Research Article

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Study hoop algebras by fuzzy (*n*-fold) obstinate filters

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Abstract: In this article, we introduce the concept of fuzzy (n-fold) obstinate filter on hoop algebras and study some of the properties. We define and study fuzzy prime filter and fuzzy n-fold implicative filter on hoop algebras. Also, the relationship between fuzzy obstinate filter and some other fuzzy filters likeness fuzzy prime and fuzzy positive implicative filters are investigated. Then we show that the quotient of this structure is a Boolean algebra and obtained some condition equivalent with fuzzy n-fold implicative filter. Finally we show that every fuzzy n-fold obstinate filter with some conditions is a fuzzy n-fold implicative filter.

Keywords: hoop algebra, fuzzy (n-fold) obstinate filter, fuzzy n-fold prime filter

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1 Introduction

Naturally ordered commutative residuated integral monoids (hoop algebra) introduced by Bosbach in [8,9]. It is well known that in various logical systems, filters play a fundamental role and filters correspond to sets of provable formulas closed with respect to Modus Pones. In last years, the hoop theory was enriched with deep structure theorems. Several researchers investigated the theory of filters on hoop algebra and in 2014, Borzooei and Aaly Kologani studies filter theory on hoop algebras and several characterizations of filters, implicative, positive implicative and prime filters are derived. Furthermore, the relation between these filters on hoop algebras is established. In the study by Namdar et al. [11], obstinate filter on hoop algebra defined and some relations between these filter and (positive) implicative filters, maximal filters, prime filters, fantastic filters, and perfect filters on hoop algebras were investigated. In the fuzzy approach, fuzzification ideas have been applied to some fuzzy logical algebras. In previous studies [2,4,6], fuzzy filters on hoop algebras were studied. In particular, several types of fuzzy filters such as fuzzy implicative filters, fuzzy positive implicative filters, fuzzy Boolean filters, and fuzzy fantastic filters were introduced. Several researchers investigated the theory of filter and n-fold filters on hoop algebra, and in 2017, Luo et al. studied n-fold filters theory on hoop algebras, and several characterizations of n-fold filters, implicative and positive implicative, are investigated. They show that if A is a n-fold (positive) implicative hoop algebra, then every filter of A is an n-fold (positive) implicative filter and A/F is an n-fold (positive) implicative hoop. Furthermore, the relation between these n-fold filters on hoop algebras is established. n-fold obstinate filter on hoop algebra defined and some relations between these filter and (positive) implicative filters, maximal filters, prime filters, fantastic filters, and perfect filters on hoop algebras were investigated [7]. In the fuzzy approach, fuzzification ideas have been applied to some fuzzy logical algebras. In the study by Alavi et al. [4], fuzzy filters on pseudo hoop algebras were studied.

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In particular, several types of fuzzy filters such as fuzzy implicative filters, fuzzy positive implicative filters, fuzzy Boolean filters, and fuzzy fantastic filters were introduced.

Nowadays, filters are tools of extreme importance in many areas of classical mathematics. Considering the notion of (n-fold) obstinate filters [7,11], we present the notion of fuzzy obstinate filters and fuzzy n-fold obstinate filters in hoop algebras. The relationship between fuzzy (n-fold) obstinate filters and other types of fuzzy filters on hoop algebras is established, and we obtained the condition equivalent to the fuzzy obstinate filter. We define and study the notion of fuzzy prime filter on $\vee$-hoop and study properties of them and show that quotient of hoop algebra with respect to fuzzy obstinate filter is a Boolean algebra.

In Section 2, some basic concepts and properties are recalled, and some new notions about the thresholds are introduced to represent fuzzy filters like fuzzy filters, which are convenient to study the properties of fuzzy filters. In Section 3, the relationship between fuzzy obstinate filters and other types of fuzzy filters on hoop algebras is established, and we obtained the condition equivalent to the fuzzy obstinate filter. We define and study the notion of fuzzy prime filter on $\vee$-hoop and study properties of them, and we show that quotient of hoop algebra with respect to fuzzy obstinate filter is a Boolean algebra. In Section 4, the relationship between fuzzy n-fold obstinate filters and fuzzy n-fold filters on hoop algebras is established, and we obtained the condition equivalent to the fuzzy n-fold obstinate filter.

We show that extension theorem of fuzzy n-fold obstinate filter on hoop algebra is established, and the preimage of a fuzzy n-fold obstinate filter μ under f is a fuzzy n-fold obstinate filter on hoop algebra. In special case, we prove that if μ_a is an n-fold obstinate filter, then μ is a fuzzy n-fold obstinate filter of A. Also if μ is a fuzzy n-fold obstinate filter of A, then level subset μ_a is an n-fold obstinate filter of A. In Section 5, we introduce the concept of fuzzy n-fold implicative filters and we prove some related results and obtained some conditions equivalent with fuzzy n-fold implicative filters and study the relation between fuzzy n-fold obstinate filter and fuzzy n-fold implicative filter. Finally, we show that the quotient of n-fold obstinate hoop algebra with respect to fuzzy filter is an n-fold obstinate hoop algebra.

2 Preliminaries

In this section, we recollect some definitions and results that will be used, not cite them every time they are used.

Definition 2.1. [3] A *hoop algebra* or *hoop* is an algebra $(A, \odot, \to, 1)$ of type (2, 2, 0) such that, for all $x, y, z \in A$:
(HP1) $(A, \odot, 1)$ is a commutative monoid,
(HP2) $x \to x = 1$,
(HP3) $(x \odot y) \to z = x \to (y \to z)$,
(HP4) $x \odot (x \to y) = y \odot (y \to x)$.

On hoop A, we define $x \leq y$ if and only if $x \to y = 1$. It is easy to see that $\leq$ is a partial order relation on A. A hoop A is *bounded* if there is an element $0 \in A$ such that $0 \leq x$, for all $x \in A$. In this case, we define a negation "$'$" on A by, $x' = x \to 0$, for all $x \in A$. If $(x')' = x$, for all $x \in A$, then the bounded hoop A is said to have the *doublenegationproperty*, or (*DNP*), for short. An element a in A is called *dense* if and only if $a' = 0$.

Definition 2.2. [4] Let A be a bounded hoop, and for any $x, y \in A$, we define $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$. If $\vee$ is the join operation on A, then A is called a $\vee$-*hoop*.

A hoop A is *prelinear* if $(x \to y) \vee (y \to x) = 1$, for all $x, y \in A$.

Proposition 2.3. [8,9] *Let A be a bounded hoop. Then the following properties hold, for all $x, y, z \in A$:*
(*i*) *$(A, \leq)$ is a meet-semilattice with $x \wedge y = x \odot (x \to y)$,*
(*ii*) *$x \leq (x \to y) \to y$,*

(*iii*) $1 \to x = x, x \odot y \leq x, y$,
(*iv*) *if* $x \leq y$, *then* $z \to x \leq z \to y$ *and* $y \to z \leq x \to z$,
(*v*) $x \vee y \to z = (x \to z) \wedge (y \to z), x \leq y \to x$.

Definition 2.4. [5,11] Let F be a nonempty subset of A such that $1 \in F$. Then for any $x, y, z \in A$:
(i) F is called a *filter*, if $x, x \to y \in F$, then $y \in F$.
(ii) F is called an *implicative filter* of A, if $x \to ((y \to z) \to y) \in F$ and $x \in F$, then $y \in F$.
(iii) F is called a *fantastic filter* of A, if $z \to (y \to x) \in F$ and $z \in F$, then $((x \to y) \to y) \to x \in F$.
(iv) F is called an *obstinate filter* of A, if F is a proper filter and for any $x, y \notin F$, $x \to y \in F$ and $y \to x \in F$.
(v) A proper filter F of a $\vee$-hoop A is called a *prime filter* of A, if $x \vee y \in F$ implies $x \in F$ or $y \in F$.
(vi) F is called an *n-fold implicative filter* of A, if $x \to ((y^n \to z) \to y) \in F$ and $x \in F$, then $y \in F$.
(vii) F is called an *n-fold obstinate filter* of A, if F is a proper filter and for any $x, y \notin F$, $x^n \to y \in F$ and $y^n \to x \in F$.

If A is a bounded hoop, then a filter is proper if and only if it is not containing 0.

Proposition 2.5. [11] *Every obstinate filter of A is a* (*prime, implicative, positive implicative, fantastic*) *filter of A*.

Definition 2.6. [7] Let A and B be two bounded hoops. A map $f : A \to B$ is called a *hoop homomorphism* if and only if for all $x, y \in A$, $f(0) = 0$, $f(1) = 1$, $f(x \odot y) = f(x) \odot f(y)$, and $f(x \to y) = f(x) \to f(y)$.

If f is a hoop homomorphism, then $f(x^n \to y) = f(x^n) \to f(y)$.

Definition 2.7. [4] Let μ be a fuzzy subset of A and $\mu(x) \leq \mu(1)$, for all $x \in A$. Then μ is called:
(*i*) *fuzzy filter* if $\mu(x \to y) \wedge \mu(x) \leq \mu(y)$.
(*ii*) *fuzzy implicative filter* if $\mu(x \to ((y \to z) \to y)) \wedge \mu(x) \leq \mu(y)$.
(*iii*) *fuzzy positive implicative filter* if $\mu(x \to (y \to z)) \wedge \mu(x \to y) \leq \mu(x \to z)$.
(*iv*) *fuzzy fantastic filter* if $\mu(z \to (x \to y)) \wedge \mu(z) \leq \mu(((y \to x) \to x) \to y)$.

Theorem 2.8. [4,10]
(i) *μ is a fuzzy implicative filter if $\mu(x' \to x) \leq \mu(x)$.*
(ii) *Let $\alpha \in [0, 1]$. Then μ be a fuzzy* (*positive implicative, fantastic, implicative*) *filter if and only if $\mu_\alpha \neq \emptyset$ is a* (*positive implicative, fantastic, implicative*) *filter.*
(iii) *A is an n-fold implicative bounded hoop if and only if* $\{1\}$ *is an n-fold implicative filter.*
(iv) *Let μ be a fuzzy filter of A and $x \leq y$. Then $\mu(x) \leq \mu(y)$.*

Theorem 2.9. [2] *Let μ be a fuzzy filters of A, and we define a relation $\sim_{\mu_{\mu(1)}}$ on A as follows*:

$$x \sim_{\mu_{\mu(1)}} y \quad \textit{if and only if} \quad x \to y \in \mu_{\mu(1)},\ y \to x \in \mu_{\mu(1)}.$$

Then $\sim_{\mu_{\mu(1)}}$ is a congruence relation on A. Also $A/\mu = (A/\mu, \wedge, \odot, \to, \mu^1)$ is a hoop algebra.

Definition 2.10. [4] Let μ be a fuzzy filter of A and $A/\mu = \{\mu^x | x \in A\}$. For all $\mu^x, \mu^y \in A/\mu$, define $\mu^x \to \mu^y = \mu^{x \to y}$, $\mu^x \odot \mu^y = \mu^{x \odot y}$, and $\mu^x : A \to [0, 1]$, which is defined by $\mu^x(y) = \mu(x \to y) \wedge \mu(y \to x)$.

Definition 2.11. [7,10]
(i) A bounded hoop A is called an *n-fold implicative bounded hoop* if $(x^n \to 0) \to x = x$ for any $x \in A$.
(ii) A bounded hoop A is called an *n-fold obstinate hoop* if $x^n = 0$ for any $x \neq 1$.

Notation: From now one, we let $(A, \odot, \to, 1)$ or A is a hoop, unless otherwise state.

3 Fuzzy obstinate filters

In this section, we introduce the notion of fuzzy obstinate filter and fuzzy prime filter on a hoop algebra and investigate some properties of them.

Definition 3.1. A fuzzy filter μ of A is called a fuzzy obstinate filter if for any $x, y \in A$,

$$(1 - \mu(x)) \wedge (1 - \mu(y)) \leq \mu(x \to y) \wedge \mu(y \to x).$$

In the following example, we show that any fuzzy filter may not be a fuzzy obstinate filter.

Example 3.2. (i) Let $(A = \{0, a, b, 1\}, \leq)$ be a chain. Define the operations $\odot$ and $\to$ on A as follows:

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

Then $(A, \odot, \to, 1, 0)$ is a bounded hoop algebra.

(1) Let $\alpha, \beta \in [0, 1]$ that $\beta < 1/2 < \alpha$ and μ be a fuzzy subset on A such that $\mu(b) = \alpha = \mu(1)$, $\mu(0) = \mu(a) = \beta$. It is clear that μ is a fuzzy filter but not a fuzzy obstinate filter. Because, $(1 - \mu(a)) \wedge (1 - \mu(0)) > \mu(a \to 0) \wedge \mu(0 \to x)$.

(2) Let λ be a fuzzy filter such that $\lambda(0) = 2/3$, $\lambda(a) = \lambda(b) = 4/5$, $\lambda(1) = 1$. It is clear that λ is a fuzzy obstinate filter of A.

(ii) Let $(A = \{0, a, b, c, 1\}, \leq)$ be a poset with $0 < c < a, b < 1$, but a and b are incomparable. Define the operations $\odot$ and $\to$ on A as follows:

$\to$	0	c	a	b	1
0	1	1	1	1	1
c	0	1	1	1	1
a	0	b	1	b	1
b	0	a	a	1	1
1	0	c	a	b	1

$\odot$	0	c	a	b	1
0	0	0	0	0	0
c	0	c	c	c	c
a	0	c	a	c	a
b	0	c	c	b	b
1	0	c	a	b	1

Then $(A, \odot, \to, 1, 0)$ is a bounded hoop algebra. Let μ be a fuzzy filter such that $\mu(0) = 1/2$, $\mu(c) = 4/7 = \mu(a)$, $\mu(b) = 5/7$, and $\mu(1) = 6/7$. It is clear that μ is a fuzzy obstinate filter.

Proposition 3.3. *Any nonempty subset F of A is an obstinate filter if and only if the characteristic function χ_F is a fuzzy obstinate filter.*

Proof. Assume that F is an obstinate filter of A. We show that χ_F is a fuzzy obstinate filter.

Let $x, y \in A$. We show that

$$(1 - \chi_F(x)) \wedge (1 - \chi_F(y)) \leq \chi_F(x \to y) \wedge \chi_F(y \to x).$$

Case one: If $x \in F$ or $y \in F$, then $(1 - \chi_F(x)) \wedge (1 - \chi_F(y)) = 0$ and

$$(1 - \chi_F(x)) \wedge (1 - \chi_F(y)) \leq \chi_F(x \to y) \wedge \chi_F(y \to x).$$

Case two: If $x \notin F$ and $y \notin F$, then $(1 - \chi_F(x)) \wedge (1 - \chi_F(y)) = 1$. Since F is an obstinate filter and obtain $x \to y \in F$ and $y \to x \in F$, then $\chi_F(x \to y) \wedge \chi_F(y \to x) = 1$. Hence,

$$(1 - \chi_F(x)) \wedge (1 - \chi_F(y)) \leq \chi_F(x \to y) \wedge \chi_F(y \to x).$$

Conversely, if χ_F is a fuzzy obstinate filter, then we show that F is an obstinate filter of A. Let $x, y \notin F$. Then $\chi_F(x) = 0 = \chi_F(y)$. Since χ_F is a fuzzy obstinate filter, thus

$$1 = (1 - \chi_F(x)) \wedge (1 - \chi_F(y)) \leq \chi_F(x \to y) \wedge \chi_F(y \to x).$$

We obtain $\chi_F(x \to y) = \chi_F(y \to x) = 1$. Therefore, $x \to y \in F$ and $y \to x \in F$. □

Theorem 3.4. *Let μ be a fuzzy obstinate filter of A for any $\alpha \in [0, 1/2]$. Then level subset $\mu_\alpha = \{x \in A | \mu(x) \geq \alpha\}$ is an obstinate filter of A.*

Proof. Let μ be a fuzzy obstinate filter of A. If $\alpha = 0$, then $\mu_\alpha = A$ and μ_α is an obstinate filter of A.

Let $\alpha \in (0, 1/2]$ and $x, y \notin \mu_\alpha$. Then $\mu(x) < \alpha$ and $1 - \alpha < 1 - \mu(x)$. Also $\mu(y) < \alpha$ and $1 - \alpha < 1 - \mu(y)$. Since μ is a fuzzy obstinate filter, thus

$$1 - \alpha < (1 - \mu(x)) \wedge (1 - \mu(y)) \leq \mu(x \to y) \wedge \mu(y \to x).$$

Since $\alpha \in (0, \frac{1}{2}]$, we obtain $\alpha < \mu(x \to y)$ and $\alpha < \mu(y \to x)$. Hence,

$$x \to y \in \mu_\alpha \quad \text{and} \quad y \to x \in \mu_\alpha$$

Therefore, level subset μ_α is an obstinate filter of A. □

In the following example, we show that the aforementioned theorem for $\alpha \in (1/2, 1]$, and the converse of Theorem 3.4, is not true, in general.

Example 3.5.
(a) In Example 3.2(i) part(2), let $\alpha = 1/3$. Then $\lambda_{1/3} = A$ and $\lambda_{1/3}$ is an obstinate filter.
(b) In Example 3.2(ii), let $\alpha = 3/4$. Then $\mu_{3/4} = \{1, a, b\}$ and $0 = b \to a \notin \mu_{3/4}$. Hence, $\mu_{3/4}$ is not an obstinate filter.
(c) In Example 3.2(i), let $\mu(a) = \mu(0) = 1/3$, $\mu(b) = \mu(1) = 2/3$, and $\alpha = 1/3$. Then $\mu_\alpha = \{0, a, b, 1\}$ is an obstinate filter, but μ is not fuzzy obstinate filter. Because $(1 - \mu(a)) \wedge (1 - \mu(0)) > \mu(a \to 0) \wedge \mu(0 \to a)$.

Theorem 3.6. *Let μ be a fuzzy filter of A. Then μ is a fuzzy obstinate filter if and only if for any $x \in A$, $1 - \mu(x) \leq \mu(x')$.*

Proof. Let for any $x \in A$, $1 - \mu(x) \leq \mu(x')$. Then $1 - \mu(y) \leq \mu(y')$. Hence, $1 - \mu(x) \leq \mu(x \to 0)$ and $1 - \mu(y) \leq \mu(y \to 0)$. By Proposition 2.3(iv), $x \to 0 \leq x \to y$ and $y \to 0 \leq y \to x$. Since μ is a fuzzy filter, $\mu(x \to 0) \leq \mu(x \to y)$ and $\mu(y \to 0) \leq \mu(y \to x)$. So $1 - \mu(x) \leq \mu(x \to y)$ and $1 - \mu(y) \leq \mu(y \to x)$. Hence,

$$(1 - \mu(x)) \wedge (1 - \mu(y)) \leq \mu(x \to y) \wedge \mu(y \to x).$$

Then μ is a fuzzy obstinate filter of A.

Conversely, let μ be a fuzzy obstinate filter of A. Then for any $x, y \in A$, $(1 - \mu(x)) \wedge (1 - \mu(y)) \leq \mu(x \to y) \wedge \mu(y \to x)$. If $y = 0$, then $1 - \mu(x) \leq \mu(x \to 0)$. Therefore, $1 - \mu(x) \leq \mu(x')$. □

In the following example, we give an example for Theorem 3.6.

Example 3.7. In Example 3.2(ii), we have $1 - \mu(0) \leq \mu(0')$, $1 - \mu(a) \leq \mu(a')$, $1 - \mu(b) \leq \mu(b')$, $1 - \mu(c) \leq \mu(c')$, and $1 - \mu(1) \leq \mu(1')$. Hence, for any $x \in A$, $1 - \mu(x) \leq \mu(x')$.

Definition 3.8. Let A and B be two hoop algebras and μ be a fuzzy subset of A and λ be a fuzzy subset of B such that $f : A \to B$ be a hoop homomorphism. Then image μ under f denoted by $f(\mu)$ is a fuzzy subset of B that for any $y \in B$:

$$f(\mu)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \varnothing \\ 0 & \text{if } f^{-1}(y) = \varnothing. \end{cases}$$

The preimage of λ under f denoted by $f^{-1}(\lambda)$ is a fuzzy set of A denoted by for any $x \in A$, $f^{-1}(\lambda)(x) = \lambda(f(x))$.

Definition 3.9. [1] Fuzzy subset μ of A has sup property if for any nonempty subset Y of A, there exists $y_0 \in Y$ such that $\mu(y_0) = \sup_{y\in Y}\mu(y)$.

Example 3.10. In Example 3.2(i), let $Y_1 = \{0, a\}$, $a \in Y_1$ such that $\mu(a) = \sup_{y\in Y_1}\mu(y)$. Also $Y_2 = \{0, b\}$, $b \in Y_2$ such that $\mu(b) = \sup_{y\in Y_2}\mu(y)$.

Proposition 3.11.
(i) *Let $f : A \to B$ be an onto hoop homomorphism. Then preimage of a fuzzy obstinate filter λ under f is a fuzzy obstinate filter of A.*
(ii) *Let $f : A \to B$ be an onto hoop homomorphism and μ be a fuzzy obstinate filter of A with sup property. Then $f(\mu)$ is a fuzzy obstinate filter of B.*

Proof.
(i) It is clear.
(ii) We show that

$$(1 - f(\mu)(y_1)) \wedge (1 - f(\mu)(y_2)) \leq f(\mu)(y_1 \to y_2) \wedge f(\mu)(y_2 \to y_1).$$

Let $y_1, y_2 \in B$ and $x_1 \in f^{-1}(y_1)$, $x_2 \in f^{-1}(y_2)$ such that $1 - \mu(x_1) = 1 - \sup_{a\in f^{-1}(y_1)}\mu(a)$ and $1 - \mu(x_2) = 1 - \sup_{a\in f^{-1}(y_2)}\mu(a)$. Then $f(\mu)(y_1 \to y_2) = \sup_{a\in f^{-1}(y_1\to y_2)}\mu(a) \geq \mu(x_1 \to x_2)$ and $f(\mu)(y_2 \to y_1) = \sup_{a\in f^{-1}(y_2\to y_1)}\mu(a) \geq \mu(x_2 \to x_1)$. Hence,

$$\begin{aligned}&(f(\mu)(y_1 \to y_2)) \wedge (f(\mu)(y_2 \to y_1))\\ \geq&(\mu(x_1 \to x_2)) \wedge (\mu(x_2 \to x_1))\\ \geq&(1 - \mu(x_1)) \wedge (1 - \mu(x_2)).\end{aligned}$$

Also, $(1 - \mu(x_1)) \wedge (1 - \mu(x_2)) = (1 - f(\mu)(y_1)) \wedge (1 - f(\mu)(y_2))$. Therefore, $f(\mu)$ is a fuzzy obstinate filter of B. □

Theorem 3.12. *Let μ, λ be two fuzzy filters of A, which satisfy $\mu \subseteq \lambda$, $\mu(1) = \lambda(1)$. If μ is a fuzzy obstinate filter, then λ is a fuzzy obstinate filter of A.*

Proof. Let μ be a fuzzy obstinate filter and λ is not fuzzy obstinate filter of A. Then by Theorem 3.6, $1 - \mu(x) \leq \mu(x')$. Also $\mu(x) \leq \lambda(x)$ and $\mu(x') \leq \lambda(x')$. Hence, $1 - \lambda(x) \leq 1 - \mu(x) \leq \mu(x') \leq \lambda(x')$. Therefore, λ is a fuzzy obstinate filter. □

Definition 3.13. Let μ be a nonconstant fuzzy filter of $\vee$-hoop A. Then μ is called a fuzzy prime filter of A if for each $\alpha \in [0, 1]$, $\mu_\alpha = \emptyset$ or μ_α is a prime filter of A.

Example 3.14. In Example 3.2(ii), let $\mu(a) = \mu(c) = \mu(0) = 1/5$, $\mu(b) = \mu(1) = 4/5$. Then for $\alpha = 3/5$, μ_α is a prime filter.

Proposition 3.15. *Let μ be a nonconstant fuzzy filter of $\vee$-hoop A. Then μ is a fuzzy prime filter of A if and only if $\mu(x \vee y) \leq \mu(x) \vee \mu(y)$, for any $x, y \in A$.*

Proof. Let μ be a fuzzy prime filter of A and for any $x, y \in A$, $\alpha = \mu(x \vee y)$. Then μ_α is a filter of A and $x \vee y \in \mu_\alpha$. If $\mu_\alpha = A$, then $x, y \in \mu_\alpha$, and thus, $\mu(x \vee y) \leq \mu(x) \vee \mu(y)$. If $\mu_\alpha \neq A$, since μ_α is a prime filter, then $x \vee y \in \mu_\alpha$ implies $x \in \mu_\alpha$ or $y \in \mu_\alpha$. Hence, $\alpha \leq \mu(x)$ or $\alpha \leq \mu(y)$. Thus, $\mu(x \vee y) \leq \mu(x) \vee \mu(y)$.

Conversely, for any $\alpha \in [0, 1]$, if $\mu_\alpha \neq \emptyset$, then μ_α is a filter. If $\mu_\alpha \neq A$ and $x \vee y \in \mu_\alpha$, then $\alpha \leq \mu(x \vee y)$. So $\alpha \leq \mu(x) \vee \mu(y)$. Hence, $x \in \mu_\alpha$ or $y \in \mu_\alpha$. Therefore, μ is a fuzzy prime filter. □

Corollary 3.16. *Let μ be a nonconstant fuzzy filter of $\vee$-hoop A. Then μ is a fuzzy prime filter of A if and only if $\mu(x \vee y) = \mu(x) \vee \mu(y)$, for any $x, y \in A$.*

Proof. Since fuzzy filter μ is order-preserving, and $x, y \leq x \vee y$, hence $\mu(x) \vee \mu(y) \leq \mu(x \vee y)$. □

Proposition 3.17.

(*i*) *Let μ be a nonconstant fuzzy prime filter of $\vee$-hoop A. Then $\mu_{\mu(1)}$ is a prime filter of A.*

(*ii*) *If A is a prelinear $\vee$-hoop and $\mu_{\mu(1)}$ is a prime filter of A, then μ is a fuzzy prime filter of A.*

Proof.

(i) Let μ be a nonconstant fuzzy filter. Then $\mu(0) < \mu(1)$ and $0 \notin \mu_{\mu(1)}$. Hence, $\mu_{\mu(1)}$ is a prime filter of A.

(ii) Since $(x \to y) \vee (y \to x) = 1 \in \mu_{\mu(1)}$, then $x \to y \in \mu_{\mu(1)}$ or $y \to x \in \mu_{\mu(1)}$ for any $x, y \in A$. By Proposition 2.3(v), $(x \vee y) \to y = x \to y \in \mu_{\mu(1)}$ or $(x \vee y) \to x = y \to x \in \mu_{\mu(1)}$. Hence, $\mu((x \vee y) \to y) = \mu(1)$ or $\mu((x \vee y) \to x) = \mu(1)$. We obtain $\mu(y) \geq \mu((x \vee y) \to y) \wedge \mu(x \vee y) = \mu(x \vee y)$ or $\mu(x) \geq \mu((x \vee y) \to x) \wedge \mu(x \vee y) = \mu(x \vee y)$. So $\mu(x) \vee \mu(y) \geq \mu(x \vee y)$. Therefore, by Proposition 3.15, μ is a fuzzy prime filter. □

Corollary 3.18.

(*i*) *Let A be a prelinear $\vee$-hoop. Then $\mu_{\mu(1)}$ is a prime filter of A if and only if μ is a fuzzy prime filter of A.*

(*ii*) *Let A be a prelinear $\vee$-hoop. Then a nonempty subset F is a prime filter of A if and only if χ_F is a fuzzy prime filter of A.*

Theorem 3.19.

(*i*) *Let μ be a fuzzy prime filter of $\vee$-hoop A. Then $\mu(x \to y) = \mu(1)$ or $\mu(y \to x) = \mu(1)$.*

(*ii*) *Let A be a prelinear $\vee$-hoop and $\mu(x \to y) = \mu(1)$ or $\mu(y \to x) = \mu(1)$. Then μ is a fuzzy prime filter of A.*

Proof.

(i) By Proposition 3.17(i), $\mu_{\mu(1)}$ is a prime filter, then $x \to y \in \mu_{\mu(1)}$ or $y \to x \in \mu_{\mu(1)}$ for any $x, y \in A$. Hence, $\mu(x \to y) = \mu(1)$ or $\mu(y \to x) = \mu(1)$.

(ii) Let A be a prelinear $\vee$-hoop and $\mu(x \to y) = \mu(1)$ or $\mu(y \to x) = \mu(1)$. Then $x \to y \in \mu_{\mu(1)}$ or $y \to x \in \mu_{\mu(1)}$. Hence, $\mu_{\mu(1)}$ is a prime filter. By Proposition 3.17(ii), μ is a fuzzy prime filter. □

Proposition 3.20. *Let A be a $\vee$-hoop and μ be a fuzzy prime filter of A and $\alpha \in [0, \mu(1))$. Then $\mu \vee \alpha$ is a fuzzy prime filter of A, where $(\mu \vee \alpha)(x) = \mu(x) \vee \alpha$, for any $x \in A$.*

Proof. Let $x, y, z \in A$ and $x \odot y \leq z$. Then $\mu(x) \wedge \mu(y) \leq \mu(z)$ and $(\mu \vee \alpha)(z) \geq (\mu(x) \wedge \mu(y)) \vee \alpha = (\mu \vee \alpha)(x) \wedge (\mu \vee \alpha)(y)$. Hence, $\mu \vee \alpha$ is a fuzzy filter of A. Since $\alpha < \mu(1)$, then $(\mu \vee \alpha)(1) = \mu(1) \vee \alpha = \mu(1) \neq (\mu \vee \alpha)(0)$, so $\mu \vee \alpha$ is a nonconstant fuzzy filter. Then $(\mu \vee \alpha)(x \vee y) = \mu(x \vee y) \vee \alpha = \mu(x) \vee \mu(y) \vee \alpha = (\mu \vee \alpha)(x) \vee (\mu \vee \alpha)(y)$. Therefore, by Corollary 3.16, $\mu \vee \alpha$ is a fuzzy prime filter. □

Theorem 3.21. *Let μ be a nonconstant fuzzy filter of $\vee$-hoop A. Then there is a fuzzy prime filter λ of A such that $\mu \leq \lambda$.*

Proof. If μ is a nonconstant fuzzy filter, then we have $\mu_{\mu(1)}$ is a proper filter of A. So there is a prime filter F such that $\mu_{\mu(1)} \subseteq F$. By Corollary 3.18 (ii), χ_F is a fuzzy prime filter. Let $\lambda = \chi_F \vee \alpha$ and $\alpha = \vee_{x \in A - F}\, \mu(x)$. Then λ is a fuzzy prime filter and $\mu \leq \lambda$. □

Theorem 3.22. *Let μ and λ be two fuzzy filters of A, which satisfy $\mu \subseteq \lambda$, $\mu(1) = \lambda(1)$. If μ is a fuzzy prime filter, then λ is a fuzzy prime filter of A.*

Proof. Let μ be a fuzzy prime filter. Then $\mu(x \to y) = \mu(1)$ or $\mu(y \to x) = \mu(1)$ for any $x, y \in A$. If $\mu(x \to y) = \mu(1)$, by $\mu \leq \lambda$ and $\mu(1) = \lambda(1)$, then $\lambda(x \to y) = \lambda(1)$. Similarly, if $\mu(y \to x) = \mu(1)$, then $\lambda(y \to x) = \lambda(1)$. Therefore, λ is a fuzzy prime filter. □

Proposition 3.23. *Every fuzzy obstinate filter μ of A such that $\mu(x) < 1/2$ is a fuzzy implicative filter, for any $x \in A$.*

Proof. Let μ be a fuzzy obstinate filter and $x \in A$ such that $\mu(x) < 1/2$. Since μ is a fuzzy filter, then

$$\mu((x)' \to x) \wedge \mu((x)') \leq \mu(x).$$

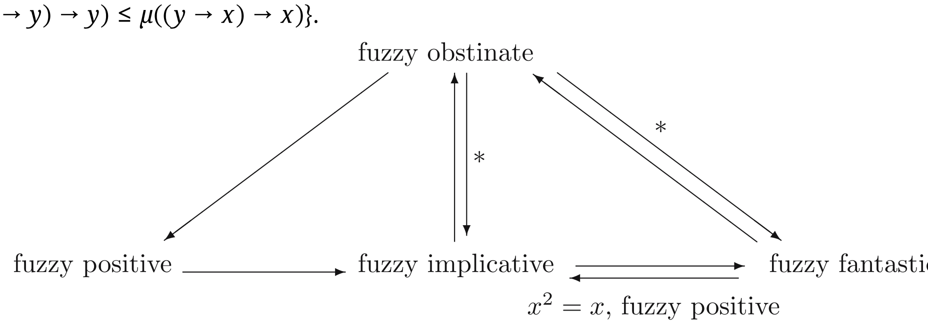
By hypothesis and Theorem 3.6, $\mu((x)' \to x) \wedge (1 - \mu(x)) \leq \mu(x)$. If $\mu((x)' \to x) \wedge (1 - \mu(x)) = 1 - \mu(x)$, then $\mu(x) \geq 1/2$, which is the hypothesis. So $\mu((x)' \to x) \wedge (1 - \mu(x)) = \mu((x)' \to x)$ and $\mu((x)' \to x) \leq \mu(x)$. Therefore, by Theorem 2.8(i), μ is a fuzzy implicative filter. □

Corollary 3.24. *Let A be a $\vee$-hoop with (DNP) and μ be a fuzzy obstinate filter of A and $\alpha \in [0, 1/2]$. Then A/μ is a Boolean algebra.*

Proposition 3.25. *Let μ be a fuzzy obstinate filter of A and $\alpha \in [0, \frac{1}{2}]$. Then:*
(*i*) *μ is a fuzzy positive implicative filter of A.*
(*ii*) *μ is a fuzzy fantastic filter of A.*
(*iii*) *μ is a fuzzy prime filter of A.*

Proof.
(i) By Proposition 2.5, Theorem 2.8(ii), and Theorem 3.4, it is clear.
(ii) By Proposition 2.5 and Theorem 2.8(ii), it is clear.
(iii) By Proposition 2.5, it is clear. □

In diagram (1), let μ be a fuzzy filter on A and $x \in A$. We show the relationship between fuzzy (positive implicative, implicative, fantastic, obstinate) filters on hoop algebra, where the condition
{*:$(x' = 0, 1 - \mu(0) \leq \mu(1))$}
{**:$\mu((x \to y) \to y) \leq \mu((y \to x) \to x)$}.



diagram(1)

4 Fuzzy *n*-fold obstinate filter

In this section, we introduce the notion of fuzzy n-fold obstinate filter on hoop algebra and investigate some properties of them.

Definition 4.1. A fuzzy filter μ of A is a fuzzy n-fold obstinate filter if for any $x, y \in A$ and $n \in \mathbb{N}$:

$$(1 - \mu(x)) \wedge (1 - \mu(y)) \leq \mu(x^n \to y) \wedge \mu(y^n \to x).$$

In particular, fuzzy 1-fold obstinate filters are fuzzy obstinate filters.

In the following example, we show that any fuzzy filter may not be fuzzy n-fold obstinate filter.

Example 4.2. Let $A = \{0, a, b, c, d, 1\}$ and define the operations $\odot$ and $\to$ on A as follows:

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	c	b	1
b	d	a	1	b	a	1
c	a	a	1	1	a	1
d	b	1	1	b	1	1
1	0	a	b	c	d	1

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	d	0	d	a
b	0	d	c	c	0	b.
c	0	0	c	c	0	c
d	0	d	0	0	0	d
1	0	a	b	c	d	1

Then $(A, \odot, \to, 1, 0)$ is a bounded hoop algebra.

(i) It is easy to see that $\mu(x) = 1/2$ for any $x \in A$ and is a fuzzy n-fold obstinate filter.

(ii) Let $\mu(0) = \mu(a) = \mu(c) = \mu(b) = \mu(d) = 1/3, \mu(1) = 2/3$. Then $\mu(x)$ is a fuzzy filter, but is not a fuzzy two-fold obstinate filter. Because $(1 - \mu(b)) \wedge (1 - \mu(0)) = 2/3 = (1 - 1/3) \wedge (1 - 1/3) \nleq \mu(a) \wedge \mu(1) = \mu(b^2 \to 0) \wedge \mu(0^2 \to b)$.

Theorem 4.3. *Let μ be a fuzzy filter of A. Then μ is a fuzzy n-fold obstinate filter if and only if for any $x \in A$, $1 - \mu(x) \leq \mu((x^n)')$.*

Proof. Let for any $x \in A$, $1 - \mu(x) \leq \mu((x^n)')$. Then $1 - \mu(y) \leq \mu((y^n)')$. Hence, $1 - \mu(x) \leq \mu(x^n \to 0)$ and $1 - \mu(y) \leq \mu(y^n \to 0)$. By Proposition 2.3(iv), $x^n \to 0 \leq x^n \to y$ and $y^n \to 0 \leq y^n \to x$. Since μ is a fuzzy filter, by Theorem 2.8(iv), $\mu(x^n \to 0) \leq \mu(x^n \to y)$, and $\mu(y^n \to 0) \leq \mu(y^n \to x)$. So, $1 - \mu(x) \leq \mu(x^n \to y)$ and $1 - \mu(y) \leq \mu(y^n \to x)$. Hence,

$$(1 - \mu(x)) \wedge (1 - \mu(y)) \leq \mu(x^n \to y) \wedge \mu(y^n \to x).$$

Thus μ is a fuzzy n-fold obstinate filter of A.

Conversely, let μ be a fuzzy n-fold obstinate filter of A. Then for any $x, y \in A$, $(1 - \mu(x)) \wedge (1 - \mu(y)) \leq \mu(x^n \to y) \wedge \mu(y^n \to x)$. If $y = 0$, then

$$1 - \mu(x) = (1 - \mu(x)) \wedge (1 - \mu(0)) \leq \mu(x^n \to 0).$$

Therefore, $1 - \mu(x) \leq \mu(x^n)'$. □

Theorem 4.4. (Extension theorem of fuzzy n-fold obstinate filters) *Let μ, λ be two fuzzy filters of A such that $\mu \subseteq \lambda$. If μ is a fuzzy n-fold obstinate filter, then λ is a fuzzy n-fold obstinate filter of A.*

Proof. Let μ be a fuzzy n-fold obstinate filter such that $\mu \subseteq \lambda$. Then by Theorem 4.3, $1 - \mu(x) \leq \mu((x^n)')$. Also $\mu(x) \leq \lambda(x)$ and $\mu((x^n)') \leq \lambda((x^n)')$. It follows that

$$1 - \lambda(x) \leq 1 - \mu(x) \leq \mu((x^n)') \leq \lambda((x^n)').$$

Hence, $1 - \lambda(x) \leq \lambda((x^n)')$, for all $x \in A$. Therefore, λ is a fuzzy n-fold obstinate filter of A. □

Theorem 4.5. *If μ is a fuzzy n-fold obstinate filter of A, then μ is a fuzzy $(n + 1)$-fold obstinate filter of A.*

Proof. Let μ be a fuzzy n-fold obstinate filter of A. Then

$$(1 - \mu(x)) \wedge (1 - \mu(y)) \leq \mu(x^n \to y) \wedge \mu(y^n \to x).$$

We show that

$$(1 - \mu(x)) \wedge (1 - \mu(y)) \leq (\mu(x^{n+1} \to y)) \wedge (\mu(y^{n+1} \to x)).$$

By Proposition 2.3(iii,iv), $x^n \to y \leq x^{n+1} \to y$ and $y^n \to x \leq y^{n+1} \to x$. Since μ is a fuzzy filter, thus $\mu(x^n \to y) \leq \mu(x^{n+1} \to y)$ and $\mu(y^n \to x) \leq \mu(y^{n+1} \to x)$. By hypothesis,

$$(1 - \mu(x)) \wedge (1 - \mu(y)) \leq \mu(x^{n+1} \to y) \wedge \mu(y^{n+1} \to x).$$

Therefore, μ is a fuzzy $(n+1)$-fold obstinate filter of A. □

By finite induction, we can prove that every fuzzy n-fold obstinate filter is a fuzzy $(n+k)$-fold obstinate filter for any integer $k \geq 0$.

The following example shows that any fuzzy $(n+1)$- fold obstinate filter may not be a fuzzy n-fold obstinate filter of A.

Example 4.6. Let $(A = \{0, a, b, 1\}, \leq)$ be a chain. Define the operations $\odot$ and $\rightharpoonup$ on A as follows:

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a.
b	0	a	b	b
1	0	a	b	1

Then $(A, \odot, \to, 1, 0)$ is a bounded hoop algebra. Define a fuzzy set on A by $\mu(b) = 8/10 = \mu(1), \mu(0) = 3/10 = \mu(a)$. By using Theorem 4.3, for $n = 2$, it is easy to see that μ is a fuzzy two-fold obstinate filter of A but is not a fuzzy 1-fold obstinate filter of A. Because $7/10 = 1 - \mu(a) \nleq \mu(a') = \mu(a) = 3/10$.

Proposition 4.7. *Any nonempty subset F of A is an n-fold obstinate filter if and only if the characteristic function χ_F is a fuzzy n-fold obstinate filter.*

Proof. Assume that F is an n-fold obstinate filter of A. We show that χ_F is a fuzzy n-fold obstinate filter.

Let $x, y \in A$. We show that

$$(1 - \chi_F(x)) \wedge (1 - \chi_F(y)) \leq \chi_F(x^n \to y) \wedge \chi_F(y^n \to x).$$

Case one: If $x \in F$ or $y \in F$, then $(1 - \chi_F(x)) \wedge (1 - \chi_F(y)) = 0$ and

$$(1 - \chi_F(x)) \wedge (1 - \chi_F(y)) \leq \chi_F(x^n \to y) \wedge \chi_F(y^n \to x).$$

Case two: If $x \notin F$ and $y \notin F$, then $(1 - \chi_F(x)) \wedge (1 - \chi_F(y)) = 1$. Since F is an n-fold obstinate filter and obtain $x^n \to y \in F$ and $y^n \to x \in F$, then $(\chi_F(x^n \to y)) \wedge (\chi_F(y^n \to x)) = 1$. Hence,

$$(1 - \chi_F(x)) \wedge (1 - \chi_F(y)) \leq \chi_F(x^n \to y) \wedge \chi_F(y^n \to x).$$

Conversely, if χ_F is a fuzzy n-fold obstinate filter, then we show that F is an n-fold obstinate filter of A. Let $x, y \notin F$. Then $\chi_F(x) = 0 = \chi_F(y)$. Since χ_F is an n-fold fuzzy obstinate filter, thus

$$1 = (1 - \chi_F(x)) \wedge (1 - \chi_F(y)) \leq (\chi_F(x^n \to y)) \wedge (\chi_F(y^n \to x)).$$

We obtain $\chi_F(x^n \to y) = \chi_F(y^n \to x) = 1$. Therefore, $x^n \to y \in F$ and $y^n \to x \in F$. □

Theorem 4.8.

(i) *Let μ be a fuzzy n-fold obstinate filter of A for any $\alpha \in [0, 1/2]$. Then level subset $\mu_\alpha = \{x \in A | \mu(x) \geq \alpha\}$ is an n-fold obstinate filter of A.*

(ii) *If $\mu_\alpha \neq \emptyset$ is an n-fold obstinate filter and $\alpha \in (1/2, 1]$, then μ is a fuzzy n-fold obstinate filter of A.*

Proof. (i) Let μ be a fuzzy n-fold obstinate filter of A. If $\alpha = 0$, then $\mu_\alpha = A$ and μ_α is an n-fold obstinate filter of A.

Let $\alpha \in (0, 1/2]$ and $x, y \notin \mu_\alpha$. Then $\mu(x) < \alpha$ and $1 - \alpha < 1 - \mu(x)$. Also $\mu(y) < \alpha$ and $1 - \alpha < 1 - \mu(y)$. Hence,

$$1-\alpha<1-\mu(x) \quad \text{and} \quad 1-\alpha<1-\mu(y).$$

Since μ is a fuzzy n-fold obstinate filter, thus

$$1-\alpha<(1-\mu(x))\wedge(1-\mu(y))\leq\mu(x^n\to y)\wedge\mu(y^n\to x).$$

Since $\alpha\in(0,1/2]$, we obtain $\alpha\leq\mu(x^n\to y)$ and $\alpha<\mu(y^n\to x)$. Hence,

$$x^n\to y\in\mu_\alpha \quad \text{and} \quad y^n\to x\in\mu_\alpha$$

Therefore, level subset μ_α is an n-fold obstinate filter of A.

(ii) Assume that for every $\alpha\in(1/2,1]$, $\mu_\alpha\neq\varnothing$ is an n-fold obstinate filter of A. We will prove that μ is a fuzzy n-fold obstinate filter of A. It is easy to prove that for all $x\in A$, $\mu(x)\leq\mu(1)$. Let $x,y\in A$. Then we show that $(1-\mu(x))\wedge(1-\mu(y))\leq\mu(x^n\to y)\wedge\mu(y^n\to x)$. If not, there exist $a,b\in A$ such that

$$\mu(a^n\to b)\wedge\mu(b^n\to a)<(1-\mu(a))\wedge(1-\mu(b)).$$

Let $\alpha_0=1/2(\{\mu(a^n\to b)\wedge\mu(b^n\to a)\}+\{(1-\mu(a))\wedge(1-\mu(b))\})$.

Then $\mu(a^n\to b)\wedge\mu(b^n\to a)<\alpha_0<(1-\mu(a))\wedge(1-\mu(b))$. We conclude that $\mu(a^n\to b)<\alpha_0$ or $\mu(b^n\to a)<\alpha_0$. Also $\alpha_0<1-\mu(a)$ and $\alpha_0<1-\mu(b)$. We consider two cases:

Case 1: If $\alpha_0>1/2$, then we conclude that $\mu(a)<1-\alpha_0<\alpha_0$ and $\mu(b)<1-\alpha_0<\alpha_0$. So $a\notin\mu_{\alpha_0}$ and $b\notin\mu_{\alpha_0}$. Also, since $\mu(a^n\to b)<\alpha_0$ or $\mu(b^n\to a)<\alpha_0$, hence $a^n\to b\notin\mu_{\alpha_0}$ or $b^n\to a\notin\mu_{\alpha_0}$. Since μ_{α_0} is an n-fold obstinate filter of A, which is contradiction.

Case 2: If $\alpha_0\leq1/2$, since $1/2\leq1-\alpha_0$, then $\mu(a^n\to b)<\alpha_0\leq1/2\leq1-\alpha_0$ or $\mu(b^n\to a)<\alpha_0\leq\frac{1}{2}\leq1-\alpha_0$. Also, $\mu(a)<1-\alpha_0$ and $\mu(b)<1-\alpha_0$, then $a\notin\mu_{1-\alpha_0}$ and $b\notin\mu_{1-\alpha_0}$. Hence, $a^n\to b\notin\mu_{1-\alpha_0}$ or $b^n\to a\notin\mu_{1-\alpha_0}$. Since $\mu_{1-\alpha_0}$ is an n-fold obstinate filter of A, which is contradiction. □

Corollary 4.9. *Let μ be a fuzzy n-fold obstinate filter of A and for any $\mu(1)\in[0,1/2]$. Then level subset $I=\{x\in A|\mu(x)=\{1\}\}$ is an n-fold obstinate filter of A.*

Definition 4.10. Let A and B be two hoop algebras and μ be a fuzzy subset of A and λ a fuzzy subset of B such that $f:A\to B$ is a hoop homomorphism. Then image μ under f denoted by $f(\mu)$ is a fuzzy set of B that for any $y\in B$: $f(\mu)(y)=\sup_{x\in f^{-1}(y)}\mu(x)$ if $f^{-1}(y)\neq\varnothing$ and $f(\mu)(y)=0$ if $f^{-1}(y)=\varnothing$.

The preimage of λ under f denoted by $f^{-1}(\lambda)$ is a fuzzy set of A denoted by for any $x\in A$, $f^{-1}(\lambda)(x)=\lambda(f(x))$.

Definition 4.11. Fuzzy subset μ of A has sup property if for any nonempty subset Y of A, and there exists $y_0\in Y$ such that $\mu(y_0)=\sup_{y\in Y}\mu(y)$.

Example 4.12. In Example 4.6, let $Y_1=\{0,a\}$, $a\in Y_1$ such that $\mu(a)=\sup_{y\in Y_1}\mu(y)$. Also $Y_2=\{0,b\}$, $b\in Y_2$ such that $\mu(b)=\sup_{y\in Y_2}\mu(y)$.

Proposition 4.13.

(*i*) *Let $f:A\to B$ be an onto hoop homomorphism algebra. Then preimage of a fuzzy n-fold obstinate filter μ under f is a fuzzy n-fold obstinate filter of A.*

(*ii*) *Let $f:A\to B$ be an onto hoop homomorphism algebra and μ be a fuzzy n-fold obstinate filter of A with sup property. Then $f(\mu)$ is a fuzzy n-fold obstinate filter of B.*

Proof. (i) Let μ be fuzzy n-fold obstinate filter of B. Then for any $x,y\in A$, we have

$(1-f^{-1}(\mu)(x))\wedge(1-f^{-1}(\mu)(y)) = (1-\mu(f(x)))\wedge(1-\mu(f(y)))\leq\mu(f(x^n\to y))\wedge(\mu(f(y^n\to x))) = (f^{-1}(\mu)(x^n\to y))\wedge(f^{-1}(\mu)(y^n\to x))$. Therefore, $f^{-1}(\mu)$ is a fuzzy n-fold obstinate filter of A.

(ii) We show that

$$(1-f(\mu)(y_1))\wedge(1-f(\mu)(y_2))\leq(f(\mu)(y_1^n\to y_2))\wedge(f(\mu)(y_2^n\to y_1)).$$

Let $y_1, y_2 \in B$ and $x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)$ such that

$$1 - \mu(x_1) = 1 - \sup_{\alpha \in f^{-1}(y_1)} \mu(\alpha), 1 - \mu(x_2) = 1 - \sup_{\alpha \in f^{-1}(y_2)} \mu(\alpha).$$

Then

$$f(\mu)(y_1^n \to y_2) = \sup_{\alpha \in f^{-1}(y_1^n \to y_2)} \mu(\alpha) \geq \mu(x_1^n \to x_2),$$

$$f(\mu)(y_2^n \to y_1) = \sup_{\alpha \in f^{-1}(y_2^n \to y_1)} \mu(\alpha) \geq \mu(x_2^n \to x_1).$$

Hence,

$$(f(\mu)(y_1^n \to y_2)) \wedge (f(\mu)(y_2^n \to y_1)) \geq (\mu(x_1^n \to x_2)) \wedge (\mu(x_2^n \to x_1)) \geq (1 - \mu(x_1)) \wedge (1 - \mu(x_2)).$$

Also,

$$(1 - \mu(x_1)) \wedge (1 - \mu(x_2)) = (1 - f(\mu)(y_1)) \wedge (1 - f(\mu)(y_2)).$$

Therefore, $f(\mu)$ is a fuzzy n-fold obstinate of B. □

Proposition 4.14. *Let A be an n-fold obstinate hoop algebra and μ be a fuzzy filter.*
(i) *If for any $x \in A, \mu(x) \leq 1/2 \leq \mu(1)$, then any fuzzy filter is a fuzzy n-fold obstinate filter of A.*
(ii) *A/μ is an n-fold obstinate hoop algebra.*

Proof. (i) Let A be an n-fold obstinate hoop algebra and $\mu(x) \leq 1/2 \leq \mu(1)$. Then for any $x, y \in A$, $(1 - \mu(x)) \wedge (1 - \mu(y)) \leq 1/2$ and $\mu(x^n \to y) \wedge \mu(y^n \to x) = \mu(1) \geq 1/2$. Hence, $(1 - \mu(x)) \wedge (1 - \mu(y)) \leq 1/2 \leq \mu(x^n \to y) \wedge \mu(y^n \to x)$. Therefore, μ is a fuzzy n-fold obstinate filter of A.

(ii) Let A be an n-fold obstinate hoop algebra. Then for any $1 \neq x \in A, x^n = 0$. Hence, for any $x \neq 1, ((\mu^x)^n) = \mu^{x^n} = \mu^0$. Therefore, A/μ is an n-fold obstinate hoop algebra. □

Corollary 4.15. *By Example* 4.2(i), *condition n-fold obstinate for hoop A is necessary in Proposition* 4.14.

5 Fuzzy *n*-fold implicative filter

In this section, we introduce the notion of fuzzy n-fold implicative filter on hoop algebra and investigate some properties of them. We obtained some equivalent conditions for fuzzy n-fold implicative filters.

Definition 5.1. Fuzzy subset μ of A is called *fuzzy n-fold implicative filter* of A if $\mu(x) \leq \mu(1)$ and

$$\mu(x \to ((y^n \to z) \to y)) \wedge \mu(x) \leq \mu(y).$$

Example 5.2. In Example 4.6, let $\mu(0) = \mu(a) = \mu(b) = \mu(c) = \mu(d) = 1/3, \mu(1) = 1/2$. Then μ is a fuzzy two-fold implicative filter

Proposition 5.3. *Every fuzzy n-fold implicative filter of A is a fuzzy filter of A.*

Proof. Let μ be a fuzzy n-fold implicative filter of A. Then $\mu(x) \leq \mu(1)$. If $z = 1$, then $\mu(x \to ((y^n \to 1) \to y)) \wedge \mu(x) \leq \mu(y)$. Then $\mu(x \to y) \wedge \mu(x) \leq \mu(y)$. □

Theorem 5.4. *μ is a fuzzy n-fold implicative filter of A if and only if $\mu_\alpha \neq \varnothing$ is an n-fold implicative filter of A, for any $\alpha \in [0, 1]$.*

Proof. Let μ be a fuzzy n-fold implicative filter of A and $x, x \to ((y^n \to z) \to y) \in \mu_\alpha$. Then $\mu(x) \geq \alpha$ and $\mu(x \to ((y^n \to z) \to y)) \geq \alpha$, so $\mu(y) \geq \alpha$. Hence, $y \in \mu_\alpha$ and $\mu_\alpha \neq \varnothing$ is an n-fold implicative filter of A. The proof of converse is similar. □

Theorem 5.5. *Let μ be a fuzzy filter of A. Then for any $x, y \in A$, the following conditions are equivalent*:
(*i*) *μ is a fuzzy n-fold implicative filter of A,*
(*ii*) $\mu((x^n \to y) \to x) \leq \mu(x)$,
(*iii*) $\mu(((x^n \to y) \to x) \to x) = \mu(1)$,
(*iv*) $\mu(((x^n)' \to x) \to x) = \mu(1)$.

Proof. (*i*) ⇒ (*ii*) Let μ be a fuzzy n-fold implicative filter of A. Then $\mu(x) \leq \mu(1)$ for any $x \in A$. By Proposition 2.3(iii), $(x^n \to y) \to x = 1 \to ((x^n \to y) \to x)$. Hence, $\mu((x^n \to y) \to x) = \mu(1 \to ((x^n \to y) \to x)) \wedge \mu(1) \leq \mu(x)$.

(*ii*) ⇒ (*i*) Let μ be a fuzzy filter of A. Then $\mu(x) \leq \mu(1)$ and $\mu(z) \wedge \mu(z \to ((x^n \to y) \to x)) \leq \mu((x^n \to y) \to x)$ for any $x \in A$.

By Proposition 2.3(v), $x \leq (x^n \to y) \to x$, then $\mu(x) \leq \mu((x^n \to y) \to x)$. By assumption, we have $\mu((x^n \to y) \to x) = \mu(x)$.

Hence, $\mu(z) \wedge \mu(z \to ((x^n \to y) \to x)) \leq \mu(x)$.

Therefore, μ is a fuzzy n-fold implicative filter of A.

(*iii*) ⇒ (*i*) Let μ be a fuzzy filter of A. Then $\mu(x) \wedge \mu(x \to ((y^n \to z) \to y)) \leq \mu((y^n \to z) \to y)$. Since $\mu((y^n \to z) \to y) \wedge \mu(((y^n \to z) \to y) \to y) \leq \mu(y)$ and by assumption $\mu(1) = (((y^n \to z) \to y) \to y)$, we have $\mu(1) \wedge \mu((y^n \to z) \to y) \leq \mu(y)$.

Hence, $\mu(x) \wedge \mu(x \to ((y^n \to z) \to y)) \leq \mu(y)$.

Therefore, μ is a fuzzy n-fold implicative filter of A.

(*iii*) ⇒ (*iv*) Put $y = 0$ in part (iii). Then $\mu(((x^n \to 0) \to x) \to x) = \mu(((x^n)' \to x) \to x) = \mu(1)$.

(*iv*) ⇒ (*iii*) By Proposition 2.3(iii), since $0 \leq y$, then $(x^n)' \leq x^n \to y$.

By Proposition 2.3(iv),

$$(x^n \to y) \to x \leq (x^n)' \to x \quad \text{and} \quad ((x^n)' \to x) \to x \leq ((x^n \to y) \to x) \to x.$$

Hence,

$$\mu(((x^n)' \to x) \to x) \leq \mu(((x^n \to y) \to x) \to x).$$

Since $\mu(((x^n)' \to x) \to x) = \mu(1)$, thus $\mu(((x^n \to y) \to x) \to x) = \mu(1)$.
(*iii*) ⇒ (*ii*) $\mu((x^n \to y) \to x) = \mu((x^n \to y) \to x) \wedge \mu(1) = \mu((x^n \to y) \to x) \wedge \mu(((x^n \to y) \to x) \to x) \leq \mu(x)$. □

Proposition 5.6. *If fuzzy set μ of A is defined by*

$$\mu(x) = \begin{cases} 0 & x \neq 1 \\ \alpha & x = 1 \end{cases}$$

for $\alpha \in (0, 1]$, then the following are equivalent:
(i) *A is an n-fold implicative hoop algebra,*
(ii) *Any fuzzy filter is a fuzzy n-fold implicative filter of A,*
(iii) *μ is a fuzzy n-fold implicative filter of A.*

Proof. (*i*) → (*ii*) Let A be an n-fold implicative hoop algebra and μ be a fuzzy filter of A. Then $(x^n \to 0) \to x = x$ and $\mu((x^n \to 0) \to x) = \mu(x) \leq \mu(x)$. By Theorem 5.5, μ is a fuzzy n-fold implicative filter of A.

(*ii*) → (*iii*) First, we show that μ is a fuzzy filter. By definition of μ, for any $x \in A$, $\mu(x) \leq \mu(1)$. We consider two case for y.

If $y = 1$, then $\mu(x \to y) \wedge \mu(x) \leq \alpha = \mu(1) = \mu(y)$

If $y \neq 1$, then we consider two following cases for x. If $x = 1$, then

$$\mu(x \to y) \wedge \mu(x) = \mu(1 \to y) \wedge \mu(1) = \mu(y) \wedge \mu(1) = \mu(y) \leq \mu(y),$$

If $x \neq 1$, then

$$\mu(x \to y) \wedge \mu(x) = \mu(x \to y) \wedge 0 = 0 \leq \mu(y).$$

Therefore, μ is a fuzzy filter, and by part (ii), is a fuzzy n-fold implicative filter of A.

$(iii) \to (i)$ Since μ is a fuzzy n-fold implicative filter, then by Theorem 2.8(iii), $\mu_1 = \{x \in A | \mu(x) \geq 1\} = \{1\}$ is an n-fold implicative filter and by Theorem 2.8(iv), and A is an n-fold implicative hoop algebra. □

Theorem 5.7.

(i) μ is a fuzzy n-fold implicative filter if and only if $\mu((x^n \to 0) \to x) \leq \mu(x)$, for any $x \in A$.

(ii) μ is a fuzzy n-fold implicative filter of A if and only if $\mu((x^n \to y) \to x) = \mu(x)$, for any $x \in A$.

Proof. (i) If μ is a fuzzy n-fold implicative filter, then by Theorem 5.5(ii), $\mu((x^n \to y) \to x) \leq \mu(x)$. If $y = 0$, then it is clear.

Conversely, let $\mu((x^n \to 0) \to x) \leq \mu(x)$. Then $\mu(((x^n)' \to x) \to x) = \mu(1)$. Therefore, by Theorem 5.5(iv), μ is a fuzzy n-fold implicative filter.

(ii) Let μ be a fuzzy n-fold implicative filter of A. Then by Theorem 5.5(2), $\mu((x^n \to y) \to x) \leq \mu(x)$. By Proposition 2.3(v), $x \leq ((x^n \to y) \to x)$, so $\mu(x) \leq \mu((x^n \to y) \to x)$. Hence, $\mu((x^n \to y) \to x) = \mu(x)$.

Conversely by Theorem 5.5 is true. □

Theorem 5.8. *Let μ be a fuzzy n-fold implicative filter of A. Then $\mu((x^n \to y) \to y) = \mu((y^n \to x) \to x)$, for any $x, y \in A$.*

Proof. Let μ be a fuzzy n-fold implicative filter of A. Then by Theorem 5.4, $\mu_\alpha \neq \emptyset$ is an n-fold implicative filter of A, for any $\alpha \in [0,1]$. Suppose $\alpha = \mu((x^n \to y) \to y)$. Then $(x^n \to y) \to y \in \mu_\alpha$.

By Proposition 2.3(ii), $y^n \leq (y^n \to x) \to x$.

By Proposition 2.3(iv), $(x^n \to y) \to y \leq (x^n \to y) \to (y^n \to x)$.

Hence, $(x^n \to y) \to ((y^n \to x) \to x) \in \mu_\alpha$.

By similar way, $x \leq (y^n \to x) \to x$, then by Proposition 2.3(iv), $((y^n \to x) \to x) \to y \leq x \to y$ and $(x \to y) \to ((y^n \to x) \to x) \leq (((y^n \to x) \to x) \to y) \to ((y^n \to x) \to x)$.

Hence, $(((y^n \to x) \to x) \to y) \to ((y^n \to x) \to x) \in \mu_\alpha$. Since μ_α is an n-fold implicative filter, by Proposition 2.3(iii), $1 \to ((((y^n \to x) \to x) \to y) \to ((y^n \to x) \to x)) \in \mu_\alpha$.

Also, $1 \in \mu_\alpha$, $(y^n \to x) \to x \in \mu_\alpha$. Then $\alpha \leq \mu((y^n \to x) \to x)$. Hence, $\mu((x^n \to y) \to y) \leq \mu((y^n \to x) \to x)$. By similar way, we have $\mu((y^n \to x) \to x) \leq \mu((x^n \to y) \to y)$. □

Proposition 5.9. *If μ is a fuzzy n-fold implicative filter of A, then μ is a fuzzy $(n+1)$-fold implicative filter of A.*

Proof. Let μ be a fuzzy n-fold implicative filter of A. Then by Proposition 2.3(iii), $x^{n+1} \leq x^n$, thus $x^n \to 0 \leq x^{n+1} \to 0$, so $(x^{n+1} \to 0) \to x \leq (x^n \to 0) \to x$. Hence, $((x^n \to 0) \to x) \to x \leq ((x^{n+1} \to 0) \to x)$, and by Theorem 5.5(iv), $\mu(1) = \mu(((x^n \to 0) \to x) \to x) \leq \mu(((x^{n+1} \to 0) \to x)) \to x$. Therefore, by Theorem 5.5(iv), μ is a fuzzy $(n+1)$-fold implicative filter of A. □

The following example shows that the converse of Proposition 5.9 is not true in general.

Example 5.10. Let $(A = \{0, a, b, 1\}, \leq)$. Define the operations $\odot$ and $\to$ on A as follows:

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	0	a.
b	0	0	a	b
1	0	a	b	1

Then $(A, \odot, \to, 1, 0)$ is a bounded hoop algebra. Define a fuzzy set in A by $\mu(0) = \mu(a) = \mu(b) = 1/4$, $\mu(1) = 3/4$. By using Theorem 5.7, for $n = 3$, it is easy to see that μ is a fuzzy three-fold implicative filter of A but is not a fuzzy two-fold implicative filter of A. Because $\mu((b^2 \to 0) \to b) = \mu(1) \nleq \mu(b)$.

Proposition 5.11. *If μ is a fuzzy filter and $\mu(x) < \mu(x')$, then μ is a fuzzy n-fold implicative filter of A.*

Proof. Let $\mu(x) < \mu(x')$. Then by Proposition 2.3(iii), $x^n \leq x$ and by Proposition 2.3(iv), $x' \leq (x^n)'$. So $\mu(x) \leq \mu(x') \leq \mu((x^n)')$. Since μ is a fuzzy filter, then

$$\mu((x^n)' \to x) \wedge \mu((x^n)') \leq \mu(x).$$

Hence, $\mu((x^n)' \to x) \leq \mu(x)$. Therefore, by Theorem 5.7(i), μ is a fuzzy n-fold implicative filter. □

Proposition 5.12. *Every fuzzy n-fold obstinate filter μ of A such that $\mu(x) \leq 1/2$ is a fuzzy n-fold implicative filter, for any $x \in A$.*

Proof. Let μ be a fuzzy n-fold obstinate filter and $x \in A$ such that $\mu(x) \leq 1/2$. Since μ is a fuzzy filter, then

$$\mu((x^n)' \to x) \wedge \mu((x^n)') \leq \mu(x).$$

By hypothesis and Theorem 4.3, $\mu((x^n)' \to x) \wedge (1 - \mu(x)) \leq \mu(x)$.

If $\mu((x^n)' \to x) \wedge (1 - \mu(x)) = 1 - \mu(x)$, then $\mu(x) \geq \frac{1}{2}$, and by hypothesis, $\mu(x) = 1/2$. Hence, $1/2 = 1 - \mu(x) \leq \mu((x^n)')$, and by fuzzy filter, $\mu((x^n)' \to x) \wedge \mu((x^n)') \leq \mu(x)$. So $\mu((x^n)' \to x) \leq \mu(x)$.

If $\mu((x^n)' \to x) \wedge (1 - \mu(x)) = \mu((x^n)' \to x)$, then $\mu((x^n)' \to x) \leq \mu(x)$.

Therefore, by Theorem 5.7(i), μ is a fuzzy n-fold implicative filter. □

The following example shows that the converse of Proposition 5.12 is not true in general.

Example 5.13. In Example 5.10, let $\mu(0) = \mu(a) = \mu(b) = 1/4$, $\mu(1) = 2/4$. Then μ is a fuzzy three-fold implicative filter of A but is not a three-fold fuzzy obstinate filter. Because $1 - \mu(0) = 3/4 \nleq \mu((0^3)') = 2/4$.

6 Conclusion

In this article, first, we investigated fuzzy obstinate filter and fuzzy prime filter in hoop algebras and studied properties of them. Then we obtained relation between fuzzy obstinate filter with fuzzy (implicative, positive implicative, fantastic, and prime) filters. The relationship between them summarized in diagram (1). Hence, we investigated fuzzy n-fold obstinate filter and fuzzy n-fold implicative filter on hoop algebras and obtained relation between fuzzy n-fold obstinate filter with fuzzy n-fold implicative filters. Thus, A is an n-fold implicative hoop algebra if and only if any fuzzy filter is a fuzzy n-fold implicative filter if and only if μ is a fuzzy n-fold implicative filter. We show that every fuzzy n-fold implicative filter is a fuzzy filter, and we investigated the following conditions equivalent for fuzzy n-fold implicative filters. In our opinion, these definitions and main results study other fuzzy n-fold filter on hoop algebra and can be extended to some other algebraic systems such as CI-algebras and EQ-algebras.

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