonhyperbolic model. The dashed branches indicate the hyperbolic contribution. Note the phase-transition-like effects for both scaling functions $S_G(\varepsilon)$ and $f(\alpha)$ which appear together at the same parameter values.

Let us finally point out the relationship of the generalized entropy function of our model with the entropy-like functions obtained for the logistic map. In our system, the appearance of the phase-transition-like effect on the r.h.s. of the scaling function seems to be puzzling at first sight since in the scaling functions of the logistic map the analog phenomenon appears on the l.h.s. The phenomenon, however, can be understood from the setting of our model, where the shortest intervals generated by the measure map instead of the intervals of biggest weight are associated with the endpoints of the unit interval (0 and 1). In this way, in the (ε, α)-plane the nonhyperbolic contribution appears *above* the hyperbolic contribution, the fact of which leads to the somewhat unexpected situation.

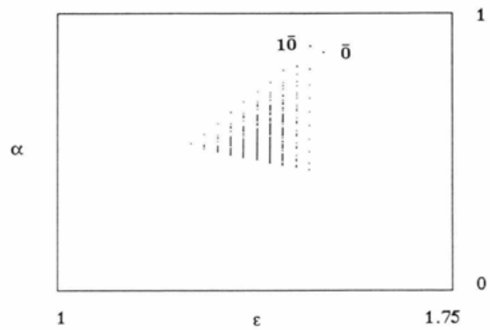

Fig. 5. Location of the periodic orbits of the nonhyperbolic system in the α-ε-plane. The regular pattern parallel to the y-axis is a consequence of the piecewise linearity of the tent map. The uppermost periodic points are labelled by their symbolic sequences.

d) Periodic Orbits

For an improved understanding of the generalized entropy function let us make a comparison with the support of the periodic orbits of the system (see Figure 5). As is easily found, the endpoints of the line N are determined by the periodic orbits with the symbol sequences $1\bar{0} = 1000000\ldots$ (left end) and $\bar{0} = 000000\ldots$ (right end point). These symbol sequences correspond to the endpoints 1 and 0 of the unit interval. For the system considered, in the limit of $n \to \infty$, these points can be expected to converge to a point located above point $\bar{0}$ (same ε), but at an even higher α-value than maximally attained by the system at level $n=10$. In this way, the existence of a line at the upper end of the support of $S_G(\alpha, \varepsilon)$ must be considered as a finite-n-artefact. Furthermore, the phase-transition for $S_G(\varepsilon)$ can be shown to vanish for $n \to \infty$. The asymptotic result, which is of interest itself, will be discussed in a forthcoming publication.

5. Conclusions

In this contribution, with the help of an appropriate model, the theoretical tools were outlined which permit us to predict and understand the different combinations of first-order phase transitions which appear for generic dynamical systems. In addition, a means for the modelling of such spectra has been provided. This may help the physicist with the construction of realistic models of experiments.

Acknowledgements

I wish to thank T. Tél and Z. Kovács warmly for stimulating discussions and for sending me their paper [22] prior to publication. Thanks also go to Profs. Eilenberger and Lustfeld for inviting me to the IFF in Julich, Germany, where part of the discussions leading to the final form of this article took place. I thank J. Parisi for reading the manuscript and G. Zumofen, ETH, for generous support for my calculations.

[1] T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia, and B. Shraiman, Phys. Rev. **A 33,** 1141 (1986).
[2] H. Fujisaka, Prog. Theor. Phys. **70,** 1264 (1983).
[3] Y. Oono and Y. Takahashi, Prog. Theor. Phys. **63,** 1804 (1980).
[4] J. P. Eckmann and I. Procaccia, Phys. Rev. **A 34,** 659 (1986).
[5] M. Sano, S. Sato, and Y. Sawada, Prog. Theor. Phys. **76,** 945 (1986).
[6] P. Szépfalusy and T. Tél, Phys. Rev. **A 34,** 2520 (1986).
[7] T. Horita, H. Hata, H. Mori, T. Morita, and K. Tomita, Prog. Theor. Phys. **80,** 923 (1988).
[8] R. Stoop, J. Peinke, J. Parisi, B. Roehricht, and R. P. Huebener, Physica **35 D,** 425 (1988).
[9] M. Kohmoto, Phys. Rev. **A 37,** 1345 (1988).
[10] T. Tél, Z. Naturforsch. **43 a,** 1154 (1988).
[11] R. Stoop, Helv. Phys. A 1, 1989.
[12] R. Stoop, On the numerical characterization of dissipative dynamical systems, Zentralstelle der Studentenschaft, Zurich 1991.
[13] R. Stoop and J. Parisi, Phys. Rev. **A 43,** 1802 (1991).
[14] R. Stoop, J. Peinke, and J. Parisi, Physica **50 D,** 405 (1991).
[15] R. Stoop, J. Parisi, and H. Brauchli, Z. Naturforsch. **46 a,** 642 (1991).
[16] R. Stoop, J. Parisi, and H. Brauchli, Helv. Phys. A (1991) in press.
[17] R. Stoop, J. Parisi, submitted.
[18] R. Stoop, J. Parisi, IUTAM and Nato Advanced Studies symposium on the 'Interpretation of Time Series', held in Warwick, GB (1991), Conference Proceedings, to be published in Physica D.
[19] R. Stoop in 'Chaos in Semiconductors', J. Peinke, J. Parisi, R. Stoop, and O. E. Roessler, Springer, Berlin, in press.
[20 a] T. Tél, Phys. Rev. **A 36,** 2507 (1987).
[20 b] M. J. Feigenbaum, I. Procaccia, and T. Tél, Phys. Rev. **A 39,** 5359 (1989).
[21] Z. Kovács and T. Tél, Phys. Rev. **A 40,** 4641 (1989).
[22] Z. Kovács and T. Tél, preprint.
[23] A. Renyi, Probability Theory, North-Holland, Amsterdam 1970.
[24] P. Grassberger and I. Procaccia, Physica **13 D,** 34 (1984).
[25] R. Artuso, E. Aurell, and P. Cvitanović, Nonlinearity **3,** 325 (1990).
[26] P. Cvitanović, in Proceedings of the Workshop in Condensed Matter, Atomic and Molecular Physics, Trieste, Italy 1986.
[27] E. Ott, W. Withers, and J. A. Yorke, J. Stat. Phys. **36,** 697 (1984).
[28] P. Grassberger, in Proceedings of the Workshop of Dynamics on Fractals and Hierarchies of Critical Exponents, Orsay, France 1986.
[29] D. Katzen and I. Procaccia, Phys. Rev. Lett. **58,** 1169 (1987).
[30] M. H. Jensen and T. Bohr, Phys. Rev. **A 36,** 4904 (1987).
[31] P. Szépfalusy, T. Tél, A. Cordás, and Z. Kovács, Phys. Rev. **A 36,** 3525 (1987).
[32] T. Horita, H. Hata, H. Mori, T. Morita, S. Kuroki, and H. Okamoto, Prog. Theor. Phys. **80,** 793 (1988).