A Lorentz Invariant Schrödinger Equation for Spin 0

E. Trübenbacher

Institut für Physik, Johannes-Gutenberg-Universität, Mainz, West Germany

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Using the concept of distributions, the square root of the operator $-\Delta+m^2$ is taken in a mathematically well defined way for one component wave functions. A new representation of proper Lorentz transformations for one component wave functions makes it possible to construct a relativistic quantum mechanics for spin 0, comprising a Lorentz invariant wave equation, a scalar product, and a positive definite density satisfying, together with a current, a continuity equation, and coupling of scalar and vector potentials.

Some interesting consequences of the theory concerning the concept of particle trajectory and velocity of propagation of the probability amplitude are discussed in detail. As an example of practical application a perturbation theory for discrete states is set up.

Introduction

Relativistic one particle wave equations are derived by giving the operator $\sqrt{-\Delta+m^2}$ a mathematically well defined meaning in configuration ($\boldsymbol{x}$) space.

It is well known that, for spin 0, $\sqrt{-\Delta+m^2}$ can be given a meaning with the help of Fourier representation of the wave function, but the corresponding wave equation is abandonned every time for various reasons, one of them being the difficulty to couple a vector potential, see e.g. [1].

Now, the mathematical tool of distributions [2] allows to set up a mathematically well defined and convenient representation of $\sqrt{-\Delta+m^2}$ and the corresponding Lorentz invariant quantum mechanics, i.e. wave equation, scalar product, positive definite density, continuity equation, coupling of scalar and vector potentials.

Whenever it was possible to take the square root of $-\Delta+m^2$ in a mathematically well defined way, the corresponding one particle theory was physically meaningful. The outstanding examples are

spin $\frac{1}{2}$: $\sqrt{-\Delta+m^2}=-i\boldsymbol{\alpha}\cdot\boldsymbol{\partial}+m\beta$, Dirac's equation,

spin 1, $m=0$: $\sqrt{-\Delta}=\sqrt{\text{curl curl}-\text{grad div}}=\text{curl}$,

after imposing the subsidiary condition $\text{div}=0$, giving Maxwell's equations for the wave function $\boldsymbol{\Phi}=\boldsymbol{E}+i\boldsymbol{B}$. (In this case $\hbar$ drops out exactly.)

Reprint requests to Prof. Dr. E. Trübenbacher, Institut für Physik, Universität Mainz, Staudingerweg 7, Postfach 3980, D-6500 Mainz.

So there is good reason, too, to take the spin 0 theory seriously. A spin 0 equation would be useful for various reasons.

1. It could fill up an unsatisfactory gap in quantum field theory. The idea of quantum field theory is to construct a many particle theory with suitable statistics by quantizing a one particle equation. In case of spin 0 particles we possess the quantum field theory of the Klein-Gordon equation only, but we lack the quantum mechanical one particle equation.
2. One particle equations with a potential can serve as approximations in cases when field theoretic methods are too cumbersome, for instance to describe bound states.
3. One particle equations complete the physical picture of a particle in a convenient way, cf. Dirac's equation for electrons or Maxwell's equations for photons.

The construction of one particle quantum mechanics for spin 0 is not possible in a straight forward manner. Indeed, one needs a generalization of the familiar representation of scalar Lorentz transformations to assure the Lorentz invariance of the scalar product. The new generators again contain distributions [3]. They show a certain parallelism to the generators of the transformation of a Dirac wave function. The Lorentz invariant spin 0 theory is complete in the sense that it comprises a Lorentz invariant scalar product, and a conserved positive definite density. Its particles, however, aquire some very unusual features.

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So the concept of particle trajectory looses its meaning unless for states of definite momentum, i.e. plane waves, and the amplitude of a particle initially located in one space point shows a non-vanishing probability for propagation with velocity $v>1$ ($\hbar=c=1$) during the very first time interval of order $1/m$, i.e. some 10^{-23} sec for pions, although there are no states with $v>1$.

Since the theory is based on established physical principles only, it seems reasonable to discuss these phenomena extensively.

The work comprises the following sections:

1. The wave equation
2. Iteration of the wave equation
3. Behaviour under Lorentz transformation
4. Continuity equation. Scalar product
5. The finite Lorentz transformation
6. Plane wave solutions
7. Coupling of potentials
8. Perturbation theory

1. The Wave Equation

a) The Wave Equation in Momentum Space

It is generally agreed that in momentum ($\boldsymbol{k}$) space one uses as a relativistic Schrödinger equation

$$i\partial_t \tilde{\Psi}(t,\boldsymbol{k})=\omega(\boldsymbol{k})\,\tilde{\Psi}(t,\boldsymbol{k}) \tag{1}$$

for the wave function $\tilde{\Psi}(t,\boldsymbol{k})$ in momentum space, with $\omega(\boldsymbol{k})=(\boldsymbol{k}^2+m^2)^{1/2}$, the energy of a free particle of mass m and momentum $\boldsymbol{k}$, as (multiplicative) energy operator in momentum space.

b) The Wave Equation in Coordinate Space

To a multiplication of two $\boldsymbol{k}$-space functions there corresponds in coordinate ($\boldsymbol{x}$) space the folding integral of the corresponding $\boldsymbol{x}$-space functions. Thus, if we denote the $\boldsymbol{x}$-space wave function by $\Psi(t,\boldsymbol{x})$ and the (formal) Fourier transform of $\omega(\boldsymbol{k})$ by $\Omega(\boldsymbol{x})$, (1) reads in $\boldsymbol{x}$-space

$$i\partial_t \Psi(t,\boldsymbol{x})=\int \mathrm{d}x'^3\,\Omega(\boldsymbol{x}-\boldsymbol{x}')\,\Psi(t,\boldsymbol{x}), \tag{2}$$

where

$$\Psi(t,\boldsymbol{x}) \quad =(2\pi)^{-3/2}\int \mathrm{d}k^3 \exp(i\boldsymbol{k}\cdot\boldsymbol{x})\,\tilde{\Psi}(t,\boldsymbol{k}), \tag{3}$$

$$\Omega(\boldsymbol{x}-\boldsymbol{x}')=(2\pi)^{-3}\int \mathrm{d}k^3 \exp(i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}'))\,\omega(\boldsymbol{k}). \tag{4}$$

The Fourier transform (4) does not exist as a proper function of $\boldsymbol{x}$. But it has a mathematically well defined meaning as a so-called tempered distribution, i.e. a linear functional of sufficiently rapidly decreasing functions $\Psi(\boldsymbol{x})$ for $\boldsymbol{x}\to\infty$ [4].

To be definite, the value of the functional Ω is defined by

$$\begin{aligned}\int \mathrm{d}x'^3\,\Omega(\boldsymbol{x}-\boldsymbol{x}')\,\Psi(\boldsymbol{x}') &=\int \mathrm{d}x'^3\int \mathrm{d}k^3\,\omega(\boldsymbol{k})\,\exp(i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}'))\,\Psi(\boldsymbol{x}')\\ &=\int \mathrm{d}k^3\exp(i\boldsymbol{k}\cdot\boldsymbol{x})\,\omega(\boldsymbol{k})\,\tilde{\Psi}(\boldsymbol{k}).\end{aligned} \tag{5}$$

Let us abbreviate expression (5) by writing simply $\Omega\Psi$. Thus (2) takes on the form

$$i\partial_t\Omega=\Omega\Psi \tag{6}$$

Remarks:

1) Ω is evidently a real Hermitean, that is symmetric operator. We have, for two wave functions Φ, Ψ:

$$\begin{aligned}\int \mathrm{d}x^3\,\Phi^*(\boldsymbol{x})\,(\Omega\Psi)(\boldsymbol{x}) &=\int \mathrm{d}x^3\,(\Omega\Psi)^*(\boldsymbol{x})\,\Psi(\boldsymbol{x})\\ &=\int \mathrm{d}x^3\,(\Omega\Psi^*)(\boldsymbol{x})\,\Psi(\boldsymbol{x}).\end{aligned} \tag{7}$$

2) There exist arbitrary powers of the distribution Ω,

$$\Omega^n(\boldsymbol{x}-\boldsymbol{x}')=(2\pi)^{-3}\int \mathrm{d}k^3\,\omega^n(\boldsymbol{k})\exp(i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')). \tag{8}$$

In particular, the inverse of Ω is given by

$$\begin{aligned}&\Omega^{-1}(\boldsymbol{x}-\boldsymbol{x}')\\ &\quad=(2\pi)^{-3}\int \mathrm{d}k^3\,\omega^{-1}(\boldsymbol{k})\exp(i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')).\end{aligned} \tag{9}$$

3) The operator $\Omega(\boldsymbol{x})$ is non-local. It has a mean range of order m^{-1}:

$$\sqrt{\langle r^2\rangle}=\left(\int \mathrm{d}x^3\,r^2\,\Omega(\boldsymbol{x})/\int \mathrm{d}x^3\,\Omega(\boldsymbol{x})\right)^{1/2}=\sqrt{3}/m.$$

2. Iteration of the Ω-Equation

Iterating (6) reproduces, of course, the Klein-Gordon equation; for

$$-\partial_t^2\,\Psi=\Omega^2\,\Psi$$

and $\Omega^2(\boldsymbol{x}-\boldsymbol{x}')=(-\Delta+m^2)\,\delta(\boldsymbol{x}-\boldsymbol{x}')$.

3. Behaviour under Lorentz Transformation

We shall prove the Lorentz invariance of (6) by showing: There exists a representation of the Lorentz group such that any solution of (6) remains a solution after transformation.

a) Commutators

Let us list a few useful commutators:

$$[x, \Omega^n] = n\,\partial_x\,\Omega^{n-2} \tag{10}$$

Especially $n=1$

$$[x, \Omega] = \partial_x\,\Omega^{-1}, \tag{11}$$

$$[\partial_x, \Omega] = 0. \tag{12}$$

Analogously for the y and z components.

The velocity operator, using (11), is given by

$$\boldsymbol{u} = \dot{\boldsymbol{x}} = i[\Omega, x] = -i\,\partial\,\Omega^{-1} = \boldsymbol{p}/\Omega, \tag{13}$$

in accordance with the classic relativistic expression for the velocity.

The second time derivative vanishes because of (12):

$$\ddot{\boldsymbol{x}} = i[\Omega, \dot{\boldsymbol{x}}] = 0. \tag{13a}$$

It is interesting to compare these results with the behaviour of the corresponding quantities in case of the Dirac equation: As is well known [5], in the latter case $\dot{\boldsymbol{x}}$ is not constant but consists of two terms, one corresponding to (13), the other being known as the so-called "trembling motion". (13a) does not show trembling motion, but we shall see (Sect. 4d) that we do have to pay a corresponding price for the reconcilitation of the principles of quantum mechanics and relativity. Finally, let us note the Hermiticity of u_i:

$$\int \mathrm{d}x^3\, \Phi^*(u_i\,\Psi) = \int \mathrm{d}x^3 (u_i\,\Phi)^*\,\Psi \tag{14}$$

for two arbitrary square integrable functions Φ, Ψ.

b) Representations of the Proper Lorentz Group for Spin 0

Under an infinitesimal Lorentz transformation in direction x, with infinitesimal velocity v,

$$\begin{aligned} t' &= t - v\,x, \qquad t = t' + v\,x', \\ x' &= x - v\,t, \qquad x = x' + v\,t', \end{aligned} \tag{15}$$

a wave function $\Psi(t, x)$ is usually transformed according to

$$\Psi'(t, x) = \Psi(t, x) + v\,L_x\,\Psi(t, x), \tag{16}$$

with the generator L_x given by

$$L_x = x\,\partial_t + t\,\partial_x. \tag{17}$$

Correspondingly, we have

$$\begin{aligned} L_y &= y\,\partial_t + t\,\partial_y, \\ L_z &= z\,\partial_t + t\,\partial_z \end{aligned}$$

for Lorentz transformations in y and z direction, respectively. Together with the generators of pure rotations,

$$\begin{aligned} M_x &= -i(y\,\partial_z - z\,\partial_y), \\ M_y &= -i(z\,\partial_x - x\,\partial_z), \\ M_z &= -i(x\,\partial_y - y\,\partial_x), \end{aligned} \tag{18}$$

the L_i represent the Lie algebra of the proper Lorentz group, i.e. the set of commutators

$$[L_x, L_y] = i\,M_z;\ [L_x, M_y] = i\,L_z;\ [M_x, M_y] = i\,M_z. \tag{19}$$

It is important to note, however, that these generators are not the only possibility to transform a spin 0 wave function. In fact, the new generators $\tilde{L}_i$, defined by

$$\begin{aligned} \tilde{L}_x &= L_x + i\,\vartheta\,\partial_x\,\Omega^{-1} = L_x - \vartheta\,u_x, \\ \tilde{L}_y &= L_y + i\,\vartheta\,\partial_y\,\Omega^{-1} = L_y - \vartheta\,u_y, \\ \tilde{L}_z &= L_z + i\,\vartheta\,\partial_z\,\Omega^{-1} = L_z - \vartheta\,u_z, \end{aligned} \tag{20}$$

where ϑ is an arbitrary parameter (real or even complex) and where we have denoted the operator $-i\,\partial_i\,\Omega^{-1}$ (the velocity operator) by u_i, together with M_i (18) again fulfill the algebra (19), i.e. the commutators

$$[\tilde{L}_x, \tilde{L}_x] = i\,M_z;\ [\tilde{L}_x, M_y] = i\,\tilde{L}_z;\ [M_x, M_y] = i\,M_z. \tag{21}$$

Their validity is easily proved with the help of (10) and (12).

Remark: On comparing with the transformation of a Dirac spinor under (15), which is effected by the generator $L_x - 1/2\,\alpha_x$, and remembering that there $\dot{x} = i[H, x] = \alpha_x$, we find that this corresponds to a value of $\vartheta = 1/2$. Indeed it will turn out in Sects. 4 and 5 that we should choose this value for ϑ. For the time being we replace (16) by

$$\Psi'(t, x) = \Psi(t, x) + v\,\tilde{L}_x\,\Psi(t, x), \tag{22}$$

for fixed but arbitrary ϑ.

c) Lorentz Invariance of the Wave Equation

α) Proper Lorentz Transformations

It is sufficient to show the Lorentz invariance for one direction, x say. So in the wave function we shall only write the arguments t, x.

Let $\Psi(t, x)$ be a solution of (6). We find that $\Psi'(t, x)$ from (22) will also be a solution if

$$(i\,\partial_t - \Omega)\,\tilde{L}_x\,\Psi$$

vanishes. But this is so, because

$$[i\partial_t - \Omega, \tilde{L}_x] = i\partial_x + \partial_x \Omega^{-1}\partial_t,$$

which, factoring out $i\partial_x \Omega^{-1}$,

$$= i\partial_x \Omega^{-1}(\Omega - i\partial_t) = 0$$

when acting on Ψ.

β) Space Inversion

For space inversion we define as usually

$$\Psi'(\boldsymbol{x}) = \Psi(-\boldsymbol{x}).$$

Using $\Omega(\boldsymbol{x}-\boldsymbol{x}') = \Omega(\boldsymbol{x}'-\boldsymbol{x})$, one easily verifies that for any solution $\Psi(\boldsymbol{x})$ of (6) also $\Psi(-\boldsymbol{x})$ is a solution.

γ) Time Inversion

It is easily verified that the usual time reversed function

$$\Psi'(t, \boldsymbol{x}) = \Psi(-t, \boldsymbol{x})^*$$

solves (6) if $\Psi(t, \boldsymbol{x})$ does.

4. Continuity Equation. Scalar Product

a) Continuity Equation

Using the identity $-\Omega + m^2\Omega^{-1} = \Delta\Omega^{-1}$, one verifies that with a solution Ψ of (6) a continuity equation is fulfilled,

$$\partial_t \sigma + \operatorname{div} \boldsymbol{j} = 0, \tag{23}$$

for the density

$$\sigma = 1/2(\Psi^*\Psi + (\boldsymbol{u}\Psi)^* \cdot (\boldsymbol{u}\Psi) + m^2(\Omega^{-1})^*(\Omega^{-1}\Psi)) \tag{24}$$

and the current

$$\boldsymbol{j} = 1/2(\Psi^*(\boldsymbol{u}\Psi) + (\boldsymbol{u}\Psi)^*\Psi). \tag{25}$$

The density is evidently positive definite. Equation (23) gives rise to a norm constant in time:

$$\int \mathrm{d}x^3\, \sigma = \text{const}. \tag{26}$$

With the help of (7) and (14), it may be given the familiar form

$$\int \mathrm{d}x^3\, \sigma = 1/2 \int \mathrm{d}x^3\, \Psi^*(1 + \boldsymbol{u}^2 + m^2\Omega^{-2})\Psi = \int \mathrm{d}x^3\, \Psi^*\Psi. \tag{27}$$

For (23) to hold it is only required that Ψ fulfills (6). As (6) holds in any Lorentz frame, it follows that (23) holds in any Lorentz frame. In the non-relativistic limit, $m\Omega^{-1} \to 1$, $\boldsymbol{u}^2 \to 0$, we regain the familiar expressions

$$\sigma = \Psi^*\Psi,$$
$$\boldsymbol{j} = 1/(2im)(\Psi^*(\partial\Psi) - (\partial\Psi^*)\Psi).$$

b) Scalar Product

Equation (24) suggests that we define a scalar product for two solutions Φ and Ψ of (6):

$$(\Phi, \Psi) = 1/2 \int \mathrm{d}x^3 (\Phi^*\Psi + (\boldsymbol{u}\Phi)^* \cdot (\boldsymbol{u}\Psi) + m^2(\Omega^{-1}\Phi)^*(\Omega^{-1}\Psi)), \tag{28}$$

but this reduces simply to (cf. (27))

$$(\Phi, \Psi) = \int \mathrm{d}x^3\, \Phi^*\Psi. \tag{29}$$

Let us investigate the behaviour of (29) under a Lorentz transformation according to (22) with arbitrary ϑ. We find, by making use of (6) and (14),

$$(\Phi', \Psi') = \int \mathrm{d}x^3\, \Phi'^*\Psi' = (\Phi, \Psi) + v(1 - 2\vartheta) \int \mathrm{d}x^3\, \Phi^*(u_x\Psi).$$

Thus, our scalar product will be Lorentz invariant if we choose

$$\vartheta = 1/2.$$

We replace (22) by

$$\Psi' = \Psi + \boldsymbol{v} \cdot (\boldsymbol{L} - (1/2)\boldsymbol{u})\Psi, \tag{30a}$$
$$\Psi'^* = \Psi^* + \boldsymbol{v} \cdot (\boldsymbol{L} + (1/2)\boldsymbol{u})\Psi^*, \tag{30b}$$
$$\boldsymbol{L} = \boldsymbol{x}\partial_t + t\partial$$

for a Lorentz transformation with infinitesimal velocity $\boldsymbol{v}$.

c) Expectation Values

The expectation value $\langle P \rangle$ of an operator P in a state Ψ is given by the scalar product of Ψ and $P\Psi$. So because of (29) we put

$$\langle P \rangle = \int \mathrm{d}x^3\, \Psi^* P \Psi. \tag{31}$$

d) Transformation of Density and Current

Although (23) is Lorentz-invariant, the quantities σ and $\boldsymbol{j}$ do not transform like a usual four vector. Rather

one finds under (30)

$$\sigma' = 1/2(\Psi'^* \Psi' + (\boldsymbol{u}\Psi')^* \cdot (\boldsymbol{u}\Psi') + m^2 (\Omega^{-1}\Psi')^* \cdot (\Omega^{-1}\Psi')) = (1 + \boldsymbol{v}\cdot\boldsymbol{L})\,\sigma - \boldsymbol{v}\cdot\boldsymbol{j} - \boldsymbol{v}\cdot\boldsymbol{S}, \quad (34)$$

$$\boldsymbol{j}' = 1/2(\Psi'^* (\boldsymbol{u}\Psi') + (\boldsymbol{u}\Psi')^* \Psi') = (1 + \boldsymbol{v}\cdot\boldsymbol{L})\,\boldsymbol{j} - \boldsymbol{v}\sigma - \boldsymbol{v}\cdot\tau, \quad (35)$$

where the three-vector $\boldsymbol{S}$ is defined by

$$S_i = (1/2)\,j_i - (1/4)\,((\boldsymbol{u}\Psi)^* \cdot (u_i \boldsymbol{u}\Psi) + (u_i \boldsymbol{u}\Psi)^* \cdot (\boldsymbol{u}\Psi)) - (1/4)\,m^2\,((\Omega^{-1}\Psi)^* \cdot (u_i \Omega^{-1}\Psi) + (u_i \Omega^{-1}\Psi)^* (\Omega^{-1}\Psi)) \quad (36)$$

and the 3×3-tensor τ by

$$\tau_{ij} = (1/4)\,(-\Psi^* (u_i u_j \Psi) - (u_i u_j \Psi)^* \Psi + (u_i \Psi)^* (u_j \Psi) + (u_j \Psi)^* (u_i \Psi)) + (1/2)\,(\Psi^* \Psi - (\boldsymbol{u}\Psi)^* \cdot (\boldsymbol{u}\Psi) - m^2 (\Omega^{-1}\Psi)^* (\Omega^{-1}\Psi))\,\delta_{ij} \quad (37)$$

(The calculation is somewhat lengthy, but does not present difficulties. For calculations like this it is useful to note the commutators

$$[u_i, L_j] = -\delta_{ij} + u_i u_j \quad (38\,a)$$

$$[\Omega^{-1}, L_i] = u_i \Omega^{-1}, \quad (38\,b)$$

which hold if acting on solutions of (6).)

If $\boldsymbol{S}$ and τ both vanish, $\sigma, \boldsymbol{j}$ transform correctly like a four-vector. They do vanish, for instance, for plane waves. In general, however, $\boldsymbol{S}$ and τ do not vanish.

Let us make two comments:

The first concerns the transformation of (23). If $\sigma, \boldsymbol{j}$ do not transform like a four vector, the right hand side of (23) will not vanish identically after transformation but only for solutions of (6). This might suggest the existence of new conservation laws. Indeed, substitution of (34) and (35) into

$$\partial_t \sigma' + \operatorname{div} \boldsymbol{j}' = 0$$

leads to

$$\boldsymbol{v}\cdot(\partial_t \boldsymbol{S} + \operatorname{div}\tau) = 0, \quad (39)$$

Equation (39) holds for arbitrary $\boldsymbol{v}$, so we have

$$\partial_t \boldsymbol{S} + \operatorname{div}\tau = 0 \quad (40)$$

and as a consequence

$$\int \mathrm{d}x^3\, S_i = \text{const}.$$

But this is a trivial result, since

$$\int \mathrm{d}x^3\, S_i = 1/2 \int \mathrm{d}x^3\, \Psi^* u_i (1 - \boldsymbol{u}^2 - m^2 \Omega^{-2})\,\Psi = 0$$

identically,

because the expression in the parentheses does so.

Equation (40) is again valid in every Lorentz frame, so the same type of argument may be applied to (40), and so on. But each time we arrive only at trivial conservation laws. A proof is outlined in the Appendix.

The second comment concerns the concept of particle trajectory.

Vanishing of $\boldsymbol{S}$, τ means that the quantity $\boldsymbol{v} = \boldsymbol{j}/\sigma$ transforms like a velocity field. So $\boldsymbol{j}/\sigma$ may be looked at as a field of (possible) particle trajectories. For non-vanishing $\boldsymbol{S}$ and τ this is no longer the case. This is the price we have to pay for the absence of trembling motion as mentioned in Section 3a.

5. The Finite Lorentz Transformation

a) The Finite Transformation of an Arbitrary Function

The finite transformation belonging to the representation (20) in direction x, say, with velocity v, is given by

$$\exp(w \tilde{L}_x) = \exp(w (L_x - \vartheta u_x)), \quad u_x = -i \partial_x \Omega^{-1}; \quad \tanh w = v. \quad (41)$$

Let us represent the function $\Psi(t, x)$ as a Fourier transform for both variables t and x:

$$\Psi(t, x) = (2\pi)^{-1} \int \mathrm{d}k_0\, \mathrm{d}k_1\, \tilde{\Psi}(k_0, k_1) \cdot \exp(-i k_0 t + i k_1 x) \quad (42)$$

with independent variables k_0, k_1.

Consequently,

$$\exp(w \tilde{L}_x)\,\Psi(t, x) = \int \mathrm{d}k_0\, \mathrm{d}k_1\, (\exp(w \tilde{K}_x)\,\tilde{\Psi}) \exp(-i k_0 t + i k_1 x) \quad (43)$$

where we have defined

$$\tilde{K}_x = K_0 \frac{\partial}{\partial k_1} + k_1 \frac{\partial}{\partial k_0} - \vartheta k_1/\omega(k_1). \quad (44)$$

In order to evaluate $\exp(w \tilde{K}_x)\,\tilde{\Psi}(k_0, k_1)$ we proceed as follows. Introduction of polar coordinates ϱ, α,

$$k_0 = -i\varrho \cos\alpha, \qquad \varrho = (k_1^2 - k_0^2)^{1/2},$$
$$k_1 = \varrho \sin\alpha, \qquad \alpha = \arctan(k_1/i k_0) \quad (45)$$

reduces the problem to the evaluation of

$$\exp(w\tilde{K}_x)\,\tilde{\Psi}(\alpha)=\exp(w(-i\partial_\alpha-\vartheta\,u(\alpha))\,\tilde{\Psi}(\alpha), \tag{46}$$

$$u(\alpha)=k_1/\omega(k_1)=(\varrho/m)\sin\alpha\cdot(1+(\varrho^2/m^2)\sin^2\alpha)^{-1/2}.$$

Let us call the unknown function (46) for the moment $g(\alpha, w)$. Obviously we have

$$g(\alpha, 0)=\tilde{\Psi}(\alpha) \tag{47}$$

and

$$\exp(w\tilde{K}_x)\,g(\alpha, 0)=g(\alpha, w). \tag{48}$$

Partial differentiation of (48) with respect to w leads to the following partial differential equation for $g(\alpha, w)$:

$$i\partial_\alpha g+\partial_w g+\vartheta\,u\,g=0 \tag{49}$$

with the boundary condition (47).

A general solution reads [6]

$$g(\alpha, w)=(i\varrho\cos\alpha+(m^2+\varrho^2\sin^2\alpha)^{1/2})^{-\vartheta}\,\varphi(\alpha-i\,w),$$

where φ is an arbitrary function.

Making use of (47), this leads to the final result

$$\exp(w\tilde{L}_x)\,\Psi(t, x)=\int dk_0\,dk_1\,[(\omega(k_1)-k_0)/(\omega(k_1')-k_0')]^{\vartheta}\cdot\tilde{\Psi}(k_0, k_1)\exp(-i\,k_0\,t'+i\,k_1\,x'), \tag{50}$$

where

$$\begin{pmatrix}k_0'\\k_1'\end{pmatrix}=\Lambda\begin{pmatrix}k_0\\k_1\end{pmatrix},\quad \Lambda=\begin{pmatrix}\cosh\omega & -\sinh\omega\\ -\sinh\omega & \cosh\omega\end{pmatrix} \tag{51}$$

and

$$\begin{pmatrix}t'\\x'\end{pmatrix}=\Lambda^{-1}\begin{pmatrix}t\\x\end{pmatrix}. \tag{52}$$

Under (50) an arbitrary function $\Psi(t, x)$ transforms under the representation generated by $\tilde{L}_x$. For $\vartheta=0$ we regain the familiar result for the transformation of a scalar function Ψ.

b) Transformations of the Solutions of the Ω-Equation

If we want to apply (50) to a solution of (6), we must represent the solution in the form (42). Equation (6) requires

$$k_0=\omega(k_1),$$

so we put

$$\tilde{\Psi}(k_0, k_1)=\tilde{\Psi}(k_1)\,\delta(k_0-\omega(k_1)).$$

Carrying out the integration over k_0 in the δ-function and applying l'Hospital's rule we find

$$\Psi'(t, x)=\int dk_1\,[\cosh w-(k_1/\omega(k_1))\sinh w]^{\vartheta}\,\tilde{\Psi}(k_1)\cdot\exp(-i\omega(k_1)\,t'+i\,k_1\,x'). \tag{53}$$

Specializing $\vartheta=1/2$, we end up with

$$\Psi'(t, x)=\int dk\,[\cosh w-(k/\omega(k_1))\sinh w]^{1/2}\,\tilde{\Psi}(k)\cdot\exp(-i\omega(k)\,t'+i\,k\,x') \tag{54}$$

with t' and x' from (52).

Let us apply (54) to a plane wave with momentum q, i.e. $\tilde{\Psi}(k)=\delta(k-q)$, $\Psi^*\Psi=1$; one finds

$$\Psi'^*\Psi'=\cosh w-(q/\omega(q))\sinh w=(1-u\,v)/(1-v^2)^{1/2}, \tag{55}$$

where $u=q/\omega(q)$, $v=\tanh w$. Equation (55) is exactly the change of normalization due to Lorentz contraction.

If we define the distribution

$$\mathscr{L}(w; x-x')=(2\pi)^{-1}\int dk\,\tilde{\mathscr{L}}(w; k)\cdot\exp(i\,k(x-x')), \tag{56}$$

where

$$\tilde{\mathscr{L}}(w; k)=[\cosh w-(k/\omega(k))\sinh w]^{1/2},$$

we can give (54) the concise form

$$\Psi'(t, x)=\int dx''\,\mathscr{L}(w; x'-x'')\,\Psi(t', x''), \tag{57}$$

t' and x' being given by (52) as before.

6. Plane Wave Solutions

Because of (12) there exist solutions of (6) for fixed momentum, i.e. plane waves. (6) requires

$$k_0=\omega(\boldsymbol{k})=(\boldsymbol{k}^2+m^2)^{1/2}$$

with a unique sign of ω. Negative frequences do not appear: for spin 0 particles there exists no Pauli principle and therefore no hole theory to prevent the particles from falling into negative energy states.

The expectation value of the velocity squared does not exceed the velocity of light (1 in our units) for arbitrary wave packets, from which we conclude that no signal can be transmitted faster than c:

$$\langle\boldsymbol{u}^2\rangle\leq 1.$$

Nevertheless, the Hamiltonian Ω being nonlocal, it is interesting to discuss the propagation of the amplitude Ψ in more detail.

For this purpose we study the solution of (6) with the initial condition $\Psi(t=0)=\delta(\boldsymbol{x})$. It is obviously given by

$$\Psi(t, \boldsymbol{x})=(2\pi)^{-3}\int dk^3\exp(-i\omega t+i\,\boldsymbol{k}\cdot\boldsymbol{x}),$$

which [7]

$$= i\partial_t (2\pi)^{-3} \int dk^3 (1/\omega) \cdot \exp(-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}) = i\partial_t (2i\Delta^{(+)}) = \partial_t (i\Delta + \Delta^{(1)}).$$

The first term, Δ, vanishes outside the light cone [8], but the second, $\Delta^{(1)}$, does not. Is this a catastrophe? Let us investigate to what extent it is one. Denoting the corresponding part of Ψ by Ψ_1,

$$\Psi_1 = \partial_t \Delta^{(1)},$$

we have [9] for $t^2 - r^2 < 0$, or $s = r - t > 0$,

$$\Psi_1 \sim \partial_t (K_1(z)/z),$$

where $K_n(z)$ is the modified Bessel function of n^{th} order and $z = m(r^2 - t^2)^{1/2} = m(s^2 + 2st)^{1/2}$.

Performing the differentiation and using recurrence relations [10], we get

$$\Psi_1 \sim t(2K_1(z) - z K_0(z))/(z^3). \tag{58}$$

Besides the factor t, the right hand side depends only on z. The functions $K_0(z)$ and $K_1(z)$ [11], have a singularity in $z = 0$, drop off rapidly for values $z > 0$ and for $z \gtrsim 1$ vanish like $\exp(-z) < \exp(-ms)$. Thus, $\Psi_1(z)/t$ describes a "layer" of thickness of the order $1/m$ around the sphere $r = t$. If we follow the development of (58) with time, we find a spherical wave of the following shape: At $t = 0$, Ψ vanishes outside $r = t = 0$.

As t starts growing, the "layer" in which $\Psi_1 \neq 0$, is being built up. But soon the exponential slope of K_0 and K_1 will dominate the factor t, and from there on, the thickness of the layer will stop increasing.

For a rough estimate let us adopt a simplified model:

$$\Psi_1(t, s) = t \exp(-m(s^2 + 2st)^{1/2}).$$

From $\partial_t \Psi_1 = 0$ we find for the time t_0 when Ψ_1 at a fixed distance $s = n/m$, $n = 1, 2, \ldots$ outside the sphere $r = t$ reaches its maximum value,

$$t_0 = (1/m)(1/n + (1 + 1/(n^2))^{1/2}).$$

Thus, after a time interval of the order $1/m$ the layer of thickness $1/m$ will be developed, but from there on it simply travels with the surface, without further increasing. (In fact, it will even shrink during the expansion of the sphere because of the term $2st$ in z.)

If in the propagation of Ψ there were really involved speeds $v > 1$, the thickness of the layer outside $r = t$ would have to grow for all times. Since this is not the case, no signal can be transmitted with $v > 1$ on a macroscopic scale. On a microscopic scale, i.e. during the development of the layer, or, correspondingly, within $t \lesssim 1/m$ (some 10^{-23} sec for pions, the lifetime of pionic resonances, for instance), there is a non-vanishing probability for an apparent $v > 1$. Unless we assume different upper limits for v for times $\lesssim 1/m$ and $\gtrsim 1/m$, respectively, we must ascribe the development of the layer to some other mechanism. (In some respect it looks as if for a spin 0 particle the influence of a finite extension cannot be neglected in certain cases. After all, the assumed initial condition is highly singular, $\delta(\boldsymbol{x})$ not being a normalizable wave function.)

7. Coupling of Potentials

a) Coupling of Scalar Potential, Radial Equation

We introduce a scalar potential V into (6) by equating $i\partial_t$ to the total energy $\Omega + V$. Thus we arrive at

$$i\partial_t \Psi = (\Omega + V)\Psi. \tag{59}$$

Let us emphasize that iterated (59) is no longer equivalent to the Klein-Gordon equation with potential. Putting, in case of a time-independent V,

$$\Psi(t, \boldsymbol{x}) = \exp(-iEt)\,\Psi(\boldsymbol{x}),$$

we get the stationary equation

$$(\Omega + V)\Psi = E\Psi. \tag{60}$$

In case of a spherical $V(r)$ one can set up a radial equation in analogy to the nonrelativistic Schrödinger theory.

Making use of the familiar ansatz

$$\Psi(t, \boldsymbol{x}) = u_l(r)/r\; Y_l^m$$

and expanding $\Omega(\boldsymbol{x} - \boldsymbol{x}') = \Omega(|\boldsymbol{x} - \boldsymbol{x}'|)$ into Legendre polynomials, one finally arrives [12] at the radial equation

$$\int_0^\infty dr'\, S_l(r, r')\, u_l(r') + V(r)\, u_l(r) = E\, u_l(r), \tag{61}$$

where the radial distribution $S_l(r, r')$ is defined by

$$S_l(r, r') = 2\pi^{-1} \int_0^\infty dk\, \omega(k)\, k\, r\, j_l(kr)\, k\, r'\, j_l(kr') \tag{62}$$

with the spherical Bessel functions $j_l(z)$ defined by [13].

b) Remark Concerning the Continuity Equation

It is interesting to observe that in the presence of a potential $V(\boldsymbol{r})$, (23) aquires a local source term. Using (59) instead of (6), one finds instead of (23)

$$\partial_t \sigma + \operatorname{div} \boldsymbol{j} = Q, \tag{63}$$

the source q being given by

$$Q = (1/2i)(\boldsymbol{u}\,\Psi)^* \cdot (\boldsymbol{u}\,V\,\Psi) - (\boldsymbol{u}\,V\,\Psi)^* \cdot (\boldsymbol{u}\,\Psi) + m^2 (\Omega^{-1}\,\Psi)^* (\Omega^{-1}\,V\,\Psi) - (\Omega^{-1}\,V\,\Psi)^* (\Omega^{-1}\,\Psi).$$

In the nonrelativistic limit, $\Omega^{-1} \to 1/m$, $\boldsymbol{u}^2 \to 0$, Q, of course, vanishes. Since

$$\int Q \, \mathrm{d}x^3 = 0,$$

from (63) still follows (26).

c) Coupling of Magnetic Field

The coupling of a magnetic field to (6) may be accomplished in the usual way according to the principle of minimal coupling, i.e. the replacement

$$\boldsymbol{p} \to \boldsymbol{p} - e\boldsymbol{A}, \quad \text{or} \quad -i\partial \to -i\partial - e\boldsymbol{A},$$

$\boldsymbol{A}$ being the vector potential of the magnetic field. If we use Coulomb gauge, for which $\boldsymbol{p}$ and $\boldsymbol{A}$ commute,

$$\boldsymbol{p} \cdot \boldsymbol{A} = \boldsymbol{A} \cdot \boldsymbol{p}, \tag{64}$$

we can argue as follows.

First we note that $-i\partial$ is equivalent to the distribution

$$\boldsymbol{p}(\boldsymbol{x} - \boldsymbol{x}') = (2\pi)^{-3} \int \mathrm{d}k^3 \, \boldsymbol{k} \, \exp(i\boldsymbol{k} \cdot (\boldsymbol{x} - \boldsymbol{x}')),$$

and $-i\partial - e\boldsymbol{A}$ is equivalent to

$$(2\pi)^{-3} \int \mathrm{d}k^3 (\boldsymbol{k} - e\boldsymbol{A}(\boldsymbol{x})) \exp(i\boldsymbol{k} \cdot (\boldsymbol{x} - \boldsymbol{x}')).$$

Consequently, and because of (64), we can couple a magnetic field to (6), replacing Ω from (4) by

$$\Omega_M(\boldsymbol{x}, \boldsymbol{x}') = (2\pi)^{-3} \int \mathrm{d}k^3 ((\boldsymbol{k} - e\boldsymbol{A}(\boldsymbol{x})^2 + m^2)^{1/2} \cdot \exp(i\boldsymbol{k} \cdot (\boldsymbol{x} - \boldsymbol{x}')). \tag{65}$$

Equation (65) may be cast into a more elaborate form.

Making use of $\partial/\partial \boldsymbol{k}$ being the generator of a translation in $\boldsymbol{k}$-space:

$$\exp\left(\boldsymbol{A} \cdot \frac{\partial}{\partial \boldsymbol{k}}\right) f(\boldsymbol{k}) = f(\boldsymbol{k} + \boldsymbol{A})$$

for arbitrary $f(\boldsymbol{k})$, and applying partial integration, we may write the wave equation with magnetic field

$$i\partial_t \Psi = \Omega_M \, \Psi = \int \mathrm{d}x'^3 \exp(ie\boldsymbol{A}(\boldsymbol{x}) \cdot (\boldsymbol{x} - \boldsymbol{x}')) \, \Omega(\boldsymbol{x} - \boldsymbol{x}') \, \Psi(\boldsymbol{x}'). \tag{66}$$

In the nonrelativistic limit we regain the Schrödinger equation in a magnetic field in Coulomb gauge. In this limit

$$\Omega(\boldsymbol{x} - \boldsymbol{x}') = m\delta(\boldsymbol{x} - \boldsymbol{x}') - (2m)^{-1} \Delta\delta(\boldsymbol{x} - \boldsymbol{x}'),$$

and the right hand side reduces to

$$m\,\Psi - (2m)^{-1} (\Delta\Psi - 2ie\boldsymbol{A} \cdot \partial\Psi + (-ie\boldsymbol{A})^2 \Psi).$$

8. Perturbation Theory for Discrete States

a) Energy Shift of First Order

Assuming that the relativistic corrections introduced by (6) are small enough we set up a perturbation theory according to familar methods, not in the potential V, though, but in the kinetic energy.

We designate the nonrelativistic limit of ω by ω_0,

$$\omega_0 = m + k^2/2m.$$

Let Ω_0 be the corresponding distribution in $\boldsymbol{x}$-space

$$\Omega_0(\boldsymbol{x} - \boldsymbol{x}') = (2\pi)^{-3} \int \mathrm{d}k^3 \, \omega_0(k) \exp(i\boldsymbol{k} \cdot (\boldsymbol{x} - \boldsymbol{x}')) \tag{67}$$

and

$$\Omega^{(1)} = \Omega - \Omega_0, \tag{68}$$

i.e.

$$\Omega^{(1)} = (2\pi)^{-3} \int \mathrm{d}k^3 \, \omega_1(\boldsymbol{k}) \exp(i\boldsymbol{k} \cdot (\boldsymbol{x} - \boldsymbol{x}')), \quad \omega_1 = \omega - \omega_0. \tag{69}$$

Let us split $\Psi(\boldsymbol{x})$ into

$$\Psi = \Psi_0 + \Psi_1, \tag{70}$$

Ψ_0 being a solution of

$$(\Omega_0 + V)\,\Psi_0 = E_0 \, \Psi_0, \tag{71}$$

i.e. of the Schrödinger equation for the eigenvalue E_0, and Ψ_1 being small of first order. Substituting (70) into (60), using (68) and (71), designating the difference $E - E_0$ by E_1 (small of first order) and neglecting the terms of second order $\Omega^{(1)}\Psi_1$, $E_1\,\Psi_1$, we find in the familiar way

$$E_1 = \int \mathrm{d}x^3 \, \Psi_0^* \, \Omega^{(1)} \, \Psi_0. \tag{72}$$

b) Application to Central Potential

$\Omega^{(1)}$ being a function of $|\boldsymbol{x} - \boldsymbol{x}'|$ just like Ω, we can apply the methods of section 7a to find in case of central potential

$$E_1 = \int_0^\infty \mathrm{d}k \, \omega_1(k) \, |\hat{u}_{nl}(k)|^2, \tag{73}$$

where we have defined $\hat{u}_{nl}(k)$ by

$$\hat{u}_{nl}(k)=(2/\pi)^{1/2}\int_0^\infty \mathrm{d}r\, u_{nl}(r)\, k\, r\, j_l(k\, r), \tag{74}$$

and $u_{nl}(r)$ is the radial part of the unperturbed, i.e. nonrelativistic wave function

$$\Psi_0(\boldsymbol{x})=r^{-1}\, u_{nl}(r)\, Y_l^m.$$

Summary

Obviously, there exist at least two essentially different one-component wave functions, transforming according to the generators L_i and $\tilde{L}_i$, respectively. We might call them "Lorentz scalar" and "Dirac scalar" function, respectively. If for a spin 0 meson one chooses a Dirac scalar wave function, one can set up a complete relativistic one-particle quantum mechanics, comprising the essential tools of wave equation, scalar product, conserved positive definite density. Scalar and vector potentials may be coupled. The nonlocal nature of the wave operator leads to some unusual features of the particle: current and density are in general not connected by a local velocity field (Sect. 4); reconcilation of nonlocality with the (somewhat singular) initial condition $\Psi(t=0)=\delta(\boldsymbol{x})\,\delta(\boldsymbol{x})$ being a nonnormalizable wave function) leads to an apparent $v>1$ for the amplitude at very short times, although no states with $v>1$ exist (section 6); in the presence of an external potential a source term appears in the continuity equation, which, however, does not destroy conservation of the norm (section 7).

Appendix

1) Both S_i and τ_{ij} are bilinear forms of the type (let us call it M for the moment)

$$M=\Sigma(A_k\,\Psi)^*\,(B_k\,\Psi) \tag{A1}$$

(k is not a covariant index; in τ_{ij}, for instance, we might put

$$A_1=-1/4, \qquad B_1=u_i\,u_j,$$
$$A_2=-(1/4)\,u_i\,u_j, \qquad B_2=1 \qquad \text{and so on.}$$

Nevertheless, in what follows we shall apply dummy index summation, omitting the Σ, to simplify writing.)

In (A1), Ψ is a solution of (6), A_k, B_k are hermitean operators with the following properties:

$$A_k\,B_k=0 \quad \text{identically (dummy index summation)} \tag{A2}$$

Let us call this sum the trace of M.

$$[\partial_t,\, A_k]=[\Omega,\, A_k]=[\boldsymbol{u},\, A_k]=0,$$
likewise for the B_k. (A3)

From (A2) it follows

$$\int \mathrm{d}x^3\, M=0. \tag{A4}$$

2) If we subject M to a Lorentz transformation (30), the result M' is of the shape

$$M'=(1+\boldsymbol{v}\cdot\boldsymbol{L})\,M-\boldsymbol{v}\cdot\alpha, \tag{A5}$$

where α is again of the type (A1); for,

$$\begin{aligned}M'=(A_k\,\Psi')^*\,(B_k\,\Psi')&=(A_k(\Psi+\boldsymbol{v}\cdot(\boldsymbol{L}-(1/2)\,\boldsymbol{u})\,\Psi))^*\\&\cdot(B_k(\Psi+\boldsymbol{v}\cdot(\boldsymbol{L}-(1/2)\,\boldsymbol{u})\,\Psi))=(A_k\,\Psi)^*\,(B_k\,\Psi)\\&+\boldsymbol{v}\cdot((A_k\Psi)^*\,(B_k(\boldsymbol{L}\Psi-(1/2)\,\boldsymbol{u}\,\Psi))\\&+(A_k(\boldsymbol{L}\Psi-(1/2)\,\boldsymbol{u}\,\Psi))^*\,(B_k\,\Psi))\end{aligned}$$

Extracting the generator $\boldsymbol{L}$, we find finally

$$\begin{aligned}M=(1+\boldsymbol{v}\cdot\boldsymbol{L})\,M&-\boldsymbol{v}\cdot(-(A_k\,\Psi)^*\,[B_k,\,\boldsymbol{L}]\,\Psi)\\&-([A_k,\,\boldsymbol{L}]\,\Psi)^*\,(B_k\,\Psi)\\&+(1/2)\,(A_k\,\Psi)^*\,(B_k\,\boldsymbol{u}\,\Psi)+(1/2)\,(A_k\,\boldsymbol{u}\,\Psi)^*\,(B_k\,\Psi))\end{aligned}$$

The term proportional to $\boldsymbol{v}$ is the α from (A5). Its trace vanishes, indeed

$$\begin{aligned}&-A_k[B_k,\,\boldsymbol{L}]-[A_k,\,\boldsymbol{L}]\,B_k+(1/2)\,A_k\,B_k\,\boldsymbol{u}\\&+(1/2)\,A_k\,\boldsymbol{u}\,B_k=-[A_k\,B_k,\,\boldsymbol{L}]+A_k\,B_k\,\boldsymbol{u}=0\end{aligned}$$

identically because of (A2).

3) Now let us assume that a Lorentz invariant continuity equation holds for two bilinear forms M, N of the type (A1),

$$\partial_t M+\operatorname{div} N=0. \tag{A6}$$

As just demonstrated, we have under Lorentz transformation

$$M'=(1+\boldsymbol{v}\cdot\boldsymbol{L})\,M-\boldsymbol{v}\cdot\alpha,\quad N'=(1+\boldsymbol{v}\cdot\boldsymbol{L})\,N-\boldsymbol{v}\cdot\beta,$$

where α, β are again of type (A1). From the Lorentz invariance of (A6) it follows

$$\begin{aligned}0=\partial_t M'+\operatorname{div} N'&=\boldsymbol{v}\cdot\partial M\\&+\boldsymbol{v}\cdot\partial_t N-\boldsymbol{v}\cdot(\partial_t\alpha+\operatorname{div}\beta)\end{aligned}$$

for arbitrary $\boldsymbol{v}$ and therefore

$$\partial_t(\alpha - N) + \operatorname{div}(\beta - IM) = 0. \tag{A7}$$

(Here I is a unit tensor, to allow writing $\boldsymbol{\partial} M = \operatorname{div}(IM)$.)

From the new continuity equation (A7) we derive the conservation law

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}x^3(\alpha - N) = 0,$$

which, because of the shape of α and N, is nothing new. Moreover, (A7) is again of the type (A6), so, starting from (40), the successive continuity equations do not furnish any non-trivial information.

List of some symbols used

$\boldsymbol{\partial}$ = gradient, $\quad \partial_i = \dfrac{\partial}{\partial x_i}$,

$\partial_x, \partial_\alpha, \partial_w$ and so on $= \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial \alpha}, \dfrac{\partial}{\partial w}$, respectively and so on.

[1] W. R. Theis, Grundzüge der Quantentheorie, B. G. Teubner, Stuttgart 1985, especially p. 202 or Ref. [2], Vol. II, p. 885.

[2] See, for instance, A. Messiah, Quantum Mechanics Vol. I, North Holland Publ. Co., Amsterdam, 2nd Printing 1964, Appendix A.

[3] This new representation is a nonlocal one. A survey of all local irreducible representations of the restricted Lorentz group is given by M. A. Naimark, Linear representations of the Lorentz group, American Math. Society Translations, Series 2, Vol. 6 (1957).

[4] See [2], especially p. 467: The Ω-distribution is obtained by choosing Messiah's $a(k) = (k^2 + m^2)^{1/2}$.

[5] M. E. Rose, Relativistic Electron Theory, Section 18 and Appendix C, John Wiley & Sons, New York 1961.

[6] J. Horn, Partielle Differentialgleichungen, Fourth Edition, §47, Walter de Gruyter, Berlin 1949.

[7] S. S. Schweber, An Introduction to Relativistic Quantum Field Theory, Sect. 13d, Row, Peterson & Co.; Evanston, Illinois, Elmsford – New York (1961).

[8] See [7], Sect. 7c.

[9] Appendix of J. Schwinger, Phys. Rev. **75**, 651 (1949).

[10] M. Abramowitz, I. Stegun, (Eds.), Handbook of Mathematical Functions, p. 376, Dover Publications Inc., New York 1965.

[11] See [10], Figure 9.8.

[12] Bateman Manuscript Project, A. Erdelyi, (Ed.), Tables of Integral Transforms, Vol. II, p. 277, Nr. 11, McGraw-Hill Book Company, New York 1954.

[13] See [10], p. 437.