On the Theory of Stochastic Heating of Particles in Magnetic Mirror Systems

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The paper presents a study of the effect of a high frequency stochastic field on particles contained in a magnetic mirror system. The r.f. field is assumed to be excited by an external source and to penetrate from one side through the mirror to only those locations in which the rf field is in resonance with the cyclotron frequency. The rf field is further assumed to have a specified correlation, and to be unaffected by changes in the particle distribution function brought about by heating.

The case in which it is possible merely to consider the effect of the perpendicular component $E_\perp + (\boldsymbol{v} \times \boldsymbol{B}_{\text{rf}})_\perp$ was studied separately from the case in which both components of $\boldsymbol{E} + (\boldsymbol{v} \times \boldsymbol{B}_{\text{rf}})$ are to be taken into consideration simultaneously; $\boldsymbol{E}$ and $\boldsymbol{B}_{\text{rf}}$ are here the electric and magnetic components of the rf field. The absorption by a small group of particles the distribution function of which has the form of a δ-function is examined. The solution is performed in one-particle approximation.

As the solution implies, in a configuration of this kind transverse heating is accompanied by diffusion of particles through the mirrors as well as by a transformation of a portion of transverse energy into longitudinal (resulting from both the transformation due to the magnetostatic field inhomogeneity and the effect of B_{rf}).

The necessity of better understanding of the mechanism of absorption under complicated conditions of plasma heating by beam instabilities arises particularly in connection with experimental studies of this phenomenon (frequently conducted in magnetic mirror fields). This problem has been discussed in several studies (for instance in [1]). In the present paper, the analysis of the mechanism of absorption is approached in such a way that the rf field is assumed to be specified, and the examination concerns its action on a small group of particles in an one-particle approximation (meaning that the effect of changes in the distribution function of particles on the character of the rf field is neglected). In agreement with the experimental data of [2], the rf field is assumed to be stochastic.

The advantage of this approach lies in the fact that a discussion of the mechanism of absorption of the rf field by particles confined in magnetic mirror systems can be caried out to a considerable detail, and thus effects that are usually neglected in the linear solution can be included. The disadvantage of our procedure is evident: it consists in the artificial concepts of our model. An obvious extension of our work would be solving the nonlinear self-consistent system of pertinent equations and examining in more detail the phenomena to which attention is called in the present paper.

A short communication about this problem is presented in Letter [3]. All calculations are performed in canonical variables action – angle whose analysis is presented in [4, 5].

1. Description of the Model under Examination

1.1. Magnetostatic Field of the System

Let us assume that the magnetostatic field is idealized by the curve shown in Fig. 1. In the region of Q_3 $(-Q_{31}, +Q_{31})$ its form is quadratic on the central line of force, and in the regions where $Q_3 > Q_{31}$, $Q_3 < -Q_{31}$, the mirror region is represented by a homogeneous field, or may fall off. Our study is in a paraxial approximation.

It is obvious that the chosen curve is only a model of far more complicated real magnetic mirror systems.

1.2. Rf Stochastic Field

It is assumed that the rf field penetrates inside the system on the left, from the region of the homogeneous magnetostatic field. It is further assumed,

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[1] J. PREINHAELTER, Plasma Phys. **10**, 713 [1968].

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[4] L. KRLÍN, Czech. J. Phys. **B 17**, 112 [1967].

[5] L. KRLÍN, Czech. J. Phys. **B 17**, 124 [1967].

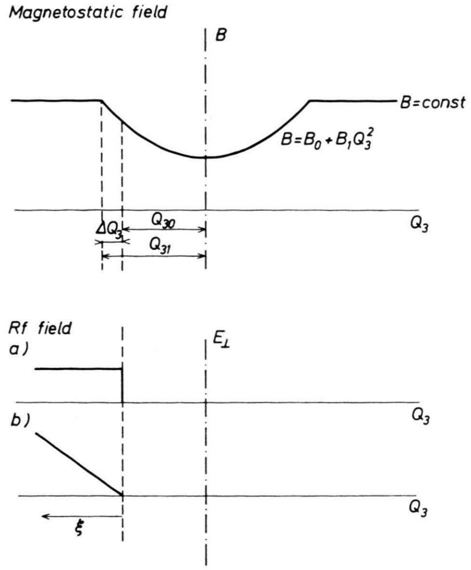


Fig. 1. Magnetostatic field and rf field.

approximately in agreement with the experiments of [2], that the intensity of the rf field at the place of resonance, $\omega = \omega_c$ (at point $Q_{30} = Q_{31} - \Delta Q_3$ in Fig. 1) tends to zero. Where further specification of the field is needed, we suppose that the wave is circularly polarized and propagates along the magnetostatic field. We suppose that in a certain region ΔQ_3 it is possible to consider a resonant interaction; this will be explained later on. The amplitude of the field is spatially variable (in what follows, two special cases will be considered). Let the rf field have a given correlation, and its space and time variations be not affected by absorption. (The experimentally established stochastic character of the rf field is for instance in the experiment [2] caused by quasilinear effects of the plasma-beam system, and it understandably calls for a separate study). The study treats (for simplicity) the effect of component $E_\perp$ alone (Chapt. 2, 3, 4) and the effects of

$$E_{\parallel} + (\boldsymbol{v} \times \boldsymbol{B})_{\parallel\, \mathrm{rf}}\,, \qquad E_\perp + (\boldsymbol{v} \times \boldsymbol{B})_{\perp\, \mathrm{rf}}$$

which are supposed to be acting simultaneously (Chapt. 5).

The assumptions made about the rf field are, of course, very artificial; the purpose of the approximation is only to point out (by analytically tractable methods) the effects whose quantitative estimate would need a much more complicated numerical analysis. In fact, the form of the field and the character of the interaction depends on the type of the wave, on the density and on the temperature of the plasma and on the shape of the magnetostatic field. In the following we use the results of the analysis of the propagation of the circularly polarized wave in the case of its absorption in a resonant region $\omega \sim \omega_c$ (ω_c is the cyclotron frequency).

The assumed forms of the envelopes of the rf field are chosen to agree qualitatively with the experimental data. The intensity of the rf field decreases, of course, to a negligible value in a certain region around the point of the resonance $\omega = \omega_c$ (except for the case of zero density, where the field retains its vacuum character). If we use the analysis [6, 7], it is possible to approximate the majority of the solutions by means of one out of the chosen types of the envelopes. It is possible to model the vacuum rf field in this way, if we neglect the change of the magnetic moment of particles in the non-resonant region.

The character of absorption is also determined by the frequency ω and the wave number k. Because of the variability of k (in the general case) the examination of the interaction is very complicated. We suppose therefore that in the first approximation the most intensive change of the magnetic moment takes place in a small longitudinal interval, where the influence of the change in k is negligible.

The influence of the component B_{rf} has a dominating effect for rather small amplitudes of the particle axial motion $Q_{3\mathrm{m}}$ in the region near to the minimum of the magnetostatic field and in the cold plasma in the whole region of high $k_{\parallel}$.

For the determination of the absorption character we use the expression from [7], or [6].

1.3. The Distribution Function

Throughout most of this study, the distribution function $f(J_1, J_2, J_3, t_0)$ is assumed to be of the form

$$f = A\, \delta(J_1 - J_{10})\, \delta(J_3 - J_{30})$$

with the amplitude of the axial oscillation of the particles satisfying the condition

$$Q_{30} \leqq Q_{3\mathrm{m}} \leqq Q_{31}\,.$$

[6] T. H. Stix, The Theory of Plasma Wawes (in Russian), Atomizdat, Moscow 1965.

[7] A. F. Kuckes, Plasma Phys. **10**, 367 [1968].

We therefore study the effect of the rf field on a small group of selected particles. The exact solution for a continuous distribution function needs of course a quasilinear approximation (see also later).

As the solution implies [9], the axial motion of the particles in the above mentioned inhomogeneous magnetostatic field has the form

$$Q_3 = Q_{3m} \sin(w_3 + \psi)$$

where Q_{3m} means the amplitude of the axial oscillation and where $\omega_c(Q_3) = \omega_{c0} + \omega_1 Q_3{}^2$; J_i, w_k are canonical variables action – angle. Here

$$Q_{3m}^2 = \frac{1}{\pi} \sqrt{\frac{1}{2 m \omega_1}} \cdot \frac{J_3}{\sqrt{J_1}} ;$$

$$\omega_{c0} J_1 = H_{\perp 0} ; \quad \frac{1}{\pi} \sqrt{\frac{\omega_1 J_1}{2 m}} \cdot J_3 = H_{\parallel 0}$$

where $H_{\perp 0}$, $H_{\parallel 0}$ are the perpendicular and longitudinal energy of the particle for $w_3 \equiv \psi \equiv 0$ and $w_i = \nu_i t$, where ν_1 means the mean cyclotron frequency and ν_3 the frequency of the axial motion.

1.4. The Stochasticity of the Rf Field

The stochasticity of the rf field is modelled by the following assumption. We suppose that for the particle that is passing through the rf field region ΔQ_3 during its axial motion the rf field keeps its phase ψ constant in the argument $\omega t + \psi$. However, this phase is stochastically changing at the beginning of each such transit of the particle. Such an assumption is, of course, not valid simultaneously for all particles. But we assume that on the average this model gives results which are not very far from reality.

Since in a real system one can expect such collisions (and their influence on the cyclotron phase of particles) we believe that this model may be realistic also in the case of a non-stochastic wave.

2. Discussion of Particle Motion in a Magnetic Mirror System under the Action of an Rf Field

At the usual mirror ratios ($R \sim 2$) a strong coupling exists in a magnetic mirror systems between the transversal motion (resulting mostly from cyclotron rotation) and the longitudinal oscillatory motion of particles between the mirrors. Let us assume that $E_{\parallel} + (\boldsymbol{v} \times \boldsymbol{B}_{\text{rf}}) = 0$ (this component causes an independent longitudinal diffusion; a general case as a more complicated one is discussed separately in Chapt. 5). The effect that occures under such conditions is as follows: the amplitude of the longitudinal motion of the particles that absorb the rf energy decreases, while the particles that transfer their transverse energy to the field, increase their axial oscillations. In the differential form this is described by the relation

$$\frac{\mathrm{d}Q_{3m}}{Q_{3m}} = - \frac{\mathrm{d}z}{z} \cos^2 2 \pi w_3 .$$

Here Q_{3m} is the amplitude of the axial oscillations $z = \sqrt{J_1} = \sqrt{m \mu / e}$, μ – the magnetic moment, $\mathrm{d}z$ and $\mathrm{d}Q_{3m}$ – the instantaneous change in z and the ensueing change in Q_{3m}, respectively, and w_3 – the phase of the axial oscillations at which the differential change has taken place. The change in $\mathrm{d}Q_{3m}$ is obviously dependent on this phase. To simplify, consider the case when z changes step-wise under the action of the δ rf field pulse. Now, whenever a change in z occurs at the center of the system, ΔQ_{3m} reaches its maximum; as long as z changes in the turning point of the axial oscillation only (for which $\mathrm{d}Q_3/\mathrm{d}t = 0$), Q_{3m} is conserved as an integral of motion. The latter case is a singular one. Therefore it is possible to say that (with the above mentioned exceptions and under the influence of only $E_{\perp}$) an incrase in magnetic moment – and hence also in mean transverse energy – is accompanied by a decrease in axial oscillations (and vice versa). In our model, this effect plays quite an important role. With growing transverse energy the diffusion coefficient decreases, because the particle stays in the rf field for a shorter time. As we shall show later, this fact causes the heating to slow down appreciably.

3. Discussion of the Diffusion Coefficient

As stated in [10] in the case that the rf field acts on particles in a stochastic sequence of pulses, the diagonal component of the diffusion tensor is given by

$$D_{P_i P_i} = \langle n \rangle \langle \Delta P_i^2 \rangle$$

where $\langle n \rangle$ is the mean pulse frequency, ΔP_i the change in the momentum P_i during a pulse, $\langle \Delta P_i^2 \rangle$ the mean value of the quadratic increment of P_i during a pulse.

In our case, $\langle n \rangle = \nu_3$ wher ν_3 is the frequency of the axial oscillations of particles between the mir-

[8] L. Krlín, IPPCZ-106, in press.

[9] M. Seidl, Plasma Phys. **6**, 597 [1965].

[10] L. Krlín, Czech. J. Phys. **B 18**, 977 [1968].

rors. The magnitude of ΔP_i, and in our case, the magnitude of the diffusion coefficient too, is given – besides by the spectrum of the rf field – by the proximity of the system to resonance (e. g. cyclotron resonance) as well as by the value of the correlation τ_c. It is clear e. g. from analysis [10] (and as a matter of fact, by intuition, too) that diffusion is most rapid at the resonance. As τ_c decreases, the difference between resonant and non-resonant diffusion diminishes.

As the solution implies, in magnetic mirror systems resonance occurs in two cases (not necessarily existing at the same time).

First, resonance occurs under the condition

$$\omega + k\,\nu_1 + l\,\nu_2 + m\,\nu_3 = 0 \tag{1}$$

where ω is the frequency of the external rf field, ν_1 is then the frequency which corresponds to the mean particle cyclotron frequency during the flight between the mirrors, ν_2 is the frequency of azimuthal drift, and ν_3 the frequency of oscillations between the mirrors; k, l, m are integers. It usually holds (and we assume so, too) that

$$\nu_1 : \nu_2 : \nu_3 = 1 : 10^{-2} : 10^{-2}.$$

ν_i are generally functions of energy; it is, therefore, evident (as we have pointed out in [10]) that if once the particle energy changes, the resonance is disturbed and slowing-down of the diffusion follows.

Besides, resonance also occurs when

$$\omega + k_{Q_3} \frac{dQ_3}{dt} - \omega_c = 0 \tag{2}$$

where k_{Q_3} is the wave number, ω_c the instantaneous cyclotron frequency, and dQ_3/dt the longitudinal particle velocity.

While the first type the resonance lasts continuously in time in the linear approximation, the second type of resonance acts exactly during an infinitely short time interval, or when the difference between the rf field and cyclotron rotation phases changes approximately by the value of π in less severe approximation.

The relation between the two types of resonance is quite complicated. But an approximate analysis reveals: as long as $\nu_3 : \nu_1 \sim 10^{-2}$ and particles move in the rf field for only a fraction of the axial period $(\sim 0{,}1/\nu_3 \div 0{,}3/\nu_3)$, the diffusion coefficient due to the second type of resonance prevails by one order of magnitude at least.

The way diffusion varies also depends on the place where the resonance $\omega = \omega_c$ occurs. The simplest situation exists at the resonance in the immediate proximity of the maximum of the system magnetostatic field and also if the interaction in the region of the field penetration ΔQ_3 may be still regarded as approximately resonant [according to our analysis, this ΔQ_3 is given by the expression

$$\Delta Q_3/Q_{30} \sim (\pi \cdot \nu_3/\Delta\omega_c)^{2/3}].$$

In this resonant case the diffusion coefficient is relatively easy to compute. For particles with $Q_{3m} < Q_{30}$ the coefficient of diffusion is zero (they do not penetrate inside the rf field region) and particles with $Q_{3m} > Q_{30} + \Delta Q_3$ escape from the system; this, however, can be included in boundary conditions.

The situation becomes more complicated when resonance sets in at a place not immediately adjacent to the maximum of the magnetostatic field. As long as particles transfer their perpendicular energy to the rf field, their amplitude increases; it can be proved that resonance of the type (2) again exists for all particles with $Q_{3m} > Q_{3m0}$, where $\omega_c(Q_{3m0}) = \omega$. But the particles pass now through the resonance region at a definite velocity, in distinction from the foregoing simple case when the parallel speed of the particle was practically zero in the interaction region, and consequently the coefficient of diffusion is reduced. To simplify the analysis the two alternatives will be considered separately.

4. Energy Absorption in the Case $E_\perp \neq 0, E_\parallel = 0, (\boldsymbol{v} \times \boldsymbol{B}_{rf}) \doteq 0$

4.1. *Energy Absorption in the Case that the Resonance Occurs in the Immediate Proximity of the Maximum of the Magnetostatic Field*

The determination of the interaction of a particle with rf field in the neighbourhood of the resonance $\omega = \omega_c$ is generally very complicated. In the simple approximation we shal suppose that in the region of the resonance in the immediate proximity of the maximum of the magnetostatic field it is possible to consider $k_\parallel$ as a constant. Then the change of z is given approximately by:

$$\Delta z \sim \int_{t_0}^{t_0+\Delta t} \cos\left(\frac{k_\parallel\, Q_{30}\, \nu_3^2}{2}\, \xi^2 + \frac{\omega_1\, Q_{30}\, \nu_3^2}{6}\, \xi^2 + \psi\right) d\xi\,.$$

Here Δt means the time interval during which the particle stays in the rf field region ΔQ_3; ψ is the phase difference between the rf field and cyclotron rotation phase. For smal values of Δt it is possible to consider the heating as a resonant process; Δt must, however, satisfy

$$\Delta t_{\max} = \min \left[\frac{1}{\nu_3} \left(\frac{2\pi}{k_{\|} Q_{30}} \right)^{\frac{1}{2}}; \left(\frac{3\pi}{\omega_1} \right)^{\frac{1}{3}} \cdot \left(\frac{Q_{30}\, \nu_3^2}{2} \right)^{-\frac{1}{3}} \right].$$

Under these conditions it is possible to suppose the resonant heating in the form

$$dz/dt = \beta \quad \text{or} \quad dz/dt = \beta\, \xi$$

where β is a constant given by the components of the rf field. The first case corresponds to that of the rectangle-shaped envelope of the rf field, the second to a linear growth (in accordance with Fig. 1).

The calculation of the coefficient of diffusion is performed by solving the particle motion in such type of rf field and its values are presented in Table 1 (the coefficient of diffusion is considered to be of the form

$$D_{11} = \langle n \rangle \langle \Delta z^2 \rangle$$

where n is the frequency of the pulses (here $\langle n \rangle = \nu_3$) and Δz an increase of z in one pulse). It follows that D_{11} falls off very rapidly with increasing particle energy. Furthermore, it falls off with increasing nonhomogeneity of the magnetostatic field (represented by ω_1) and also grows with decreasing Q_{30}. All these results are valid only if $x_0/Q_{30} \ll 1$.

Diffusion is obtained by integrating the equation

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial z} \left[D_{11}(z) \frac{\partial f}{\partial z} \right] \tag{3}$$

with the initial condition (as we have already mentioned):

$$f(z, t = t_0) = A\, \delta(z - z_0)\, \delta(J_3 - J_{30})$$

and the boundary condition:

$$f(z = z_{0\,\min},\ J_3 = J_{30\,\min},\ t) = 0\,, \tag{3a}$$

where $z_{0\,\min}$, $J_{30\,\min}$ define the loss-cone

$$\sin\alpha = \left(\frac{B_{\max}}{B_0} \right)^{\frac{1}{2}} = \frac{1}{2\sqrt{m}} (2\, e\, B_0\, J_{10\,\min})^{\frac{1}{2}} \cdot \left[\omega_0\, J_{10\,\min} + \frac{1}{\pi} J_{30\,\min} \left(\frac{\omega_1 J_{10\,\min}}{2\, m} \right)^{\frac{1}{2}} \right]^{-\frac{1}{2}}$$

for $J_{30\,\min}$ given by

$$(Q_{30} + \Delta Q_3)^2 = \frac{1}{\pi} (2\, m\, \omega_1\, J_{10\,\min})^{-\frac{1}{2}} \cdot J_{30\,\min}\,.$$

For the diffusion process, boundary condition (3a) means a total absorption. This signifies in our case that the particles whose z drops to $z_{0\,\min}$ escape from the system and are lost for the subsequent process of heating.

Now, let us consider directly the absorbed energy $Q_\perp(t)$ defined by

$$Q_\perp(t) = \int H_\perp\, f(z, t)\, dz$$

where $H_\perp$ is the perpendicular kinetic energy of a particle.

In general, the perpendicular energy of a particle is defined by

$$H_\perp = \omega_c\, z^2 + \Delta H(z, \boldsymbol{A}, \Phi)$$

where $\boldsymbol{A}$, Φ is the vector or the scalar potential of the rf field. As long as only Φ is applied, than the perpendicular kinetic energy is given by

$$H_\perp = \omega_c\, z^2.$$

For the case of a circularly polarized wave

$$H_\perp = \omega_c\, z^2 - \sqrt{2\, \omega_c/m} \cdot z\, e\, A_{\rm rf}\,.$$

If $A_{\rm rf}$ is only a small perturbation of the whole perpendicular part of the Hamiltonian, than for particles with

$$m\, v_\perp / (e\, A_{\rm rf}) \gg 1$$

it is possible to calculate the energy (with the same error) from the Hamiltonian without the rf field.

$H_\perp$ of course depends on the longitudinal coordinate Q_3. For simplicity we shall use the minimum energy that occurs for $Q_3 = 0$ and $\omega_c = \omega_{c\,\min}$. Then the energy $Q_\perp(t)$ is determined by

$$Q_\perp(t) = \int \omega_{c\,\min} \cdot z^2\, f(z, t)\, dz$$

which in the region without the rf field is valid exactly and in the interaction region with the above mentioned error. Here $f(z, t)$ is the solution of Eq. (3) with the stated initial and boundary conditions. Denoting the coefficient of diffusion as

$$D_{11} = D_0 / z^n$$

we arrive at $Q(t)$ given by the general expression

$$(m = 2 + n)$$

$$Q(t) < \omega_{c\,\min} A\, z_0^{(1-n)/2} \frac{1}{m^2 D_0 (t-t_0)} \left\{ 4^{-\frac{7+n}{2m}} [m^2 D_0 (t-t_0)]^{\frac{7+n}{2m}} \Gamma\left[\frac{7+n}{2m}+1\right] + \frac{2 z_0^{m/2}}{m} 4^{-\frac{7+n}{2m}+\frac{3}{2}} \cdot [m^2 D_0(t-t_0)]^{-\frac{7+n}{2m}+\frac{3}{2}} \cdot \Gamma\left[\frac{7+n}{2m}+\frac{3}{2}\right]\right\} \tag{4}$$

(here Γ is gamma-function).

The values obtained for the limiting case of $t \to \infty$, $z_{0\,\min} \to 0$ are reviewed in the Table. (For comparison the table also lists the case in which – though coupling between the individual degrees of freedom is considered – it is assumed that it affects only the escape of the particles and not the magnitude of the coefficient of diffusion; the latter is taken to be constant.)

The solution results in a simple conclusion: in the limit $t \to \infty$ the heating is so slow that it may be neglected. (Practically speaking, we may write the most rapid heating as

$$\Delta Q_\perp(t) < Q_\perp(t_0)\,(\Delta H_\perp/H_{\perp 0})^{4/11} \cdot (2\,\pi)^{2/11} \cdot \bar{n}^{2/11}$$

where $\Delta H_{\perp 0}$ is the maximum rate of growth of the transverse energy during one transit through the rf field and $\bar{n}$ the average number of particle transits through the rf field). Because of our limit $z_{0\,\min} \to 0$ the real increment $\Delta Q_\perp$ is still smaller.

The physical reasons for the slowing-down of the heating are obvious – due to the dependence $D_{11}(z)$, the particles diffuse very slowly towards higher z and are more likely to escape from the system.

4.2. *Energy Absorption in the Case when the Resonance Occurs at an Arbitrary Point of the System*

This case though far more realistic, is also more complicated than the previous one. The particles that prolong their axial oscillations beyond the original resonant region, are kept in the system until the time when their amplitude grows to Q_{31}. With respect to our assumption they are continuously moving in the rf field, and thus it is necessary to determine their coefficient of diffusion, too. Unlike in the case of $\mathrm{d}Q_3/\mathrm{d}t = 0$, these particles pass through their region of resonance always with a definite velocity. The maximum possible momentum change therefore decreases, and so does the coefficient of diffusion.

In what folows it is necessary to resort to an approximation quite different from that of the foregoing case. We shall suppose that it is possible to neglect the change of the axial velocity of the particles. It is clear that for the given mirror geometry this approximation would be valid only to a certain increase in energy. In fact, the average energies must satisfy

$$\Delta H_\perp/H_{\perp 0} \sim (Q_{31}/Q_{30})^2 - 1$$

where Q_{31} is the maximum axial amplitude of particles. Off this limit most of the particles reach the region, where it is necessary to consider the change of the axial velocity. This effect, of course, weakens if the point of resonance approaches the centre of the system.

If our approximation is valid, it is possible to use for the change of the perpendicular energy the expression [7]

$$W = \pi\, e^2 E_\perp E_\perp^* / (2\, m\, \omega_c'\, v_\parallel)$$

where $\omega_c' = \partial \omega_c / \partial Q_3$ and $v_\parallel = \mathrm{d}Q_3/\mathrm{d}t$ must be in resonance:

$$\omega - \omega_c(Q_{3r}) - k_\parallel \left.\frac{\mathrm{d}Q_3}{\mathrm{d}t}\right|_{Q_3 = Q_{3r}} = 0\,.$$

The solution was carried out in [7] by means of an improper integral. The path of a particle is in our case always limited; nevertheless we stil can use the expression, because the greatest part of the energy increase takes place in the neighbourhood of the resonance.

The diffusion coefficient (in the variable z) has therefore the form

$$D_{11} = \frac{\nu_3}{z^2} \left(\frac{\pi\, e^2 E_\perp E_\perp^*}{2\, m\, \omega_c'\, \omega_c\, v_\parallel} \right)^2.$$

Of course, in the exact solution it is necessary to consider the diffusion in both momentums J_1, J_2

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial t} f(J_1, J_3, t)\,.$$

If we negect the influence of change in J_3 and $v_\parallel$ we shall consider $v_{\parallel 0}$ as a parameter (in fact the change of $v_\parallel$ would partially encrease the influence of z), then the coefficient has the form

$$D_{11} = D_0 / z^2.$$

For the boundary condition $f(z,t)_{z=0} = 0$

$$\Delta Q_\perp(t) \sim t^{\frac{1}{3}}.$$

The resonance near the minimum of the magnetostatic field therefore is speeding up the absorption. For the purpose of comparison, the table also shows the case of total reflection with the assumption of $D_{11} = \text{const}$ as follows from the solution [8]. This is the fastest form of heating.

4.3. Heating with Continuous Distribution Function

The calculations made so far were related to diffusion of the δ-function distribution. In reality, however, the distribution function which has to be examined, will be of the continuous form $F(z, J_3, t)$. We obtain this form by integration of our previous function

$$F(z, J_3, t) = \int f(z, z_0, J_{30}, t) \, \mathrm{d}z_0 \, \mathrm{d}J_{30} .$$

The rate of absorption is then under our approximation formally described as

$$\frac{\mathrm{d}Q_\perp}{\mathrm{d}t} = \int \omega_c z^2 \frac{\partial F}{\partial t} \, \mathrm{d}z \, \mathrm{d}J_3 .$$

When we for instance consider that the absorption is influenced only by a distribution in z, we obtain

$$\frac{\mathrm{d}Q_\perp}{\mathrm{d}t} = \int_z \int_{J_{30}} \int_{z_0} \omega_c z^2 \frac{\partial}{\partial z} \left[D_{11}(z) \frac{\partial f}{\partial z} \right] \mathrm{d}z_0 \, \mathrm{d}J_{30} \, \mathrm{d}z \quad (5)$$

which is already near to the customary quasilinear form. In this type of the quasilinear theory the rate of absorption decreases with the diffusion of the distribution function, while the resonant diffusion coefficient is influenced only by the change of the spectrum andd oes not depend on the energy. In our case the rate of absorption further falls off due to the nonlinear character of the diffusion coefficient; the dependence of the diffusion coefficient on the energy causes in fact it's effective reduction.

The form of (5) is analogous to the first equation of the generalized quasilinear system. Nevertheless $D_{11}(z)$ (which is in our approximation determined from our model) must be in an exact solution calculated from the analysis of the wave propagation through a hot plasma with a – supposed stationary – distribution function f.

5. Energy Absorption in the Case $E_\perp \neq 0$, $E_\parallel \neq 0$; the Effect of Coupling $\sqrt{J_1} (\partial A_\varphi / \partial Q_3)_{\mathrm{rf}}$

We have assumed in the foregoing solutions that the effect of $E_\parallel$ may be neglected. It is clear that in the majority of cases, this assumption is not exactly satisfied. Neither the energy transformation in the magnetic component of the rf field can be neglected generally. Otherwise, the solution would include only the effect of $(E + [\boldsymbol{v} \times \boldsymbol{B}])_{\perp \mathrm{rf}}$.

With the joint action of both components the situation becomes extremely complicated. The particles diffuse in part in J_1, with ensueing diffusion in J_3 (due to the transformation) and in addition also independently in J_3 (through $E_\parallel + [\boldsymbol{v} \times \boldsymbol{B}]_{\parallel \mathrm{rf}}$).

The exact solution of the problem is thus given by the solution of the equation

$$\frac{\partial f}{\partial t} = D_1 \frac{\partial f}{\partial J_1} + D_{11} \frac{\partial^2 f}{\partial J_1^2} + D_3 \frac{\partial f}{\partial J_3} + D_{33} \frac{\partial^2 f}{\partial J_3^2} + D_{13} \frac{\partial^2 f}{\partial J_1 \, \partial J_3}$$

(here we neglect the dependence on the radial space coordinates). As all coefficients are also functions of J_1, J_3, the solution of the problem is in this case very difficult.

Relying on a physical notion, we shall present only an approximate solution.

The effect of $E_\parallel \neq 0$ will manifest itself in the following way: To start with, independent diffusion in J_3 joins the dependent change of J_3 (caused by a change of J_1). This independent diffusion will change the number of particles present in region ΔQ_3, as well as the magnitude of the coefficient of diffusion for the diffusion in J_1. The effect of $E_\parallel$ namely will make it possible for the particles with large J_1 to be passing through the rf field for a considerably loger time than at $E_\parallel = 0$.

Generally speaking, the effect of coupling $\sqrt{J_1} (\partial A_\varphi / \partial Q_3)_{\mathrm{rf}}$ where $(\partial A_\varphi / \partial Q_3)_{\mathrm{rf}}$ corresponds to the magnetic component of the rf field, cannot be neglected, either. Even though this force causes no change in the total particle energy, it contributes – through the effect of the magnetic component of the rf field – to the transformation between transverse and longitudinal energies. Consider, e. g. a circularly polarized wave which propagates almost in a vacuum. Then for resonant particles

$$\frac{\mathrm{d}Q_\perp}{\mathrm{d}t} : \frac{\mathrm{d}Q_\parallel}{\mathrm{d}t} = \frac{\omega}{k_\parallel v_\parallel} ;$$

$$(\mathrm{d}Q_\parallel / \mathrm{d}t)_{\mathrm{transf}} = \frac{1}{2} \frac{\Delta B_{\max}}{B_{\min}} \cdot \frac{\mathrm{d}Q_\perp}{\mathrm{d}t}$$

where $\mathrm{d}Q_\perp / \mathrm{d}t$ is the change of the perpendicular energy, and $\mathrm{d}Q_\parallel / \mathrm{d}t$ the change of the longitudinal energy, produced by the same wave; $(\mathrm{d}Q_{\parallel}/\mathrm{d}t)_{\mathrm{transf}}$ is the rate of the energy transformed in the magnetostatic field, it the rf field would be acting near the maximum of the axial amplitude of the particle motion. From this it is obvious that near $B_{\min}$ predominates the influence of the change of the longitudinal energy caused by the wave. The particles with negligible longitudinal energy absorb a much larger proportion of the perpendicular than of the longitudinal energy. Nevertheless, the influence of $\mathrm{d}Q_{\parallel}/\mathrm{d}t$ will appear in the longitudinal diffusion.

Next we shall consider the joint effect of $E_{\parallel}$ and $v_{\perp} \times B_{\perp \mathrm{rf}}$ with the common coefficient of diffusion D_{33}. D_{33} being a complicated function of transverse and longitudinal particle energies, we shall attempt to present at least a qualitative description of two alternative models which somewhat clear up the whole problem.

Assume first that the action of the rf field is such that the diffusion coefficiencs $D_{\perp}$, $D_{\parallel}$ are different from zero for only a narrow resonant region of amplitude $Q_{3\mathrm{m}}$ (in the region $2\,\overline{\Delta Q_3}$) which does not immediately join the maximum or the minimum of the field. Consequently, the particles will become isolated in this region of amplitudes and will not be able to penetrate with their $Q_{3\mathrm{m}}$ outside $Q_{3\mathrm{m}} \pm \overline{\Delta Q_3}$. It is therefore possible to assume that in transverse energy, the particle are subject to diffusion with the mean coefficient of diffusion (due to $D_{\parallel} \neq 0$)

$$D_{11}^{(1)} = \frac{1}{\Delta z}\int_0^{\Delta z} D_{11}(z)\,\mathrm{d}z \qquad \text{for} \quad \Delta z \to \infty\,.$$

According to Table 1, the absorbed energy will be

$$\Delta Q_{\perp}(\Delta t) \sim 2\,D_{11}^{(1)}\,A\,\omega_c\,\Delta t\,.$$

Second let us suppose that D_{33} acts in the whole of the system and D_{11} again only in a narrow region of $\overline{\Delta Q_3}$. Let us first suppose that the particles do not escape from the system. Then the system in variable Q_3 can be considered (approximately at least) as recurrent; let $\tau_{\mathrm{c\,rec}}$ be the mean time of recurrence during which the particle amplitude reaches again to the region $\overline{\Delta Q_3}$. In such a case, $D_{11}^{(2)}$ will be given by the expression

$$D_{11}^{(2)} = \frac{1}{\Delta z}\int \frac{1}{\tau_{\mathrm{c\,rec}}}\cdot\frac{1}{\nu_3}\cdot D_{11}(z)\,\mathrm{d}z \qquad \text{for} \quad \Delta z \to \infty\,.$$

The effect of the escape will be accounted for by the decrease in the value A (considered always constant in the foregoing discussion). The decrease will be found by taking the diffusion coefficient for longitudinal diffusion constant throughout the whole of the system and independent of energy. However, instead of diffusion in longitudinal velocity we shall consider diffusion in $Q_{3\mathrm{m}}$ and take the distribution in $Q_{3\mathrm{m}}$ uniform with a total number of particles A throughout the whole system. The coefficient of diffusion is now defined as

$$D_{33} = \lim_{\tau \to 0} \frac{1}{\tau}\,\langle \Delta Q_{2\mathrm{m}}^{3} \rangle$$

Under these assumptions and for slow diffusion it will approximately hold for the absorbed energy (cf. [8]) that (l is the length of the system)

$$Q_{\perp}(t) = Q_{\perp}(t)^{(0)} \exp\left(-\frac{\pi}{l^2}\,D_{33}\,t\right)$$

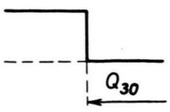
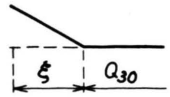
Course of the rf field	$D = D_0/z^n$	Case of total absorption $\Delta Q(t)$ for $t \to \infty$	Resonance $\omega = \omega_c$ exists below the maximum of the magnetostatic field $\Delta Q(t)$ for $t \to \infty$
	$2\pi\sqrt{\frac{2m}{\omega_1}}\,\beta^2\,\frac{x_0}{Q_{30}}\,\frac{z_0^{2/3}}{z^{5/3}}$	$\omega_c A\, z_0^{4/3}\cdot 4^{-13/11}\cdot\left[\left(\frac{11}{3}\right)^2 D_0(t-t_0)\right]^{2/11}$	$\sim (t^{1/3} - t^{1/3})$
	$\frac{2^3}{124}\,\pi^3\sqrt{\frac{2m}{\omega_1}}\,\beta^2\,\frac{x_0^3}{Q_{30}}\,\frac{z_0^{18/5}}{z^{13/5}}$	$\omega_c A\, z_0^{9/5}\cdot 4^{-24/23}\cdot\left[\left(\frac{23}{5}\right)^2 D_0(t-t_0)\right]^{1/23}$	
The diffusion coefficient is constant	$D(z) = D_0$	$4/\sqrt{\pi}\,(z_0 - z_{0\,\mathrm{min}})\sqrt{D_0}\,A\,\omega_c(t-t_0)^{1/2}$	Case of total reflection $\sim 2\,D_0\,A\,\omega_c(t-t_0)$

Table 1. The coefficients of diffusion, and the asymptotic expression of absorbed energy. $Q_{3\mathrm{m}} = Q_{30} + x_0$ is the amplitude of axial oscillation, $\omega_1 = e\,B_1/m$, the phase of rf field changes stochastically after $\tau_c \sim 1/\nu_3$, where ν_3 is frequency of axial motion.

where $Q_\perp(t)^{(0)}$ is the absorption for

$$E_\| \equiv (\partial A_\varphi/\partial Q_3)_{\mathrm{rf}} \equiv 0 .$$

Accordingly, the absorbed energy will be characterized by a relative maximum in a certain time $t_{\mathrm{opt}} = l^2/\pi^2 D_{33}$.

From these two extreme cases the following is obvious: the longitudinal component of diffusion [caused either by $E_\| \neq 0$, or by $(\partial A_\varphi/\partial Q_3)_{\mathrm{rf}} \neq 0$] weakens the nonlinear effect which was described in the previous chapters. Simultaneously, of course, it causes diffusion along the field lines. As long therefore as D_{33} is not negligible in the region of mirrors, it is posible to suppose that after reaching a maximum of the absorbed energy the total perpendicular energy will decrease.

6. Conclusion

On a simple model we have examined the effect of nonlinearities of particle motion in magnetic mirror systems, the influence of diffusion along the field lines and the effect of the rf field inhomogeneity on the rate of energy absorption. The assumption of a negligible effect of $E_\|$, $v_\perp \times B_{\perp \mathrm{rf}}$ allowed us to make the study sufficiently analytic. As the solution indicates, a marked slowing-down of heating occurs due to the effect of nonlinearities and diffusion. The configuration in which resonance sets in close to the minimum value of B_{stat} seems to be more advantageous with respect to heating. The simultaneous effect of $E_\perp \neq 0$, $E_\| \neq 0$, $v_\perp \times B_{\perp \mathrm{rf}} \neq 0$ could not be examined otherwise than qualitatively; as our considerations reveal, in this case a relative maximum of absorbed energy forms at a certain optimum time.

The situation in both cases is changed, whenever there exists a source of particles which compensates the particle diffusion through the mirror and keeps the total number of particles constant. Then in all cases is the growth of perpendicular energy faster [from $Q_\perp(t) \sim t^{\frac{1}{3}}$ to $Q_\perp(t) \sim t$].

The simplicity of the chosen model enables us to find a simple analytic solution of our problem. In fact the described effects could appear – in spite of their importance, as we believe – only in the corresponding quasilinear solution in the velocity-space configuration which includes also the nonlinearity in the axial motion of the particles.

We suppose that the above mentioned effects are some among the limiting factors of heating. It is possible that they are comparable with the commonly accepted quasilinear change of the derivatives $\partial f/\partial v_\perp$, $\partial f/\partial v_\|$. An exact solution of this problem will help to answer the important question, what is the rate of heating of a confined, space-limited volume of plasma in magnetic mirror systems.

We believe that the described effects are important also in the problem of eventual confinement of plasma by rf fields.

The next step of the calculation will therefore be to solve the quasilinear self-consistent approximation.