

Computer Averaging of Time-variable Signals by Autocorrelation

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A general method is given for a computer accumulation of signals with unpredictable and variable drift with respect to the time-scale. By calculating the autocorrelation function of each individual scan via two forward Fourier transformations any time instability can be removed. Coaddition of the autocorrelation functions leads to the expected improvement in signal-to-noise ratio. Some examples for various signal shapes are given.

The detection of signals buried in noise is a general problem in all experimental sciences. A widely employed and very powerful technique for improving the signal-to-noise (S/N) ratio is the coaddition of independent experimental data in a computer memory (computer average transients CAT). This method can be used in situations in which a series of signals is obtained over a period of time. Theoretically, the S/N ratio increases with the square root of the number of accumulated scans. This optimum value can only be obtained when signal components in all individual scans are identical. In many cases the signal components are not completely identical in principle or for instrumental reasons. Among the different possibilities for signal variation the time-instability or drift of signals plays an important role. The time-delay variation of nerve action potentials after an excitation or of a radar echo of a moving object may serve as typical examples. Simple coaddition of such instable signals would not lead to an overall improvement of the S/N ratio in the sum of the individual scans and the question to be answered here is whether there is any other method for the retrieval of small time-variable signals hidden by noise.

If a function of t is shifted by a time delay τ then the corresponding Fourier transform is multiplied by a factor $\exp(i 2\pi \nu \tau)$ [1].

$$S(t) \xrightarrow{\text{FT}} s(\nu),$$

$$S(t \pm \tau) \xrightarrow{\text{FT}} \exp(\pm i 2\pi \nu \tau) s(\nu).$$

Calculating the power spectrum from the real and imaginary parts of the frequency domain spectrum one obtains

$$p(\nu) = [\cos(2\pi \nu \tau) s(\nu)]^2 + [\sin(2\pi \nu \tau) s(\nu)]^2 = s^2(\nu),$$

$$A(f) = \mathcal{F}\{p(\nu)\}.$$

Because the power spectrum $p(\nu)$ and autocorrelation function $A(f)$ are a Fourier pair (Wiener-Khinchine theorem) [2] a Fourier transformation leads to the autocorrelation function which is completely independent of τ [3]. Therefore, by transforming the time domain information into a power spectrum or the autocorrelation function, individual experiments can be coadded even if the signal component is affected by a variable time delay. Figure 1 demonstrates schematically the procedure for a coaddition of time-shifted rectangular pulses.

As the autocorrelation process corresponds to a convolution of a function with itself, the autocorrelation of a rectangular function is a triangular function. Figure 2 shows a selection of functions and their autocorrelations.

A direct recalculation of the basic signal from its autocorrelation function is not possible because a function $f(x)$ or its Hilbert transform or any linear combination of both results in identical autocorrelation functions. In spite of this fundamental ambiguity, in most practical situations the expected line-shape of the signal shape is known and the basic problem is first of all to find out if there is a signal at all and subsequently to measure its width e.g. Fig. 3 shows the computer simulation of such a situation.

Noisy Lorentzian peaks of constant width but variable time-delay are accumulated after trans-

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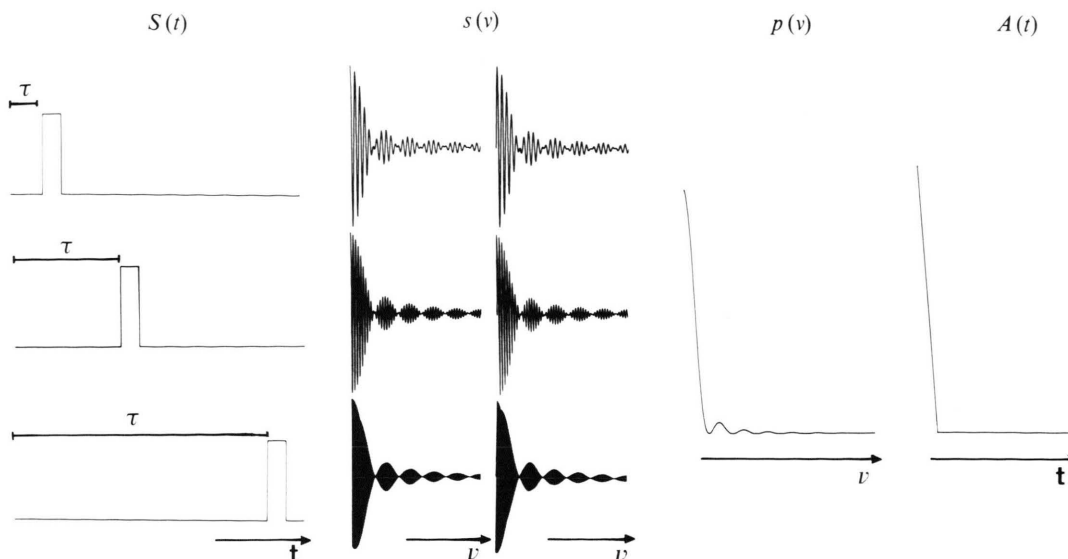


Fig. 1. Coaddition of rectangular signals. The signals with identical shape but variable time delay $S(t)$ lead after Fourier transformation to spectra $s(v)$ with different modulation frequencies given by $\exp(i 2 \pi v \tau)$. The power spectrum $p(v)$ and its Fourier transform, the autocorrelation spectrum $A(t)$ are independent of τ .

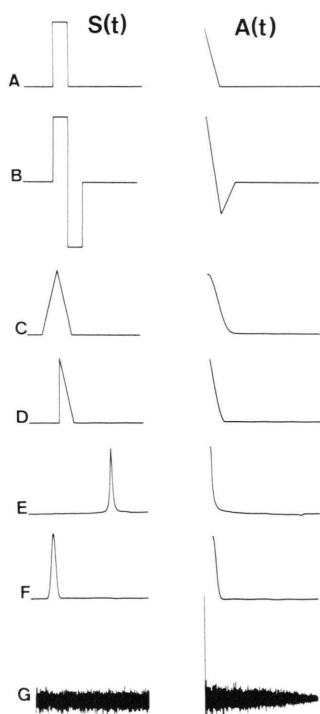


Fig. 2. A selection of different types of signals $S(t)$ and their autocorrelation functions $A(t)$.

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|----------------------|-------------------------|
| A) rectangular pulse | B) switched rectangular |
| C) triangular pulse | D) sawtooth |
| E) Lorentzian | F) Gaussian |
| G) white noise | |

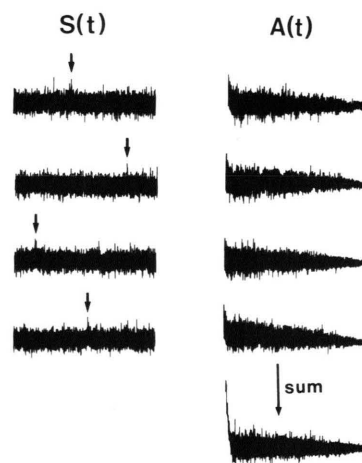


Fig. 3. Coaddition of noisy Lorentzian signals with variable time delay. A coaddition of signals $S(t)$ with Lorentzian line shape but different time delays cannot lead to a S/N ratio improvement. After a coaddition of 16 of the corresponding autocorrelation spectra $A(t)$ the presence of a signal can be detected unambiguously. In all autocorrelation spectra the first data point was omitted.

forming to power spectra in the frequency domain. Due to the mathematical procedure and the finite time window of the digitized data the noise amplitude in the corresponding autocorrelation function is not constant but decreases linearly with increasing f values. The accumulated signal information in the

sum of autocorrelation functions is centered at the origin and in the case of a Lorentzian line the width is twice that of the original signal [4].

This result demonstrates impressively that a summation of time variable signals can be coadded after calculating the autocorrelation function and leads to a remarkable improvement in the S/N ratio. An extension of the method to more complex signals is under investigation.

Experimental

All calculations were performed on a Bruker Aspect 2000 computer system.

Acknowledgements

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