

The Problem of Radiation Reaction in Classical Electrodynamics

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A new covariant theory of the classical radiating electron is compared with other radiation reaction theories: On the one hand, the new theory can be deduced from Caldirola's finite-differences theory by suitable approximations; on the other hand, the Lorentz-Dirac theory and the theory of Mo and Papas are shown to be approximative forms of the new theory. The latter is free from the difficulties of the other theories: there are no undetermined internal oscillations, no runaway solutions and pre-acceleration effects, and radiation reaction exists in the case of one-dimensional motion. The general energy-momentum balance is studied, and the implications of the existence of radiation reaction in a static constant force field are discussed with regard to the principle of equivalence.

I. Introduction and Survey of Results

Recently, a new equation of motion for the classical radiating electron has been proposed¹, which was expected to avoid most of the discrepancies inherent in the presently accepted Lorentz-Dirac theory of radiation reaction^{2,3}. Contrary to the basic equation in the Lorentz-Dirac-Rohrlich theory

$$m c^2 \dot{u}^{\lambda} = K^{\lambda} + \frac{2}{3} Z^2 [\ddot{u}^{\lambda} + (\dot{u} \dot{u}) u^{\lambda}], \quad (\text{I}, 1)$$

the new equation of motion has no runaway solutions and no pre-acceleration. But also if we compare the new theory with other theories of the radiating electron found in the literature, it is readily recognized that most of the disadvantages of those theories can be avoided by the present proposal.

From a general mathematical point of view, it can be said that the new theory is based upon a covariant differential-difference equation and requires therefore new mathematical tools in managing concrete problems. But this point is left for future work. In this paper we are first concerned with a study of the relations between Caldirola's finite-differences theory⁴ on the one hand and the theory of Mo and Papas⁵ (resp. the theory of Lorentz-Dirac-Rohrlich) on the other hand. Due to its intermediate position between the finite-differences theory, which allows for undetermined radiationless self-oscillations, and the differential equation of the Lorentz-Dirac theory with its runaway solutions, the new differential-difference theory prohibits both the self-oscillations and the runaway solutions.

Nevertheless, the new theory exhibits radiationless modes, but these must be excited by suitable external fields varying very rapidly, with periods of order 10^{-23} sec. It is supposed, therefore, that the microscopic stationary behaviour of the electron can be described with the new equation. However, radiation and the according friction force are present whenever the external force varies slowly in time compared to the elementary time interval mentioned above. Therefore, the macroscopic behaviour of the radiation-damped electron is expected to be described correctly by the new theory.

Special interest is devoted to a consistent local approximation of the exact non-local equation of motion. It is found in this respect that the Lorentz-Dirac equation (I, 1) can be obtained only by a rather dubious procedure. A more consistent approximation procedure yields the theory of Mo and Papas. The question of a consistent and physically reasonable approximation of the equation of motion is closely connected with the suitable choice of the reference point, about which the necessary expansions of the non-local quantities are to be performed. Since two reference points are possible, there exist two possibilities for the expansions: one delivers the Lorentz-Dirac theory, the other the theory of Mo and Papas.

As is well known, there is no radiation reaction in the Mo-Papas theory in the case of linear motion. This was considered as a disadvantage of this theory; but since in the present context the Mo-Papas theory arises only as an approximation to a higher level theory, which clearly exhibits radiation damping in the indicated case, one should not chalk up this point to the Mo-Papas equation.

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From this point of view, there arises also a new aspect of the lack of radiation reaction in the case of hyperbolic motion, which occurs in free fall of the radiating electron. Some authors have argued that Einstein's principle of equivalence is valid even for radiating particles, because the generally accepted Lorentz-Dirac theory produces (due to the lack of radiation damping in free fall) the same trajectory, which is also followed by a neutral particle. But in the present context the Lorentz-Dirac theory appears only as an approximation, which furthermore cannot add something new to the neutral particle trajectory in the case of free fall. So it is found that the Lorentz-Dirac theory cannot be decisive for the validity problem of Einstein's principle of equivalence. Indeed, this principle does not hold in the higher level theory of the radiating electron presented in this paper.

II. The New Equation

In a recent paper¹ we have proposed the following generalization of the Lorentz-Dirac equation (I, 1):

$$m c^2 [\hat{u}^\lambda - (\hat{u} u) u^\lambda] = K^\lambda. \quad (\text{II, 1})$$

The motivation for this proposition was the fact that runaway solutions and pre-acceleration are excluded in a theory based upon (II, 1). If the radiating electron is interacting with an external electromagnetic field $F^{\mu\nu}$, the force $\{K^\lambda\}$ was assumed to be

$$K^\lambda_{(s)} = Z F^{\mu\lambda}_{(s-\Delta s)} \cdot u_{\mu(s)} \equiv Z \hat{F}^{\mu\lambda} u_{\mu}. \quad (\text{II, 2})$$

Since we are considering now the electron in interaction with an arbitrary force field $\{K^\lambda\}$ (e.g. gravitational field), we first have to give a precise meaning to the four-force $\{K^\lambda\}$, if the force three-vector $\mathbf{K}$ is known. As a natural, non-local generalization of the usual four-force

$$\{K^\lambda\} = \left\{ \frac{\mathbf{K} \cdot \mathbf{v}/c}{\sqrt{1-\mathbf{v}^2/c^2}}; \frac{\mathbf{K}}{\sqrt{1-\mathbf{v}^2/c^2}} \right\}, \quad (\text{II, 3})$$

where $\mathbf{K}$ and $\mathbf{v}$ refer to the same event on the world line, we now choose

$$\{K^\lambda_{(s)}\} = \left\{ \frac{\mathbf{K}_{(s-\Delta s)} \cdot \mathbf{v}_{(s)}/c}{\sqrt{1-\mathbf{v}_{(s)}^2/c^2}}; \frac{\mathbf{K}_{(s-\Delta s)}}{\sqrt{1-\mathbf{v}_{(s)}^2/c^2}} \right\}. \quad (\text{II, 3}')$$

This is consistent with (II, 1) because of

$$K^\lambda_{(s)} \cdot u_{\lambda(s)} = 0.$$

In the following, we are dealing sometimes with one-dimensional problems $[\mathbf{K}_{(s)} = (0, 0, K_{(s)})]$.

Therefore we apply to (II, 1) and (II, 3) the ansatz

$$\{u^\lambda_{(s)}\} = \{\text{Cosh } w_{(s)}; 0, 0, \text{Sinh } w_{(s)}\} \quad (\text{II, 4})$$

and obtain after some elementary calculations

$$m c^2 \hat{w} \text{Cosh } (\Delta w) = \hat{K}, \quad (\text{II, 5})$$

where

$$\hat{w} := \dot{w}_{(s-\Delta s)} \equiv \left. \frac{dw_{(s)}}{ds} \right|_{s-\Delta s}; \quad \Delta w := w_{(s)} - w_{(s-\Delta s)} \\ \hat{K} := K_{(s-\Delta s)}.$$

Though a preliminary judgment on the differential-difference Eq. (II, 5) allows for the conclusion that this equation will yield a physically reasonable theory, nowhere in literature could be found a hint on a treatment of this type of equation. So, irrespective of the physics standing behind, Eq. (II, 5) seems to represent also a purely mathematical challenge.

For the physicist, the striking features of (II, 5) are the following:

1. If at a certain time the force on the particle vanishes ($\hat{K}=0$), then the acceleration vanishes, too, at the same time ($\hat{w}=0$); this means exclusion of runaway solutions and pre-acceleration.

2. The neutral particle limit ($\Delta s \rightarrow 0$) is Newton's equation of motion for a non-radiating particle, written in the variable w :

$$m c^2 \dot{w}_{(s)} = K_{(s)}. \quad (\text{II, 6})$$

3. The invariant acceleration for the radiating particle as calculated from (II, 5) is always smaller in amount than that for the neutral particle, taken from (II, 6):

$$|m c^2 \hat{w}| = \frac{|\hat{K}|}{\text{Cosh } (\Delta w)} \leq |\hat{K}| = m c^2 |\hat{w}|_{\text{neutr.}} \quad (\text{II, 7})$$

So we have always a radiation reaction in the expected direction (contrary to the Lorentz-Dirac theory and that of Mo and Papas).

4. If we want to regain the one-dimensional form of the Lorentz-Dirac equation, we write (II, 5) explicitly as

$$m c^2 \dot{w}_{(s-\Delta s)} \text{Cosh } [w_{(s)} - w_{(s-\Delta s)}] = K_{(s-\Delta s)} \quad (\text{II, 5}')$$

and expand the quantities, which refer to the "advanced" point $\{z^\lambda_{(s)}\} \equiv \{z^\lambda_{(s-\Delta s)}\}$ on the world line, about the point $\{z^\lambda_{(s)}\}$:

$$\dot{w}_{(s-\Delta s)} = \dot{w}_{(s)} - \Delta s \cdot \ddot{w}_{(s)} + \dots, \\ K_{(s-\Delta s)} = K_{(s)} - \Delta s \cdot \dot{K}_{(s)} + \dots \quad (\text{II, 8})$$

Observing¹ the constraint for Δs

$$m c^2 \Delta s = \frac{2}{3} Z^2 \quad (\text{II}, 9)$$

and retaining only terms of maximal order $(\Delta s)^0$, one finds easily from (II, 5')

$$m c^2 \dot{w}_{(s)} - \frac{2}{3} Z^2 \dot{w}_{(s)} = K_{(s)} + O_{(\Delta s^1)}. \quad (\text{II}, 10)$$

Neglection of all terms containing Δs^n with $n \geq 1$ leads immediately to the one-dimensional Lorentz-Dirac equ.

$$m c^2 \dot{w}_{(s)} - \frac{2}{3} Z^2 \dot{w}_{(s)} = K_{(s)}, \quad (\text{II}, 10a)$$

We do not think that the approximation procedure below point 4. is a correct one. As a support of this view, expand the corresponding quantities in (II, 5') about $\{\hat{z}^\lambda_{(s)}\}$, rename $\hat{s} = s - \Delta s$ by s and find

$$m c^2 \dot{w}_{(s)} = K_{(s)} + O_{(\Delta s^1)}. \quad (\text{II}, 10')$$

So we see that the two approximative forms (II, 10) and (II, 10') are different and obviously depend upon whether the expansion is performed about $\{z^\lambda_{(s)}\}$ or $\{\hat{z}^\lambda_{(s)}\}$. It is interesting to observe that an expansion about the advanced point $\{\hat{z}^\lambda_{(s)}\}$ does not lead to the unphysical effects, not even in higher orders. Accounting for the $(\Delta s)^1$ -terms in (II, 10') yields

$$m c^2 \dot{w}_{(s)} [1 + \frac{1}{2} (\Delta s)^2 \dot{w}_{(s)}^2] \cong K_{(s)}. \quad (\text{II}, 10'')$$

If we take this equation as the approximated equation of motion, then we do not have to bother ourselves with the unphysical effects. Consequently, it is better to write the equation of motion (II, 5') as

$$m c^2 \dot{w}_{(s)} \cdot \text{Cosh}[w_{(s+\Delta s)} - w_{(s)}] = K_{(s)} \quad (\text{II}, 5'')$$

in order to indicate that expansions have to be performed about the suitable reference point [now $\{z^\lambda_{(s)}\}$ in (II, 5'')]. We shall come back to this point later, when the present theory is compared with those of Caldirola and of Mo and Papas.

These general remarks on the equation of motion (II, 5) will do for our purpose here, and we proceed now to study the motion of the radiating electron in a static, homogeneous force field K , parallel to the direction of the force. As a realization of this situation one can think of free fall.

III. Simple Application: Hyperbolic Motion

Assume K in (II, 5) to be a constant throughout the motion and try the ansatz

$$w_{(s)} = g \cdot s; \quad g = \text{const.} \quad (\text{III}, 1)$$

This yields

$$m c^2 \cdot g \cdot \text{Cosh}(g \Delta s) = K (= \text{const}). \quad (\text{III}, 2)$$

If the external force K is produced by an electrical field E , we can put $K = ZE$. The constant invariant acceleration g [$= \text{sign}(K) \cdot \sqrt{V - (\dot{u} \dot{u})}$] has to be determined in terms of the force K from Equation (III, 2). Figure 1 shows a plot of the function $g_{(K)}$ in reduced units.

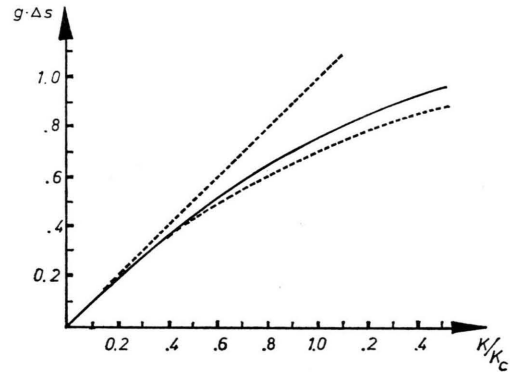


Fig. 1. The invariant acceleration g (solid curve), measured in units of $(\Delta s)^{-1}$, is plotted versus the reduced force K/K_c , where $K_c = m c^2 / \Delta s = \frac{2}{3} Z^2 / \Delta s^2$. The dotted curve results from Caldirola's finite-differences theory. The tangent to the curved lines in the origin is the corresponding result for the Lorentz-Dirac theory.

From this figure, one concludes that the acceleration of the extended particle (curved line), as described by (II, 5), is always less than the acceleration for a non-radiating particle, respectively less than the acceleration for the Lorentz-Dirac particle, which is represented in both cases by the straight line. This is what one would expect on account of radiation reaction. But the deviation of the curved line from its tangent in the origin is not relevant until the external force K assumes a horrible strength. For instance, if the relative difference between the neutral particle acceleration and the radiating particle acceleration in the same force field K is prescribed to be one percent, the force K must satisfy the requirement $K \approx 0.1 K_c$, and this is the interaction force of two Coulomb singularities at a distance of roughly four times the classical radius r_c . It is clear, that such field strengths are not accessible to experiment.

A further striking feature of Fig. 1 is the absence of any sort of breakdown in the region $K \approx K_c$, which stands in contrast to the earlier results of the finite-size model of the radiating electron⁶. Never-

theless, the present result does not contradict the former one, because in the former model the breakdown phenomena were solely due to the purely external force (E^2 -effect). However, in the present model we have not gone so far in our assumptions on the four-force $\{K^\lambda\}$ in (II, 1) to include such a purely external term, but we have only chosen a slight modification of the usual Lorentz force. The breakdown in the integro-differential formulation of the Lorentz-Dirac theory⁷ is not reproduced by the present generalization of the Lorentz-Dirac equation.

Now we assume that the constant force K is zero in the past, i. e.

$$K(s) = \begin{cases} 0; & -\infty < s < 0, \\ K (= \text{const} \neq 0); & 0 \leq s < \infty. \end{cases} \quad (\text{III, 3})$$

Clearly, a solution of (II, 5) for this case is

$$\dot{w}(s) = \begin{cases} 0; & -\infty < s < 0, \\ g_{(K)}; & 0 \leq s < \infty \end{cases} \quad (\text{III, 4})$$

with the same constant acceleration $g_{(K)}$ as in (III, 2). Once more one realizes that no preacceleration arises in the present model (see Fig. 2), contrary to the Lorentz-Dirac theory.

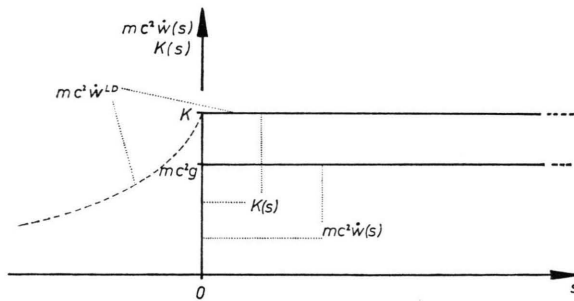


Fig. 2. Acceleration $m c^2 \dot{w}(s)$ and external force $K(s)$ versus proper time s . The Lorentz-Dirac acceleration $m c^2 \dot{w}^{LD}$ is non-zero for $s < 0$ (pre-acceleration). In the present theory, there is no pre-acceleration ($m c^2 \dot{w}(s) = 0$ for $s < 0$) and the resulting acceleration is smaller than in the neutral particle case ($m c^2 g < K$) by virtue of radiation reaction.

IV. Comparison with other Theories of the Radiating Electron

a) Caldirola's Finite-differences Theory

The introduction of a fundamental time interval into the electron's equation of motion in demand is, of course, not a new idea. All the older attempts in this respect can be found in the summary representation of Erber⁸. But in recent time, too, there are

people who hope to overcome the weak points of the Lorentz-Dirac theory by resorting to the introduction of an elementary time interval (e. g. Ingraham⁹ or Cohn¹⁰; note however Kapusta¹¹).

The most interesting theory of this sort seems to be the finite-differences theory of Caldirola⁴, which was developed twenty years ago and revived recently^{12, 13}. A comparison between Caldirola's theory and the present one is especially instructive.

As the relativistic generalization of^{14, 15}

$$\frac{m}{\tau_0} [\mathbf{v}(t) - \mathbf{v}(t - \tau_0)] = \mathbf{K}(r, v, t) \quad (\text{IV, 1})$$

Caldirola postulated the following equation of motion for the radiating electron

$$\frac{m c^2}{s_0} \{ -u^\lambda_{(s-s_0)} + [u_{(s)} \cdot u_{(s-s_0)}] u^\lambda_{(s)} \} = K^\lambda_{(s)}. \quad (\text{IV, 2})$$

Clearly, the left-hand side of (IV, 2) yields the Lorentz-Dirac form (I, 1) after expansions of the type $u^\lambda_{(s-s_0)} = u^\lambda_{(s)} - s_0 \cdot \dot{u}^\lambda_{(s)} + \dots$. The connection between mass m and length interval s_0 is here

$$\frac{1}{2} m c^2 s_0 = \frac{2}{3} Z^2 \quad (\text{IV, 3})$$

instead of (II, 9). Thus, one finds $s_0 = 2 \Delta s$. Equation (IV, 2) can equally well be written as

$$m c^2 \left\{ \frac{u^\lambda_{(s)} - u^\lambda_{(s-s_0)}}{s_0} - \left[u^\nu_{(s)} \frac{u_\nu(s) - u_\nu(s-s_0)}{s_0} \right] u^\lambda_{(s)} \right\} = K^\lambda_{(s)}. \quad (\text{IV, 2'})$$

Expanding about the "advanced" point $\{\tilde{z}^\lambda_{(s)}\} \equiv \{z^\lambda_{(s-s_0)}\}$ (see the corresponding statements at the end of Sect. II.) yields at once the new Equation (II, 1). So we see that the difference of both theories consists in the substitution of the derivatives $\{\hat{u}^\lambda\}$ in (II, 1) by the finite-differences expression $[u^\lambda_{(s)} - u^\lambda_{(s-s_0)}]/s_0$ in (IV, 2').

A further difference exists in the force $\{K^\lambda\}$; whereas Caldirola puts $K^\lambda \rightarrow Z F^{\mu\lambda} u_\mu$, we have preferred $K^\lambda \rightarrow Z \hat{F}^{\mu\lambda} u_\mu$, for this choice ensures the non-existence of pre-acceleration ($\hat{F}^{\mu\lambda} = 0 \Rightarrow \hat{u}^\lambda = 0$).

Caldirola's motivation for the proposition of his new equation was the same as ours; the new equation was required to exclude runaway solutions and pre-acceleration. Indeed, both theories have achieved this goal, but in Caldirola's theory one is forced to deal with a new degree of freedom (internal motions, spin), for which dynamical laws are missing.

Our present theory stands between the Lorentz-Dirac theory (differential equation) and Caldirola's theory (finite-differences equation) and it is therefore a differential-difference equation excluding runaway solutions on the one hand and also internal motions on the other hand.

Let us proceed to compare simple solutions of the basic equations of motion in both theories. Confining ourselves to one-dimensional motion, Caldirola's equation (IV, 2) becomes with the substitution (II, 4)

$$(m c^2/s_0) \sinh[w_{(s)} - w_{(s-s_0)}] = K_{(s)}, \quad (\text{IV}, 3)$$

where Eq. (II, 3) in the form

$$\{K_{(s)}^{\lambda}\} = K_{(s)} \{\sinh w_{(s)}; 0, 0, \cosh w_{(s)}\}$$

has been used. From (IV, 3) one can easily find the general solution of the Caldirola theory by

$$w_{(s)} = w_{(s-s_0)} + \text{ArSinh}[s_0 K_{(s)}/m c^2]. \quad (\text{IV}, 4)$$

If the initial value for w is known ($w_i = w_{(s_i)}$) together with the force at times $s_n = s_i + n s_0$, the quantity $w_n (\equiv w_{(s_i + n s_0)})$ is not uniquely determined by (IV, 4)

$$w_n = w_{n-1} + \text{ArSinh}[s_0 K_{(s_i + n s_0)}/m c^2], \quad (\text{IV}, 5)$$

for one can always add a function $w_{(s)}$, which satisfies $w_{(s_n)} = w_{(s_{n-1})} = 0$. This is the additional degree of freedom consisting of the internal motion.

But now consider an everlasting constant force $K_{(s)} = K = \text{const.}$ Putting $w_{(s)} = \tilde{g} \cdot s$, we obtain from (IV, 3) for the invariant acceleration $\tilde{g}$ quite similar as in the preceding section

$$(m c^2/s_0) \sinh(\tilde{g} s_0) = K, \quad (\text{IV}, 6)$$

which is the old result of Lanz¹⁶ (dotted curve in Fig. 1; observe $s_0 = 2 \Delta s$). From Fig. 1 we recognize again, that the result of the present differential-difference theory (solid curve) is situated between the result of the Lorentz-Dirac differential theory (straight line) and that of Caldirola's finite-differences theory (dotted curve). Thus, the same difficulties of discerning experimentally between the various theories persist with respect to the present one and Caldirola's theory (see Section III).

b) The Theory of Mo and Papas

Starting with the assumption that the radiation reaction term $\frac{2}{3} Z^2 (\dot{u} \dot{u}) u^\lambda$ in the Lorentz-Dirac equation (I, 1) should be expressible by the external

field $F^{\mu\nu}$, Mo and Papas⁵ have postulated the following equation of motion for the radiating electron

$$m c^2 \dot{u}^\lambda + \frac{2}{3} \frac{Z^3}{m c^2} (F^{\mu\nu} \dot{u}_\mu u_\nu) u^\lambda = Z F^{\mu\lambda} u_\mu \quad (\text{IV}, 7)$$

$$+ \frac{2}{3} \frac{Z^3}{m c^2} F^{\mu\lambda} \dot{u}_\mu.$$

Indeed, if one approximates the real trajectory of the radiating electron by its neutral particle limit

$$m c^2 \dot{u}^\lambda = Z F^{\mu\lambda} u_\mu \quad (\text{IV}, 8)$$

and inserts from here partly into the radiation reaction term $\frac{2}{3} Z^2 (\dot{u} \dot{u}) u^\lambda$ of the Lorentz-Dirac theory, one easily recovers the second term on the left of Equation (IV, 7). The second term on the right of this equation appears as an ad hoc expression. Clearly, a theory based on (IV, 7) does not exhibit the unphysical effects of runaway solutions and pre-acceleration and therefore seems to provide a quite reasonable description of the radiating electron. Even if Shen¹⁷ points out that the difference of the results of both theories is masked by quantum effects and is therefore experimentally not verifiable, one should nevertheless be highly interested in a consistent classical theory of the radiating electron; such demand is surely not met by the causality-violating Lorentz-Dirac theory.

In order to give a plausible explanation for the Eq. (IV, 7), the neutral particle limit (IV, 8) has been applied and it might seem therefore that (IV, 7) is only an approximation for the Lorentz-Dirac theory. However, in rederiving (IV, 7) from our new Eq. (II, 1), written as

$$m c^2 [\hat{u}^\lambda - (\hat{u} u) u^\lambda] = Z \hat{F}^{\mu\lambda} u_\mu, \quad (\text{IV}, 9)$$

one shall realize at once that (IV, 7) stands on the same level of approximation as does the Lorentz-Dirac equation. The latter one has been derived from (IV, 9) in the earlier paper¹. To pursue now the arising of (IV, 7) from (IV, 9), we observe first that expansions in (IV, 9) should be performed about the "advanced" point $\{\hat{z}^\lambda_{(s)}\}$, as has been stated repeatedly in the foregoing considerations. Therefore,

$$u^\lambda_{(s)} = \hat{u}^\lambda_{(s)} + \Delta s \hat{u}^\lambda_{(s)} + \dots$$

and hence

$$m c^2 \hat{u}^\lambda - m c^2 \Delta s (\hat{u} \hat{u}) \hat{u}^\lambda \cong Z \hat{F}^{\mu\lambda} \hat{u}_\mu + Z \Delta s \hat{F}^{\mu\lambda} \hat{u}_\mu. \quad (\text{IV}, 10)$$

Suppressing herein the bar on the dynamical quantities and applying (II, 9), one readily finds

$$m c^2 \dot{u}^\lambda - \frac{2}{3} Z^2 (\dot{u} \dot{u}) u^\lambda = Z F^{\mu\lambda} u_\mu + \frac{2}{3} (Z^3/m c^2) F^{\mu\lambda} \dot{u}_\mu. \quad (\text{IV}, 11)$$

Now contract this equation with $\{u_\lambda\}$ to obtain

$$-\frac{2}{3} Z^2 (\dot{u} \dot{u}) = \frac{2}{3} (Z^3/m c^2) F^{\mu\lambda} \dot{u}_\mu u_\lambda. \quad (\text{IV}, 12)$$

Thus, (IV, 11) and (IV, 12) lead immediately to the Mo-Papas equation (IV, 7).

It must be stressed that the neutral particle limit (IV, 8) has not been used in the foregoing derivation of (IV, 7); nothing else has been done than in the derivation of the Lorentz-Dirac equation, too. But the present result (IV, 7) is much more satisfactory.

Of course, one can express also in the new theory (IV, 9) the radiation reaction term $\mathcal{R} = -m c (u \ddot{u})$ (see below) in terms of the external force. To this end, contract Eq. (IV, 9) with $\{\hat{u}_\lambda\}$ to obtain

$$-m c^2 (u \hat{u}) (u \ddot{u}) = Z \hat{F}^{\mu\lambda} u_\mu \hat{u}_\lambda \quad (\text{IV}, 9')$$

and therefore the equation of motion can also be written as

$$m c^2 \hat{u}^\lambda + \frac{Z (\hat{F}^{\mu\nu} u_\mu \hat{u}_\nu)}{(u \hat{u})} u^\lambda = Z \hat{F}^{\mu\lambda} u_\mu, \quad (\text{IV}, 9'')$$

which is obviously a non-local generalization of the Mo-Papas theory.

Now, the equation of Mo and Papas (IV, 7) has been derived from (IV, 9), but (IV, 9) itself was derived from Caldirola's equation (IV, 2) (apart from the Δs -shift in the external force). Is therefore (IV, 7) derivable from Caldirola's equation directly?

The answer is negative: To prove this, abbreviate in Eq. (IV, 2')

$$\{\tilde{u}^\lambda_{(s)}\} := \{u^\lambda_{(s-s_0)}\} \quad (\text{IV}, 13)$$

and find

$$m c^2 \left\{ \frac{u^\lambda - \tilde{u}^\lambda}{s_0} - \left[u^\nu \cdot \frac{u_\nu - \tilde{u}_\nu}{s_0} \right] u^\lambda \right\} = Z F^{\mu\lambda} u_\mu \quad (\text{IV}, 14)$$

$$\text{or} \quad \frac{m c^2}{s_0} \{-\tilde{u}^\lambda + (u \tilde{u}) u^\lambda\} = Z F^{\mu\lambda} u_\mu. \quad (\text{IV}, 14')$$

Contracting (IV, 14') with $\{\tilde{u}_\lambda\}$ yields

$$\frac{m c^2}{s_0} \{-1 + (u \tilde{u})^2\} = Z F^{\mu\lambda} u_\mu \tilde{u}_\lambda. \quad (\text{IV}, 15)$$

For weakly curved world lines we can put

$$\begin{aligned} -1 + (u \tilde{u})^2 &= \{(u \tilde{u}) + 1\} \\ &\quad \cdot \{(u \tilde{u}) - 1\} \approx 2 \{(u \tilde{u}) - 1\}. \end{aligned}$$

Therefore

$$\frac{m c^2}{s_0} \{(u \tilde{u}) - 1\} = \frac{1}{2} Z F^{\mu\lambda} u_\mu \tilde{u}_\lambda$$

and (IV, 14) becomes with this result

$$m c^2 \frac{u^\lambda - \tilde{u}^\lambda}{s_0} + \frac{1}{2} Z (F^{\mu\nu} u_\mu \tilde{u}_\nu) u^\lambda = Z F^{\mu\lambda} u_\mu. \quad (\text{IV}, 16)$$

It would be meaningless in this equation to expand about $\{z^\lambda_{(s)}\}$; rather we expand again about $\{\tilde{z}^\lambda_{(s)}\} := \{z^\lambda_{(s-s_0)}\}$ and substitute $F^{\mu\lambda} \rightarrow \tilde{F}^{\mu\lambda}$ in order to avoid additional derivative terms in $F^{\mu\lambda}$. Then one finds

$$m c^2 \tilde{u}^\lambda + \frac{1}{2} Z s_0 (\tilde{F}^{\mu\nu} \tilde{u}_\mu \tilde{u}_\nu) \tilde{u}^\lambda = Z \tilde{F}^{\mu\lambda} \tilde{u}_\mu + s_0 Z F^{\mu\lambda} \tilde{u}_\mu. \quad (\text{IV}, 17)$$

But this equation is not identical with (IV, 7), because the s_0 -dependent coefficients in front of the additional terms (with respect to the neutral particle limit) are not the same.

Finally, we have to spend a few words on the hyperbolic motion within the framework of the Mo-Papas theory. As Hushilt and Baylis¹⁸ have observed, the Mo-Papas equation (IV, 7) reduces in the one-dimensional case (II, 4), where one easily verifies

$$(F^{\mu\nu} \dot{u}_\mu u_\nu) u^\lambda = F^{\mu\lambda} \dot{u}_\mu = (F^{03} \dot{u}) u^\lambda,$$

to the one-dimensional form (II, 6) of the neutral particle limit (IV, 8). If therefore the electron moves in a purely electric field $\{F^{0k}; k=1, 2, 3\}$, there is no radiation reaction present in the Mo-Papas theory, and the electron moves exactly like the neutral (i. e. non-radiating) particle. We therefore obtain in the hyperbolic motion case the same invariant acceleration g as in the Lorentz-Dirac theory. Hushilt and Baylis have considered this as a serious objection to the Mo-Papas theory and the author has subscribed to this point of view in an earlier paper¹⁹. But in the present context, the Mo-Papas equation appears as an approximation of the higher level theory (II, 1), which clearly exhibits radiation reaction in the special cases of motion indicated above. In regard of its approximative character, the just mentioned lack of the Mo-Papas theory seems to be excusable and should therefore no longer be regarded as an objection.

V. General Energy-Momentum Balance

Intuitively, one would associate three sorts of energy-momentum with the radiating electron accelerated by an external force field: kinetic energy-momentum of the bound self-fields constituting the "electron", energy-momentum of the electromagnetic radiation escaping from the accelerated charge, and finally the energy-momentum transfer of the external forces. However, the Lorentz-Dirac equation (I, 1) contains four terms and this has led to confusion about the fourth term (Schott term: second derivative of four-velocity u^λ). As a consequence, people believed for a certain time, that the electron in hyperbolic motion does not radiate, but this error has been elucidated definitely by Rohrlich^{20, 21}. The meaning of the Schott term as part of the energy-momentum content of the bound velocity fields surrounding the source of the Liénard-Wiechert potentials was not fully understood until recently^{19, 22-24} (see also Reference²⁵).

Regarding the new equation (II, 1), it is a matter of ease to identify the three terms of this equation with the three sorts of energy-momentum mentioned above.

a) Work of the External Forces

Clearly, we have to identify the work $dA/d\tau$ of the external force per unit proper time as

$$\frac{dA}{ds} = \frac{1}{c} \frac{dA}{d\tau} = K^0 = \frac{\hat{\mathbf{K}}_{(s)} \cdot \mathbf{v}_{(s)}/c}{\sqrt{1 - (\mathbf{v}_{(s)}/c)^2}} \quad (\text{V, 1})$$

$$= \frac{\mathbf{K}_{(s-\Delta s)} \cdot \mathbf{v}_{(s)}/c}{\sqrt{1 - (\mathbf{v}_{(s)}/c)^2}}.$$

β) Kinetic Energy-momentum

As a generalization of the usual expression

$$P_b^{\mu(s)} = m c u^{\mu(s)} \quad (\text{V, 2a})$$

let us choose here

$$P_b^{\mu(s)} = m c u^{\mu(s-\Delta s)} \equiv m c \hat{u}^\mu. \quad (\text{V, 2b})$$

This choice is in line with the foregoing statements on the Schott term, because we can recover this term, if we expand the bound four-momentum (V, 2b) with respect to Δs [observe (II, 9)]

$$P_b^{\mu(s)} \cong m c u^{\mu(s)} - \frac{1}{c} \frac{2}{3} Z^2 \dot{u}^\mu(s). \quad (\text{V, 2c})$$

According to (V, 2b) we define the kinetic energy T of the particle in arbitrary motion as

$$T_{(s)} = m c^2 [u^0_{(s-\Delta s)} - 1] \equiv m c^2 [\hat{u}^0 - 1]. \quad (\text{V, 2d})$$

For motions with constant four-velocity this yields back the standard result from relativistic point mechanics ($T_{(s)} \rightarrow m c^2 [u^0_{(s)} - 1]$). It seems very plausible that one uses now the advanced quantity $u^0_{(s-\Delta s)}$ instead of $u^0_{(s)}$, because, if one thinks the self-fields of the electron created on the world line of a representative point within a finite-size structure of extension Δs , these fields contribute to the energy-momentum of the extended structure when they have reached its surface. But then they carry the energy-momentum of the state of motion at time Δs before, when they originated on the representative world line (cf. Reference¹⁹).

γ) Radiated Four-momentum

Of course, we put

$$dP_{\text{rad}}^\mu/ds = -m c (u \hat{u}) u^\mu, \quad (\text{V, 3a})$$

which reduces in lowest order to the standard result³

$$dP_{\text{rad}}^\mu/ds \cong m c \Delta s (u \ddot{u}) u^\mu = -\frac{1}{c} \frac{2}{3} Z^2 (\dot{u} \dot{u}) u^\mu. \quad (\text{V, 3b})$$

The generalized invariant radiation rate

$$\mathcal{R} = \frac{1}{c^2} \frac{dW_{\text{rad}}}{dt}$$

becomes now

$$\mathcal{R} = u_\mu (dP_{\text{rad}}^\mu/ds) = -m c (u \hat{u}) \quad (\text{V, 4a})$$

and is independent of the chosen inertial frame, as is the case for the old result³

$$\mathcal{R} \cong \Delta s m c (u \ddot{u}) = -\frac{1}{c} \frac{2}{3} Z^2 (\dot{u} \dot{u}), \quad (\text{V, 4b})$$

which again arises here as the lowest order approximation.

An especially concrete interpretation of the invariant rate $\mathcal{R}$ is obtainable if it is expressed in terms of the external force. Equation (V, 4a) yields together with formula (IV, 9')

$$\mathcal{R} = \frac{1}{c} Z \frac{\hat{F}^{\mu\lambda} u_\mu \hat{u}_\lambda}{(u \hat{u})}. \quad (\text{V, 4c})$$

In the instantaneous rest system characterized by $\{u^\lambda\} = \{1; 0, 0, 0\}$ this expression becomes

$$\begin{aligned} (\{F^{0k}; k=1, 2, 3\} = \mathbf{E}) \\ c^2 \mathcal{R} \Rightarrow -Z(\hat{\mathbf{E}} \cdot \hat{\mathbf{v}}), \end{aligned} \quad (\text{V, 4d})$$

respectively, if (II, 3') is used,

$$c^2 \mathcal{R} \Rightarrow -(\hat{\mathbf{K}} \cdot \hat{\mathbf{v}}). \quad (\text{V, 4e})$$

So we see that in this special coordinate frame the radiated energy per unit time is exactly given by the ordinary work (per unit time) an interval $\Delta\tau = \Delta s/c$ before the particle comes to rest. Now we consider the invariant rate (V, 4a) in the specially chosen coordinate system. With $\{u^i_{(s_r)}\} = \{1; 0, 0, 0\}$ one easily verifies

$$\begin{aligned} c^2 \mathcal{R} &= -m c^3 (u \hat{u}) \Rightarrow -c \frac{d}{ds} (m c^2 \hat{u}^0) \quad (\text{V, 4f}) \\ &= -\frac{d}{d\tau} (m c^2 \hat{\gamma}), \end{aligned}$$

where $\hat{\gamma} = (1 - \hat{\mathbf{v}}^2/c^2)^{-1/2}$, $\hat{\mathbf{v}} = \mathbf{v}_{(s_r - \Delta s)}$ and s_r is the proper time, where the particle comes to rest. Hence, with (V, 4d) and (V, 4f)

$$\frac{d}{d\tau} (m c^2 \hat{\gamma}) = \frac{d}{d\tau} T_{(s_r)} = Z(\hat{\mathbf{E}} \cdot \hat{\mathbf{v}}). \quad (\text{V, 4g})$$

In this special coordinate system* the particle radiates electromagnetic energy ($c^2 \mathcal{R}$) on expense of kinetic energy ($dT/d\tau$). Clearly, the external work $dA/d\tau$ is zero in this case on account of (V, 1). In an arbitrary frame the energy balance reads (observe $u^\lambda = dz^\lambda/ds = 1/c [dz^\lambda/d\tau]$)

$$\frac{dT}{d\tau} + c^2 \mathcal{R} \frac{dz^0}{d\tau} = \frac{dA}{d\tau},$$

or by use of the laboratory time $z^0 (= ct)$

$$dT/dt + c^2 \mathcal{R} = dA/dt.$$

The work per unit time of the external force (dA/dt) is transferred to the change of the kinetic energy (dT/dt) and to the emitted radiation energy ($c^2 \mathcal{R}$).

There is however an important difference between the exact formulae (V, 3a; V, 4a) and their approximations (V, 3b; V, 4b). This difference consists in the scalar $(u \hat{u})$ not being negative definite as is the case of its lowest order approximation $(u \dot{u})$ in the approximative formulae (V, 3b; V, 4b). Writing the equation of motion (II, 1) in the form

$$dP_b^\mu/d\tau + dP_{\text{rad}}^\mu/d\tau = K^\lambda, \quad (\text{V, 5})$$

* Observe the astonishing resemblance of (V, 4g) with the energy equation of the neutral particle limit $m c^2 \ddot{u}^\lambda = K^\lambda$, which can be written in an arbitrary frame as $d/dt (m c^2 \gamma) = (\mathbf{K} \cdot \mathbf{v})$ ($dt = d\tau$ in (V, 4g), because the particle is at rest in s_r).

we see that the radiation recoil $dP_{\text{rad}}^\mu/d\tau$ being always a retarding (friction) force is not guaranteed. Is this to be considered a disadvantage of the present theory? We do not think so; rather, we take this as an indication that the present theory might give a hint on the microscopic behaviour of the electron in microscopic fields. Indeed, it is easy to convince oneself, that the radiation recoil is almost always a retarding friction force, if the electron moves in macroscopic fields, i. e. fields which do not vary appreciably over time intervals of order $\Delta\tau = \Delta s/c$ ($\cong 10^{-23}$ sec). To see this, regard the scalar $(u \hat{u})$ in (V, 3a) or (V, 4a) in the rest frame of the particle, where $\{u^i_{(s_r)}\}$ has the form $\{1; 0, 0, 0\}$. In this case, one has

$$(u \hat{u}) \Rightarrow \hat{u}^0 = \frac{1}{2} \frac{\hat{\gamma}^4}{c^3} \frac{d\hat{\mathbf{v}}^2}{dt}, \quad (\text{V, 6})$$

where all terms with a bar refer to the proper time $s = s_r - \Delta s$ [see the notation below (V, 4f)]. So we recognize that the radiation recoil is a retarding friction force only if $d\hat{\mathbf{v}}^2/dt < 0$; i. e. if the absolute value of the ordinary velocity $\mathbf{v}$ is decreasing a time interval $\Delta\tau = \Delta s/c$ before the particle comes to rest. If $d\hat{\mathbf{v}}^2/dt > 0$ [$\Rightarrow (u \hat{u}) > 0$] were valid, then the particle would accelerate a time interval $\Delta\tau$ before it would come to rest, and in this case the invariant radiation rate $\mathcal{R}$ would indeed be negative. But since $d\mathbf{v}^2/dt > 0$ at time $s_r - \Delta s$ and $\mathbf{v}^2 = 0$ at s_r , there must be a maximum (or several ones) of the absolute value of $\mathbf{v}$ in the interval $s_r - \Delta s < s < s_r$. Of course, this might occur even in macroscopic motions. But for such motions one can expect that the maxima (or minima) of $\mathbf{v}^2$ are separated by time intervals much larger than $\Delta\tau = \Delta s/c$; and since $\mathcal{R} > 0$ is possible only in an interval Δs around an extremum of $\mathbf{v}^2$, there would be $\mathcal{R} > 0$ for a much longer time than $\mathcal{R} < 0$ is possible. So the total emitted energy $c^2 \int \mathcal{R} dt$ is always positive for macroscopic motions.

These statements can be made clearer in the case of one-dimensional motion. Insert (II, 4) into the expression (V, 4a) for the radiation rate $\mathcal{R}$ and find with the abbreviations below (II, 5)

$$\mathcal{R} = m c \hat{w} \text{Sinh}(\Delta w). \quad (\text{V, 7})$$

From here one concludes readily that $\mathcal{R} < 0$ if $\hat{w}$ has opposite sign with respect to Δw . The latter is only possible in a Δs -neighbourhood of a (relative) extremum of the function $w_{(s)}$, which is uniquely as-

sociated to the corresponding extremum of the ordinary velocity $\mathbf{v}$. Since such an extremum is characterized by the change of sign of $\dot{w}_{(s)}$ we recognize from the one-dimensional equation of motion (II, 5), that $R < 0$ occurs in a Δs -neighbourhood of the change of sign of the external force $K_{(s)}$. Since these changes of sign of $K_{(s)}$ are assumed to be separated by times much larger than Δs (for macroscopic motions only), the total emitted energy is positive in general.

However, if the external forces are changing very rapidly, it might occur that the scalar $(u \hat{u})$ becomes positive for a relatively long time and therefore the electron can gain considerable energy-momentum from its own radiation field **. It might even be, that there is no radiation at all, if the electron oscillates at certain frequencies. To elaborate this point a little bit further, reconsider the one-dimensional equation of motions (II, 5). Assume the external force $\hat{K}_{(s)}$ to be such that the resulting motion has

$$\Delta w \equiv w_{(s)} - w_{(s-\Delta s)} = 0 \quad (\text{V}, 8)$$

for all s . Then (II, 5) reduces to the neutral particle limit (II, 6), and because (V, 8) implies

$$\dot{w}_{(s)} = \dot{w}_{(s-\Delta s)} \equiv \hat{w}, \quad (\text{V}, 9)$$

we see then from the neutral particle limit equation (II, 6) that $K_{(s)}$ must be a periodic function in s with period $\Delta\tau = \Delta s/c \approx 10^{-23}$ sec. Since in this case the radiating particle moves exactly like the non-radiating one, the invariant radiation rate must vanish, which is fulfilled on account of (V, 7, 8).

We see that also in the present theory there are radiationless modes quite similar as in the non-relativistic model of Bohm and Weinstein¹⁴ or in the relativistic version of Caldirola⁴. But the main difference is that the radiationless oscillations of the present theory must be excited by a suitable external force (the force-free electron can only move with constant four-velocity¹), whereas in the other theories just mentioned the radiationless self-oscillations exist without presence of an external force.

VI. Some Remarks on the Principle of Equivalence

If we think of the constant force K in (III, 2) to be of gravitational nature, there arises a certain

problem *** connected with the principle of equivalence in gravitation theory. Following Rohrlich^{3, 21}, the principle of equivalence states that "the equations of motion of a non-rotating test body in free fall in a gravitational field be independent of the energy content of that body". Now, it is well known that the vector

$$I^\lambda = \frac{2}{3} Z^2 \{ \ddot{u}^\lambda + (\dot{u} \dot{u}) u^\lambda \}, \quad (\text{VI}, 1)$$

which occurs in the Lorentz-Dirac equation as

$$m c^2 \dot{u}^\lambda = K^\lambda + I^\lambda \quad (\text{VI}, 2)$$

vanishes in the case of hyperbolic motion ($\{I^\lambda\} = 0$) and thus the Lorentz-Dirac equation reduces in the case of free fall, which is assumed to be described by hyperbolic motion, to the neutral particle limit

$$m c^2 \dot{u}^\lambda = K^\lambda. \quad (\text{VI}, 3)$$

So it seems that the principle of equivalence is fulfilled even for a radiating electron: the neutral particle and the electron fall equally fast in a static homogeneous gravitational field despite the fact that the charged particle emits electromagnetic energy and momentum.

One argues³ that if this would not be so, there would arise a paradox: A freely falling observer is connected with a comoving inertial coordinate system, relative to which Maxwell's electrodynamics is valid in its special-relativistic form. The basic equations of the latter theory have as a special solution a static Coulomb field, called electron, which is consequently at rest relative to the freely falling observer. If the electron would not drop to the earth as fast as the neutral particle, the comoving inertial observer would find the neutral particle accelerated relative to the charged one without presence of a force. Thus he could distinguish between his own free fall and a gravitation free situation. But this would violate the principle of equivalence as stated formerly by Einstein and put now in the definitive form above by Rohrlich.

Since the Lorentz-Dirac equation (VI, 2) seems to be able to avoid such a contradiction, one concludes that the principle of equivalence is indeed fulfilled for charged particles; and one is left only with the problem of radiation (which observer does the freely falling charged particle see radiating?).

** In quantum-mechanical language one would phrase this as "emission and reabsorption of photons". Observe that reabsorption refers here to the radiation field and not to the bound field, which is connected with the Schott term.

*** This problem was recently²⁶ even called the "equivalence principle paradox".

There has been spent a lot of work^{20, 21, 27-29}, about this question, but none seems to us to be conclusive. We do not argue here against the above mentioned results but refer to a footnote of Rohrlich²¹ himself concerning the conception of a test particle: "By definition of 'test particle' one must ignore here the effect of the particle's own field on its motion (electromagnetic as well as gravitational). But the electromagnetic self-energy is included as part of its mass which is not supposed to enter its equation of motion."

Well, if the effect of the particle's own field on its motion has to be neglected, why has then the Lorentz-Dirac equation (accounting for the particle's self-interaction with its own radiation field) been studied in connection with the principle of equivalence?

Our point of view is that the principle of equivalence is only valid if no other than gravitational interactions are involved. If this is the case, one can geometrize the gravitational interactions and arrives at General Relativity, which incorporates the principle of equivalence in form of the geodesic postulate. But as soon as other than gravitational interactions are involved (e. g. electromagnetic self-interactions) the principle of equivalence is overcharged to make statements about the equation of motion for the test particles. It seems to be quite incidental that the Lorentz-Dirac theory (or the Mo-Papas theory as well), which is incorporating the self-interaction of the radiating charge with its own field, produces no deviation of the charged particle's trajectory in static homogeneous gravitational fields with respect to the trajectory of a neutral particle. One might suppose that this missing of a deviation effect is due to the

fact that the non-local character of the Lorentz-Dirac theory, which is best seen in its integro-differential formulation³, drops out completely in the case of hyperbolic motion, and consequently the Lorentz-Dirac electron in hyperbolic motion might be considered as test particle in the proper sense. Hereby one should exclude any sort of non-locality, and linked with it the existence of radiation, for a test particle by definition.

So the Lorentz-Dirac theory (being an approximation in the framework of the present finite-size theory) actually represents the neutral particle limit in the special force field under consideration.

Thus, having neglected radiation reaction in using an approximation equation of motion for the radiating electron, one finds the principle of equivalence fulfilled even for radiating particles. But accounting for the finite-size of the electron means that one can no longer regard the electron as a test particle and consequently the principle of equivalence is not fulfilled for the real electron as soon as one leaves the point-like approximation.

Parenthetically, we mention that the principle of equivalence, as stated by Rohrlich³ in its strong form, is not valid for true gravitational fields^{3, 30}, even if one works in the point-like approximation; and finally the real electron has spin and therefore cannot serve for this reason, too, as a test particle for Einstein's principle of equivalence.

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