

Lie Series and Canonical Transformations in Complex Phase Space

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Dedicated to Prof. H. G. Schöpf on the Occasion of his 60th Birthday

This paper considers the Lie series representation of the canonical transformations in a complex phase space. A condition is given which selects the canonical mappings from the Lie transformations associated with a complex-valued generating function. Some special types of mappings and some simple algebraic tools are discussed.

1. Introduction

Recent advances in the theory of nonlinear dynamical systems have stimulated an intensive research around the problem of the existence of a quantum mechanical counterpart to the classical chaotic behaviour (see e.g. [1–3]). In this connection the transition from quantum mechanics to classical mechanics becomes very important since the chaotic behaviour is defined in a strict sense only for a classical system (e.g. by a positive Lyapunov exponent). A deeper principle for this transition would be desirable which is based on algebraic structures. It is well known [4–5] but not often used that classical and quantum mechanics may be embedded in the same mathematical formulation. The difference between the two theories does not lie in their mathematical structures but rather in their different physical interpretations. In a recent paper [6] we have found a connection between the entropy of an ideal Fermi or Bose system and two special classes of canonical transformations in a complex phase space. From the physical point of view such a connection is a very obscure matter and will bring additional confusion into the relation between classical and quantum mechanics. However, the entropy is a very important quantity which played a great role in the development of quantum theory [7] and may help to find an answer to some open physical questions. For instance, we hope that the transition from quantum mechanics to statistics by means of the substitution $i/t \rightarrow 1/k_B T$ can be better substantiated.

The aim of this paper is to demonstrate that the two classes of transformations in question arise in a natu-

ral way by using the Lie series representation of the canonical transformations. Lie algebraic methods and Lie series in a real phase space have found widespread application e.g. to expansions of solutions of Hamilton's equations or to reductions of Hamiltonians to normal form [8–9]. An extensive survey can be found in the references of Steinberg [10]. On the other hand, for the complex phase space Strocchi [4] has given the differential characterization of the canonical transformation only, and this contribution gives additional advice on Lie series characterization.

From this point of view our paper has a more technical content.

2. Lie Transformations and their Canonical Part

In [6] we have considered the coordinate transformations in a linear complex vector space:

$$z_k \rightarrow w_k \quad \text{with} \quad w_k = w_k(z_j, z_j^*) \\ \text{and} \quad w_k^* = w_k^*(z_j, z_j^*), \quad k, j = 1, \dots, f,$$

where z_k and w_k are complex-valued functions. The dependence upon an additional parameter is possible but is not considered here. Such a transformation is a canonical one if the new coordinates fulfill the conditions:

$$\{w_j, w_k\} = 0, \quad \{w_j, w_k^*\} = \delta_{jk}, \quad (2.1)$$

where the Poisson bracket of two complex-valued functions $A = A(z_k, z_k^*)$ and $B = B(z_k, z_k^*)$ is defined by

$$\{A, B\} \equiv \sum_{k=1}^f \left(\frac{\partial A}{\partial z_k} \frac{\partial B}{\partial z_k^*} - \frac{\partial A}{\partial z_k^*} \frac{\partial B}{\partial z_k} \right). \quad (2.2)$$

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We restrict our considerations to such functions A, B which are analytic in the $2f$ variables z_k, z_k^* [11–12] because only in this case they are also analytic in the $2f$ real variables $\text{Re}(z_k)$ and $\text{Im}(z_k)$. $\text{Re}(\cdot)$ and $\text{Im}(\cdot)$ indicate the real and imaginary part of a complex quantity. The term analytic is used in the usual sense that a convergent power series expansion exists. The complex Poisson bracket has the same properties as its real counterpart.

Let A be a specified function depending on z_k and z_k^* . Then with A is associated a linear differential operator (the Lie operator [8], the Lie derivative [10]):

$$\hat{X}_A \equiv \sum_{k=1}^f \left(\frac{\partial A}{\partial z_k} \frac{\partial}{\partial z_k^*} - \frac{\partial A}{\partial z_k^*} \frac{\partial}{\partial z_k} \right). \quad (2.3)$$

It is straightforward to calculate the commutator of two such operators

$$[\hat{X}_A, \hat{X}_B] = \hat{X}_{[A, B]}, \quad (2.4)$$

i.e. the mapping $A \rightarrow \hat{X}_A$ is a Lie algebra homomorphism from the Poisson bracket algebra of the underlying functions to the Lie operator algebra. The properties of the Poisson bracket imply the following set of properties for the Lie operator (a, b are complex numbers):

$$\begin{aligned} (\hat{X}_A)^* &= -\hat{X}_{A^*}, \quad \hat{X}_A B = -\hat{X}_B A, \\ \hat{X}_C (a \cdot A + b \cdot B) &= a \cdot \hat{X}_C A + b \cdot \hat{X}_C B, \\ \hat{X}_C (A \cdot B) &= (\hat{X}_C A) B + (\hat{X}_C B) A, \\ \hat{X}_C \{A, B\} &= \{\hat{X}_C A, B\} + \{A, \hat{X}_C B\}. \end{aligned} \quad (2.5)$$

Lie series are defined by infinite operator power series, where the exponential series

$$\exp(\hat{X}_A) \equiv \sum_{k=1}^{\infty} (1/k!) \cdot (\hat{X}_A)^k, \quad (\hat{X}_A)^0 \equiv 1 \quad (2.6)$$

are of particular interest. We shall call $\exp(\hat{X}_A)$ a Lie transformation associated with the complex-valued generating function $A(z_k, z_k^*)$. There are some basic properties of the Lie transformations which are important for their applications [10]:

$$\begin{aligned} \exp(\hat{X}_C) (a \cdot A + b \cdot B) &= a \cdot \exp(\hat{X}_C) A + b \cdot \exp(\hat{X}_C) B, \\ \exp(\hat{X}_C) (A \cdot B) &= (\exp(\hat{X}_C) A) \cdot (\exp(\hat{X}_C) B), \\ \exp(\hat{X}_C) \{A, B\} &= \{\exp(\hat{X}_C) A, \exp(\hat{X}_C) B\}, \\ \exp(\hat{X}_C) F(z_k, z_k^*) &= F(\exp(\hat{X}_C) z_k, \exp(\hat{X}_C) z_k^*). \end{aligned} \quad (2.7)$$

The proofs of these properties are based upon the power series definition and can be adopted from

Steinberg [10]. Lie transformations are very helpful tools for studying the canonical transformations (see Theorem 1 in [8] for the real case) in the phase space. However, there are differences because the generating function in our case may be a complex-valued function. To demonstrate these differences, we consider an infinitesimal Lie transformation of the complex coordinates z_k . By setting $A = -\varepsilon \cdot \phi(z_k, z_k^*)$ for the generating function and neglecting higher order terms in the small parameter ε one obtains

$$\begin{aligned} w_k &= \exp(\hat{X}_{-\varepsilon\phi}) z_k \\ &= \exp(-\varepsilon \cdot \hat{X}_\phi) z_k = z_k - \varepsilon \cdot \hat{X}_\phi z_k + O(\varepsilon^2). \end{aligned}$$

Moreover, with $\phi = S(z_k, z_k^*) + i \cdot \Omega(z_k, z_k^*)$, where S and Ω are real functions, we obtain

$$\begin{aligned} w_k &= z_k - \varepsilon \hat{X}_S z_k - i \varepsilon \hat{X}_\Omega z_k \\ &= z_k + \varepsilon \frac{\partial S}{\partial z_k^*} + i \varepsilon \frac{\partial \Omega}{\partial z_k^*}. \end{aligned}$$

A comparison with the general form of an infinitesimal canonical transformation given in [6] shows that this infinitesimal Lie transformation is not a canonical one. Therefore we must restrict the generating function ϕ , especially the real part S , by some additional conditions. These additional conditions are given by the following theorem:

Theorem 1: The Lie transformations

$$\begin{aligned} \text{(i)} \quad w_k &= \exp(-\hat{X}_\phi) z_k, \\ \text{(ii)} \quad w_k &= \exp(\hat{X}_\psi) z_k^*, \end{aligned} \quad (2.8)$$

providing they converge, are canonical mappings if and only if the generating functions fulfill the conditions

$$\begin{aligned} \text{(i)} \quad \delta_{kj} &= -\exp(\hat{X}_{\phi^*}) \hat{X}_{z_j^*} \exp(-\hat{X}_{\phi^*}) \cdot \exp(-\hat{X}_\phi) \cdot z_k, \\ \text{(ii)} \quad \delta_{kj} &= -\exp(-\hat{X}_{\psi^*}) \hat{X}_{z_j} \exp(\hat{X}_{\psi^*}) \cdot \exp(\hat{X}_\psi) \cdot z_k^*. \end{aligned} \quad (2.9)$$

Comment: The negative sign in (2.8(i)) may be dropped, but we use it to find a more direct connection to the formulas in [6]. The conditions (2.9) can be represented in a more compact form by using the adjoint operators [8].

Proof: We simply use (2.1) and the Poisson bracket preservation property in (2.7) to find for the case (i):

$$\begin{aligned} \{w_j, w_k\} &= \{\exp(-\hat{X}_\phi) z_j, \\ \exp(-\hat{X}_\phi) z_k\} &= \exp(-\hat{X}_\phi) \{z_j, z_k\} = 0. \end{aligned}$$

This follows from the Poisson bracket $\{z_j, z_k\} = 0$.

$$\begin{aligned}\{w_j, w_k^*\} &= \{\exp(-\hat{X}_\phi) z_j, \exp(\hat{X}_{\phi^*}) z_k^*\} \\ &= \exp(\hat{X}_{\phi^*}) \{\exp(-\hat{X}_{\phi^*}) \exp(-\hat{X}_\phi) z_j, z_k^*\} \\ &= -\exp(\hat{X}_{\phi^*}) \{z_k^*, \exp(-\hat{X}_{\phi^*}) \exp(-\hat{X}_\phi) z_j\} \\ &= -\exp(\hat{X}_{\phi^*}) \hat{X}_{z_k^*} \exp(-\hat{X}_{\phi^*}) \exp(-\hat{X}_\phi) z_j \\ &\stackrel{!}{=} \delta_{kj}.\end{aligned}$$

The last equality proves condition (i). Moreover, for the case (ii):

$$\begin{aligned}\{w_j, w_k\} &= \{\exp(\hat{X}_\psi) z_j^*, \exp(\hat{X}_\psi) z_k^*\} \\ &= \exp(\hat{X}_\psi) \{z_j^*, z_k^*\} = -\exp(\hat{X}_\psi) \{z_j, z_k\}^* = 0\end{aligned}$$

and

$$\begin{aligned}\{w_j, w_k^*\} &= \{\exp(\hat{X}_\psi) z_j^*, \exp(-\hat{X}_{\psi^*}) z_k\} \\ &= \exp(-\hat{X}_{\psi^*}) \{\exp(\hat{X}_{\psi^*}) \exp(\hat{X}_\psi) z_j^*, z_k\} \\ &= -\exp(-\hat{X}_{\psi^*}) \{z_k, \exp(\hat{X}_{\psi^*}) \exp(\hat{X}_\psi) z_j^*\} \\ &= -\exp(-\hat{X}_{\psi^*}) \hat{X}_{z_k} \exp(\hat{X}_{\psi^*}) \exp(\hat{X}_\psi) z_j^* \\ &\stackrel{!}{=} \delta_{kj}.\end{aligned}$$

This completes the proof.

Basically, the relations (2.9) are differential conditions which restrict the form of the generating functions ϕ, ψ . At first we study the infinitesimal transformations belonging to case (i). With $\phi \rightarrow \varepsilon \cdot \phi$ the condition (2.9(i)) yields, up to the first order in ε :

$$\delta_{kj} \stackrel{!}{=} \delta_{kj} + 2\varepsilon \cdot \frac{\partial^2 \operatorname{Re}(\phi)}{\partial z_j \partial z_k^*} + 0(\varepsilon^2).$$

The solution can be found without any difficulty and one obtains the separable functions $\operatorname{Re}(\phi) = \sum_k S_k(z_k, z_k^*)$ discussed in [6], i.e. the S_k are harmonic functions. On the other hand the transformation (2.8(ii)) does not contain an infinitesimal borderline case, i.e. this class of mappings is not connected with the identity. Also for the differential characterization given in [4] one obtains these two types, i.e. transformations which are connected with the identity and those which are not. As a matter of principle, the generating function is always determined by (2.9), however, up to an arbitrary complex constant only.

3. Special Types of Canonical Transformations

In this section we consider some special types of mappings which fulfill the conditions of theorem 1. Our

first example is given by the two families of transformations discussed in [6].

Let $\phi = \phi(I_k)$, $I_k = z_k^* z_k$ be the generating function of case (i). Then the corresponding Lie operator has the form

$$\hat{X}_\phi = \sum_{k=1}^f \left(\frac{\partial \phi}{\partial I_k} \right) \cdot \left(z_k^* \frac{\partial}{\partial z_k^*} - z_k \frac{\partial}{\partial z_k} \right).$$

This operator acts on the phase angles of the complex coordinates z_k only. The calculation of condition (2.9(i)) is easy to perform and yields

$$\begin{aligned}\delta_{kj} &= \left(\delta_{kj} + 2 z_k z_j^* \cdot \frac{\partial^2 \operatorname{Re}(\phi)}{\partial I_j \partial I_k} \cdot \exp \left(\frac{\partial \phi^*}{\partial I_j} - \frac{\partial \phi^*}{\partial I_k} \right) \right) \\ &\quad \cdot \exp \left(2 \frac{\partial \operatorname{Re}(\phi)}{\partial I_k} \right).\end{aligned}$$

The analysis of these conditions leads to the function

$$\phi(I_k) = \frac{1}{2} \cdot S_B(I_k) + i \Omega(I_k), \quad S_B, \Omega \text{ real}, \quad (3.1)$$

where $\Omega(I_k)$ is an arbitrary function and S_B is given by

$$\begin{aligned}S_B &= \sum_{k=1}^f a_k \left[\left(1 + \frac{I_k}{a_k} \right) \ln \left(1 + \frac{I_k}{a_k} \right) \right. \\ &\quad \left. - \left(\frac{I_k}{a_k} \right) \ln \left(\frac{I_k}{a_k} \right) \right].\end{aligned} \quad (3.2)$$

The a_k are parameters. An analogous calculation is possible also for the transformation of case (ii) in theorem 1 and yields

$$\psi(I_k) = \frac{1}{2} \cdot S_F(I_k) + i \Omega(I_k), \quad (3.3)$$

where S_F is defined by

$$\begin{aligned}S_F &= - \sum_{k=1}^f a_k \left[\left(1 - \frac{I_k}{a_k} \right) \ln \left(1 - \frac{I_k}{a_k} \right) \right. \\ &\quad \left. + \left(\frac{I_k}{a_k} \right) \ln \left(\frac{I_k}{a_k} \right) \right].\end{aligned} \quad (3.4)$$

Equations (3.1) and (3.3) are the generating functions for the Bose/Fermi type transformations discussed in [6]. Classical mechanics (in the usual sense) is connected with equations of motion which can be expressed for a Hamiltonian system in form of an infinitesimal canonical transformation and with the solutions of these equations of motion. The infinitesimal limit is not realizable for the Fermi type transformations, i.e. we can not use a dynamical analogy, e.g. in terms of an equation of motion, to interpret these mappings. This suggests that the transformation apparatus of classical mechanics contains more basic physical information than is generally supposed.

Some other types of canonical transformations can be found directly from condition (2.9):

Remark 1: Each function $\phi(z_k, z_k^*)$ which fulfills the operator identity

$$\exp(-\hat{X}_{\phi^*}) \exp(-\hat{X}_{\phi}) = 1 \quad (3.5)$$

generates a canonical mapping of the type (2.8 (i)).

Remark 2: Each function $\psi(z_k, z_k^*)$ which fulfills the identity

$$\exp(\hat{X}_{\psi^*}) \exp(\hat{X}_{\psi}) = -1 \quad (3.6)$$

generates a canonical mapping of the type (2.8 (ii)).

The proof is apparent by using $\hat{X}_{z_j^*} \cdot z_k = -\delta_{jk}$. However, (3.5/6) are special cases only. This can be demonstrated by considering the transformation properties of the absolute squares of the coordinates

$$w_k w_k^* = (\exp(-\hat{X}_{\phi}) z_k) (\exp(\hat{X}_{\phi^*}) z_k^*).$$

Moreover, with the product preservation property of (2.7),

$$w_k w_k^* = \exp(-\hat{X}_{\phi}) (z_k \exp(\hat{X}_{\phi}) \exp(\hat{X}_{\phi^*}) z_k^*).$$

Using the conjugate equation to (3.5), one obtains

$$w_k w_k^* = \exp(-\hat{X}_{\phi}) (z_k z_k^*) \quad (3.7)$$

and analogously for (3.6)

$$w_k w_k^* = -\exp(\hat{X}_{\psi}) (z_k z_k^*). \quad (3.8)$$

Instead of (3.5/6) we can use (3.7/8) to characterize this type of transformations. The Bose/Fermi type transformations (3.1/3) with $S_{B/F} \neq 0$ are not members of this class. A simple example of (3.5) is given by any pure imaginary function $\phi^* = -\phi$. From these special functions one can construct e.g. the unitary transformations. The generating function is

$$\phi(z_k, z_k^*) = i \sum_j \sum_k A_{jk} z_j^* z_k = i z^+ \hat{A} z, \quad (3.9)$$

where $\hat{A} = \hat{A}^+$ is a hermitian matrix. Then (2.8 (i)) becomes

$$\begin{aligned} w_1 &= \sum_{n=0}^{\infty} (1/n!) \left(-i \sum_j \sum_k A_{jk} \left(z_j^* \frac{\partial}{\partial z_k^*} - z_k \frac{\partial}{\partial z_j} \right) \right)^n \cdot z_1 \\ &= \sum_{n=0}^{\infty} (1/n!) \left(i \sum_j \sum_k A_{jk} z_k \frac{\partial}{\partial z_j} \right)^n \cdot z_1 \\ &= \sum_k (\delta_{1k} z_k + i A_{1k} z_k + (i^2/2) \sum_j A_{1j} A_{jk} z_k + \dots) \\ &= \sum_k (\exp(i \hat{A})_{1k} z_k, \end{aligned}$$

or in a compact form

$$W = \exp(i \hat{A}) z = \hat{U} z. \quad (3.10)$$

This is the usual representation of the unitary transformation, i.e. by means of hermitian generators as practised in quantum mechanics. Up to the factor i the generating function has the structure of an expectation value in quantum mechanics. The double aspect of the phase space functions is well known (see e.g. [13]).

On the one hand they represent dynamical variables (measurable quantities), and on the other hand they generate the canonical mappings. Since quantum mechanics selects the unitary transformations, the corresponding observables have the same structure as (3.9). But then the main problem is the physical substantiation for the use of the unitary transformation. For instance one can use the invariance of the normalization condition; however, we think that this is a technical argument only. A general complex-valued function with $\text{Re}(\phi) \neq 0$ and $\text{Im}(\phi) \neq 0$ corresponds to a non-unitary transformation and therefore to a non-hermitian observable. This yields some problems with the standard interpretation of quantum mechanics; however, the conception of extended variables [14] is possible.

A partial combination of the unitary transformations with (3.1/3) is possible. Let \hat{A} be a linear combination of some other operators \hat{A}_{μ} :

$$\hat{A} = \sum_{\mu=1}^n a_{\mu} \hat{A}_{\mu},$$

where a_{μ} are parameters. Then the commuting operators $[\hat{A}_{\alpha}, \hat{A}_{\beta}] = 0$ within this generator set can be diagonalized simultaneously. With their help one can form a function (3.9) which depends on the absolute squares $z_k^* z_k$ only, and this function can be used for $\Omega(I_k)$ in (3.1/2).

In [6] we have noted that a Bose-like mapping can be represented by two Fermi-like mappings. This possibility can be studied directly from (2.8). One finds

$$\exp(-\hat{X}_{\phi}) z \stackrel{!}{=} \exp(\hat{X}_{\psi_2}) \exp(-\hat{X}_{\psi_1^*}) z,$$

or the operator identity

$$\exp(-\hat{X}_{\phi}) = \exp(\hat{X}_{\psi_2}) \exp(-\hat{X}_{\psi_1^*}). \quad (3.11)$$

Let us consider the case that the Fermi-like mappings are generated by the same function $\psi_1 = \psi_2 \equiv \psi$. Then

$$\exp(-\hat{X}_{\phi}) = \exp(\hat{X}_{\psi}) \exp(-\hat{X}_{\psi^*}).$$

Moreover, with the Baker-Campbell-Hausdorff formula:

$$\begin{aligned} \exp(-\hat{X}_\phi) &= \exp(\hat{X}_\psi - \hat{X}_{\psi^*} - [\hat{X}_\psi, \hat{X}_{\psi^*}]/2 \\ &\quad - [\hat{X}_\psi, [\hat{X}_\psi, \hat{X}_{\psi^*}]]/12 \\ &\quad - [\hat{X}_{\psi^*}, [\hat{X}_\psi, \hat{X}_{\psi^*}]]/12 + \dots). \end{aligned}$$

Using (2.4) one obtains:

$$\operatorname{Re}(\phi) = \{\psi, \psi^*\}/2 + \dots, \quad (3.12)$$

$$\operatorname{Im}(\phi) = -2 \operatorname{Im}(\psi) + \{\operatorname{Re}(\psi), \{\psi, \psi^*\}\}/6i + \dots,$$

i.e. commuting or canonical functions ψ, ψ^* generate a Bose-type transformation with a pure imaginary generating function only (ϕ is determined up to a constant only!). A nontrivial real part of ϕ arises from such functions ψ, ψ^* which form an algebraic structure with a basic relation of the type: $\{\psi, \psi^*\} = f(z_k, z_k^*)$.

In this connection the study of those functions ψ, ψ^* which generate (3.1/2) may be an interesting problem.

4. Some Simple Algebraic Properties

Let us consider some algebraic aspects. The Poisson bracket plays an important role in the computations of the Lie transformations. Therefore we study their action on special functions.

Let $F(z_k, z_k^*)$ be a homogeneous function depending on the $2f$ variables z_k, z_k^* :

$$F(a z_k, a^* z_k^*) = a^m a^{*n} \cdot F(z_k, z_k^*). \quad (4.1)$$

We shall call this function homogenous of the degree (m, n) . Then Euler's theorem gives

$$\sum_k z_k^* \frac{\partial F}{\partial z_k^*} = n F, \quad \sum_k z_k \frac{\partial F}{\partial z_k} = m F,$$

or by subtraction

$$\sum_k \left(z_k^* \frac{\partial}{\partial z_k^*} - z_k \frac{\partial}{\partial z_k} \right) F = (n - m) F.$$

The left hand side is the Lie operator associated with the unitary invariant

$$J = \sum_k z_k^* z_k, \quad \text{i.e.} \quad \hat{X}_J F = (n - m) F. \quad (4.2)$$

This is an eigenvalue type equation, where the eigenvalues are given by the degree difference $(n - m)$.

Now let $\mathcal{P}_{m,n}$ be the space of all polynomials which are homogeneous of the degree (m, n) . Since the Pois-

son bracket involves multiplication and differentiations, we have for any two polynomials $p_{ik} \in \mathcal{P}_{i,k}$, $p_{jm} \in \mathcal{P}_{j,m}$:

$$\{p_{ik}, p_{jm}\} \in \mathcal{P}_{i+j-1, k+m-1}$$

or more generally

$$\{\mathcal{P}_{i,k}, \mathcal{P}_{j,m}\} \subseteq \mathcal{P}_{i+j-1, k+m-1},$$

If the polynomial $p_{ik} \in \mathcal{P}_{i,k}$, then the complex conjugate polynomial $p_{ik}^* \in \mathcal{P}_{k,i}$.

The space $\mathcal{P}_{1,1}$ has a special significance because of

$$\{\mathcal{P}_{1,1}, \mathcal{P}_{j,m}\} \subseteq \mathcal{P}_{j,m}, \quad (4.4)$$

Moreover, if $p_{11} \in \mathcal{P}_{1,1}$ is any real polynomial, then from the Lie series follows

$$\exp(i \hat{X}_{p_{11}}) \mathcal{P}_{j,m} \subseteq \mathcal{P}_{j,m},$$

i.e. the canonical transformation generated by the quadratic functions (3.9) maps $\mathcal{P}_{j,m}$ into $\mathcal{P}_{j,m}$. This property is important for the construction of invariant functions with $F(w_k, w_k^*) = F(z_k, z_k^*)$. A trivial example is given by the generating function

$$\phi = 2\pi i \cdot \sum_k z_k z_k^* = 2\pi i \cdot J,$$

which generates a phase shift of 2π . Using (4.2), one obtains

$$\begin{aligned} \exp(-\hat{X}_\phi) F(z_k, z_k^*) &= \exp(2\pi i(m-n)) \\ &\quad \cdot F(z_k, z_k^*), \end{aligned}$$

i.e. the homogeneous functions (4.1) are invariant functions if the degree difference can be represented by an integer:

$$m - n = k \in \mathbb{Z}.$$

The relation (4.4) may be formally completed by

$$\{\mathcal{P}_{1,1}, \mathcal{P}_{2-j, 2-m}\} \subseteq \mathcal{P}_{2-j, 2-m}, \quad (4.5)$$

$$\{\mathcal{P}_{2-j, 2-m}, \mathcal{P}_{j,m}\} \subseteq \mathcal{P}_{1,1}. \quad (4.6)$$

With the help of (4.4–6) one can construct some representations of Lie algebras. Typical examples are the commutation relations of the $SU(2)$ generators. (4.4) contains as a special case:

$$\{\mathcal{P}_{1,1}, \mathcal{P}_{1,1}\} \subseteq \mathcal{P}_{1,1}.$$

We consider two real elements $A, B \in \mathcal{P}_{1,1}$ with $A = z^+ \hat{A} z$, $B = z^+ \hat{B} z$, where $z \in C^f$. Then one obtains

$$\{A, B\} = z^+ [\hat{A}, \hat{B}] z, \quad (4.7)$$

i.e. the direct connection between the Poisson bracket and the matrix commutator (see also [4–5]). For $z \in C^2$ and $\hat{A}, \hat{B} \rightarrow (\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3)$, where σ_k are the hermitian Pauli spin matrices, the usual $\mathfrak{su}(2)$ algebra can be found. In its raising and lowering operator form one obtains, using

$$L_3 = (z_1^* z_1 - z_2^* z_2)/2, L_+ = z_1^* z_2, L_- = z_1 z_2^*,$$

the commutation relations

$$\begin{aligned} \{L_3, L_+\} &= L_+; \{L_3, L_-\} = -L_-; \\ \{L_+, L_-\} &= 2L_3. \end{aligned} \quad (4.8)$$

Using the homomorphism $A \rightarrow \hat{X}_A$ of (2.4), one finds the corresponding Lie operator algebra. Increasing the dimension of the phase space C^f , one can construct the various higher dimensional representations. This is a particularity because the complex phase space and the representation space become identical. On the other hand, from (4.5–6) follows

$$\{\mathcal{P}_{1,1}, \mathcal{P}_{0,2}\} \subseteq \mathcal{P}_{0,2}, \{\mathcal{P}_{2,0}, \mathcal{P}_{0,2}\} \subseteq \mathcal{P}_{1,1},$$

and with $z \in C^1$ one finds for example ($z^* z \equiv I \in \mathcal{P}_{1,1}$)

$$\begin{aligned} \{I, z^2\} &= -2z^2; \{I, z^{*2}\} = 2z^{*2}; \\ \{z^{*2}, z^2\} &= -4I. \end{aligned}$$

After the transformation $I \rightarrow 2K_3, z^{*2} \rightarrow 2K_+, z^2 \rightarrow 2K_-$ one obtains the $\mathfrak{su}(1,1)$ ($= \mathfrak{sp}(2, \mathbb{R}) = \mathfrak{so}(2,1)$) algebra in its raising and lowering operator form (see e.g. [15])

$$\begin{aligned} \{K_3, K_+\} &= K_+; \{K_3, K_-\} = -K_-; \\ \{K_+, K_-\} &= -2K_3, \end{aligned} \quad (4.9)$$

This is a realization in a 1-dimensional phase space, and it must be underlined that the $\mathfrak{su}(1,1)$ in this form can be obtained starting from any canonical quantity w with $\{w, w^*\} = 1$. The raising and lowering operator form arises from the complex Poisson bracket in a natural way. To show this, let $H(z_k, z_k^*)$ be a real function and

$$\hat{X}_H \phi_\mu = \mu \cdot \phi_\mu = \{H, \phi_\mu\} \quad (4.10)$$

the eigenvalue problem of the associated Lie operator with a real eigenvalue μ . Using (2.5), one finds the complex conjugate equation

$$\hat{X}_H \phi_\mu^* = -\mu \cdot \phi_\mu^* = \{H, \phi_\mu^*\}, \quad (4.11)$$

i.e. the complex conjugate function ϕ_μ^* corresponds to the negative eigenvalue $-\mu$: $(\phi_\mu)^* = \phi_{-\mu}$.

From (4.10–11) one obtains

$$\{H, \phi_\mu^* \phi_\mu\} = 0 \quad \Leftrightarrow \quad \mu \in \mathbb{R},$$

and the Jacobi identity associated with H, ϕ_μ, ϕ_μ^* yields

$$\{H, \{\phi_\mu, \phi_\mu^*\}\} = 0.$$

This equation can be fulfilled by $\{\phi_\mu, \phi_\mu^*\} = f(H)$, where f is a real function. For linear functions $f(H)$, H, ϕ_μ, ϕ_μ^* form an algebraic structure with respect to the Poisson bracket. Using the homomorphism $A \rightarrow \hat{X}_A$, ϕ_μ and ϕ_μ^* become raising and lowering operators. However, one needs a scalar product so that \hat{X}_H becomes a hermitian operator with real eigenvalues μ . There is an interesting analogy to the construction of the Bargmann-Hilbert space [16–17]. To simplify matters, we consider the 1-dimensional case only. We look for a scalar product in the form

$$\langle f, g \rangle = i a \cdot \int f^*(z, z^*) \cdot g(z, z^*) dz \wedge dz^*, \quad (4.12)$$

where a is a real parameter. Then the inner product has the property $\langle f, g \rangle^* = \langle g, f \rangle$, i.e. $\langle f, f \rangle$ is real. In the Bargmann space of entire analytic functions one needs an additional real positive weight function $\varrho(z, z^*)$, which can be dropped in (4.12) because the functions f, g depend upon both variables z and z^* . In terms of coordinates we obtain with the aid of $dz = dq + i\beta \cdot dp$: $dz \wedge dz^* = 2i\beta \cdot dp \wedge dq$, and the integration is performed over the whole phase space, i.e. the whole complex plane. Further on, we consider functions with a finite norm only: $\langle f, f \rangle < \infty$. Using this inner product, we can characterize the Lie operator associated with a complex-valued function $A(z, z^*)$. With (2.5) one finds

$$\langle f, \hat{X}_A g \rangle = \langle \hat{X}_{A^*} f, g \rangle + i a \int \hat{X}_A (f^* g) dz \wedge dz^*.$$

The last integral vanishes if $\{A, f^* g\} = 0$, however, we can find also the following form:

$$\begin{aligned} &\int \hat{X}_A (f^* g) dz \wedge dz^* \\ &= \int \left(\frac{\partial}{\partial z^*} \left(\frac{\partial A}{\partial z} \cdot f^* g \right) - \frac{\partial}{\partial z} \left(\frac{\partial A}{\partial z^*} \cdot f^* g \right) \right) dz \wedge dz^*. \end{aligned}$$

Moreover, with the Stokes theorem:

$$\begin{aligned} &\int \hat{X}_A (f^* g) dz \wedge dz^* \\ &= - \int_{\partial C} \left(f^* g \cdot \frac{\partial A}{\partial z} \right) dz - \int_{\partial C} \left(f^* g \cdot \frac{\partial A}{\partial z^*} \right) dz^*, \end{aligned}$$

where ∂C is the boundary of the complex phase space. We assume for the class of functions in question that

these boundary integrals vanish, i.e.

$$\langle f, \hat{X}_A g \rangle = \langle \hat{X}_{A^*} f, g \rangle, \quad (4.13)$$

Then the real functions $A = A^*$ are connected with a hermitian Lie operator. Let us study the 1-dimensional realization of the $\mathfrak{su}(1,1)$, given by (4.9). \hat{X}_{K_3} is a hermitian operator with respect to the scalar product (4.12), and from $K_+ = (K_-)^*$ follows that the raising and lowering operators \hat{X}_{K_+} , \hat{X}_{K_-} will then be adjoint to each other:

$$\langle f, \hat{X}_{K_+} g \rangle = \langle \hat{X}_{K_-} f, g \rangle.$$

The spectra of the generators and of elements of the enveloping algebra (Casimir operators) can be obtained using the classical approach, which comes from the standard treatment of angular momentum.

One can find also a realization of the $\mathfrak{su}(2)$ in a 1-dimensional phase space by the functions

$$L'_3 = z^* z/2; \quad L'_+ = i z^{*2}/2; \quad L'_- = i z^2/2,$$

which fulfill the commutation relations (4.8). But now we have $(L'_+)^* = -L'_-$, i.e. the corresponding raising and lowering operators are not adjoint to each other with respect to (4.12). The representation structure has radically changed, and the situation is analogous to the discussion about non-self-adjoint representations in [15]. However, there is a realization of $\mathfrak{su}(2)$ in terms of Lie operators in a 1-dimensional phase space with $L'_+ = L_-$ (see appendix).

The scalar product defined by (4.12) has the following property with respect to the multiplication operator:

$$\langle f, A \cdot g \rangle = \langle A^* \cdot f, g \rangle, \quad (4.14)$$

where $A(z, z^*)$ is an arbitrary complex-valued function. The combination of (4.13) and (4.14) can be used to find e.g. an operator representation of the Heisenberg-Weyl algebra

$$\{z, z^*\} = 1, \quad \{z, 1\} = \{z^*, 1\} = 0$$

The corresponding mapping is given by

$$z \rightarrow \hat{L} = \hat{X}_z + \frac{1}{2} \cdot z \cdot \hat{1}; \quad z^* \rightarrow \hat{R} = \hat{X}_{z^*} + \frac{1}{2} \cdot z^* \cdot \hat{1},$$

$1 \rightarrow \hat{1}$, where $\hat{L} = (\hat{R})^+$ and the commutation relation $[\hat{L}, \hat{R}] = \hat{1}$ can be obtained by a direct calculation. If the lowering operator acts on a real ground state function $g_0(z^* \cdot z)$, we have $\hat{L} g_0 = (\partial/\partial z^* + \frac{1}{2} \cdot z) g_0 = 0 \Leftrightarrow g_0 \sim \exp(-z^* z/2)$, i.e. g_0 is proportional to the root of the weight function in the Bargmann space.

5. Concluding Remarks

In this paper, we have investigated the Lie series representation of the canonical transformations in a complex phase space. Compared to the standard results in the real phase space there are some differences by reason of the complex nature of the generating functions. In quantum mechanics the Lie operator associated with a complex-valued function corresponds to a non-unitary operator, e.g. to a raising or lowering type operator. The special types of mappings (3.1–4) show that both the real and imaginary parts of the generating functions are physically important. It would be interesting to find some other generating functions of the Fermi type (2.8(ii)). On the basis of (3.12) we suppose that there is a connection to the algebraic properties discussed in Section 4.

As a matter of fact, both classical and quantum mechanics are associated with an algebraic structure called a Lie ring. This often yields a close connection between classical and quantum mechanics. An example is given by the reduction of Hamiltonians to the Birkhoff-Gustavson normal form [18]. However, a better understanding of the restriction of all possible observables $f(z_k, z_k^*)$ to the set of the quadratic functions $z^+ \hat{A} z$ or the selection of the unitary transformations from the full canonical group is one of the basic problems for the transition from classical to quantum mechanics (see also [19]).

Intuitively one can understand the restriction of the number of independent basic physical observables of classical mechanics as a consequence of the Planck constant. Moreover, from qualitative discussions concerning the measurement of dependent observables Caianiello [20] has derived a line element in a real 8-dimensional extended phase space which generalizes the invariant of $\text{SO}(3,1)$. Introducing complex coordinates, his line element is actually the unitary invariant of the $\text{U}(3,1)$. Basically, such an invariant acts as an auxiliary condition [21] in the classical extended phase space, and the canonical transformation which preserves this condition is the corresponding unitary group. Therefore we suppose a connection between the process of measurement and the unitary transformation.

We can not give here a final answer to the problem of the restriction of the number of independent basic observables in the transition of classical to quantum mechanics. But we hope that the use of the unitary transformations can be substantiated by a deeper physical principle.

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Appendix: Lie Operator Realization of $su(2)$

The functions

$$\begin{aligned} L_+ &= \sqrt{\beta/I - I} \cdot z^* = \sqrt{\beta - I^2} \exp(-i\theta), \\ L_3 &= I = z^* z, \\ L_- &= \sqrt{\beta/I - I} \cdot z = \sqrt{\beta - I^2} \exp(i\theta), \end{aligned} \quad (A.1)$$

where β is a real positive constant and $\theta = \arg(z)$ is the phase angle of z , form an $su(2)$ algebra with respect to the Poisson bracket in a 1-dimensional complex phase space. But the corresponding commutation relations (4.8) are valid for $\beta > I^2$ only, i.e. for real values of the roots in $L_{+/-}$. In this case we have $L_+ = (L_-)^*$.

Using the mapping $A \rightarrow \hat{X}_A$, one finds with $(z, z^*) \rightarrow (I, \theta)$:

$$\begin{aligned} \hat{X}_{L_3} &= i \partial / \partial \theta, \\ \hat{X}_{L_+} &= -\exp(-i\theta) \left(\sqrt{\beta - I^2} \frac{\partial}{\partial I} + \frac{iI}{\sqrt{\beta - I^2}} \cdot \frac{\partial}{\partial \theta} \right), \\ \hat{X}_{L_-} &= \exp(i\theta) \left(\sqrt{\beta - I^2} \frac{\partial}{\partial I} - \frac{iI}{\sqrt{\beta - I^2}} \cdot \frac{\partial}{\partial \theta} \right). \end{aligned} \quad (A.2)$$

The integration in the scalar product (4.12) must be performed over a circular area with $(\text{Re}(z))^2 + (\text{Im}(z))^2 \leq \sqrt{\beta}$. For $I^2 = \beta$ the functions $L_{+/-}$ become zero and we have only one operator \hat{X}_{L_3} which

generates the $SU(1)$ subgroup. For larger values of I^2 the roots in $L_{+/-}$ have a purely imaginary value, i.e.

$$\begin{aligned} L_+ &\rightarrow K_+ = i\sqrt{I^2 - \beta} \exp(-i\theta), \\ K_- &= (K_+)^*, \quad K_3 = L_3. \end{aligned}$$

A simple calculation shows that these functions form the $su(1, 1)$ algebra (4.9).

From eq. (A.1) follows

$$L_+ L_- = \beta - I^2 \langle = \rangle \beta = L_+ L_- + L_3^2, \quad (A.3)$$

i.e. the parameter β has the structure of the associated hermitian form of $SU(2)$. According to the mapping of $A \rightarrow \hat{X}_A$, β corresponds to the Casimir operator

$$\hat{C} = (\hat{X}_{L_+} \cdot \hat{X}_{L_-} + \hat{X}_{L_-} \cdot \hat{X}_{L_+})/2 + (\hat{X}_{L_3})^2.$$

The eigenvalue problem

$$\hat{C} \psi(I, \theta) = \sigma \psi(I, \theta)$$

with $\psi = \Phi(I) \cdot \exp(-i\theta)$ and $x \equiv I/\sqrt{\beta}$ leads to the equation

$$\left[(1-x^2) \frac{d^2}{dx^2} - 2x \frac{d}{dx} + \sigma - \frac{m^2}{(1-x^2)} \right] \Phi(x) = 0.$$

This is the Legendre differential equation, i.e. the eigenvalues are $\sigma = l(l+1)$, and $\Phi(x)$ is given by the $P_l^m(x)$. We note that the algebra (4.8) may be expressed by a single complex function, e.g. L_+ , because $L_- = (L_+)^*$ and $L_3 = \sqrt{\beta - L_+ L_+^*}$, so that the last bracket in (4.8) provides the interesting relation

$$\{L_+, L_+^*\} = 2\sqrt{\beta - L_+ \cdot L_+^*}. \quad (A.3)$$

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