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Research Article

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Eigenvalues of complex unit gain graphs and gain regularity

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Abstract: A complex unit gain graph (or $\mathbb{T}$-gain graph) $\Gamma = (G, \gamma)$ is a gain graph with gains in $\mathbb{T}$, the multiplicative group of complex units. The $\mathbb{T}$-outgain in Γ of a vertex $v \in G$ is the sum of the gains of all the arcs originating in v. A $\mathbb{T}$-gain graph is said to be an a-$\mathbb{T}$-regular graph if the $\mathbb{T}$-outgain of each of its vertices is equal to a. In this article, it is proved that a-$\mathbb{T}$-regular graphs exist for every $a \in \mathbb{R}$. This, in particular, means that every real number can be a $\mathbb{T}$-gain graph eigenvalue. Moreover, denoted by $\Omega(a)$ the class of connected $\mathbb{T}$-gain graphs whose largest eigenvalue is the real number a, it is shown that $\Omega(a)$ is nonempty if and only if a belongs to $\{0\} \cup [1, +\infty)$. In order to achieve these results, non-complete extended p-sums and suitably defined joins of $\mathbb{T}$-gain graphs are considered.

Keywords: gain graph, eigenvalues, index, T-regularity

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1 Introduction

Let $\vec{E}_G$ be the set of arcs of a nonempty simple graph G with vertex set $V_G = \{u_1, u_2, \ldots, u_n\}$. We write $u_i \sim u_j$ whenever u_i and u_j are adjacent. Each pair $\{u_i, u_j\} \subseteq V_G$ such that $u_i \sim u_j$ determines the arc e_{ij} going from u_i to u_j and the opposite arc e_{ji}. Denoted by $\mathbb{T}$ the multiplicative group $\{z \in \mathbb{C} | |z| = 1\}$, a *complex unit gain* or $\mathbb{T}$*-gain graph* is a pair $\Gamma = (G, \gamma)$, where $\gamma : \vec{E}_G \to \mathbb{T}$ is a *gain function*, i.e., a map satisfying $\gamma(e_{ij}) = \gamma(e_{ji})^{-1}$ for each $e_{ij} \in \vec{E}_G$. We usually refer to G as the *underlying graph* of Γ and $\gamma(\vec{E}_G)$ as its set of *gains*. Empty graphs can be thought as $\mathbb{T}$-gain graphs equipped with the empty gain function $\varnothing \to \mathbb{T}$.

In this article, the complex conjugate of a complex number z will be denoted by z^*. Let $M_n(\mathbb{C})$ be the set of $n \times n$ complex matrices. The *adjacency matrix* $A(\Gamma) = (a_{ij}) \in M_n(\mathbb{C})$ of a $\mathbb{T}$-gain graph $\Gamma = (G, \gamma)$ is defined by

$$a_{ij} = \begin{cases} \gamma(e_{ij}) & \text{if } v_i \sim v_j, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

If $v_i \sim v_j$, then $a_{ij} = \gamma(e_{ij}) = \gamma(e_{ji})^{-1} = \gamma(e_{ji})^* = a_{ji}^*$. Consequently, $A(\Gamma)$ is Hermitian and its eigenvalues $\lambda_1(\Gamma) \geqslant \cdots \geqslant \lambda_n(\Gamma)$, i.e., the roots of the polynomial $p_\Gamma(\lambda) = \det(\lambda I - A(\Gamma))$, are real. The largest eigenvalue $\lambda_1(\Gamma)$ gives the *index* of Γ.

Both the combinatorial and the spectral theory of $\mathbb{T}$-gain graphs embody those of simple graphs, signed graphs, and mixed graphs (as defined in [16]): these objects can be seen as $\mathbb{T}$-gain graphs whose gains are, respectively, in the subsets $\{1\}$, $\{\pm 1\}$, and $\{1, \pm i\}$ of $\mathbb{T}$. This is surely one of the reasons why, in the wake of [27], there has been a renewed and growing interest over the last decade for the Hermitian matrices associated with $\mathbb{T}$-gain graphs and their spectra (see, for instance, [5,6,8,20,23,25,26,31,34,35]). Clearly, every $\mathbb{T}_n$-gain graph,

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where $n \in \mathbb{N}$ and $\mathbb{T}_n$ denotes the group of n-th roots of unity, can be regarded as a complex unit gain graph, and many results in [2] proved for $\mathbb{T}_4$-gain graphs can be easily generalizable to $\mathbb{T}$-gain graphs. The spectral theory of $\mathbb{T}$-gain graphs turns out to be useful to achieve results on the eigenvalues of gain graphs over a fixed (not necessarily abelian) finite group G (see, for instance, [13]).

Let $\mathcal{E}_1$, $\mathcal{E}_{\pm 1}$, and $\mathcal{E}_{\mathbb{T}}$ (resp. $\mathcal{I}_1$, $\mathcal{I}_{\pm 1}$, and $\mathcal{I}_{\mathbb{T}}$) denote the sets of real numbers which are eigenvalues (resp. indices) of simple graphs, signed graphs, and $\mathbb{T}$-gain graphs. An algebraic number a is said to be *totally real* if it is a root of a real-rooted monic polynomial with integer coefficients [30], whereas it is said an *almost Perron number* if it satisfies $a \geqslant |b|$ for each conjugate b of a [24, Exercise 11.1.12]. The set of algebraic numbers which are totally real (resp. almost Perron) will be denoted by $\mathbb{TR}$ (resp. $\mathbb{AP}$).

Thanks to the achievements of Estes [15, Theorem 1] and Hoffman [21], one sees that $\mathcal{E}_1 = \mathcal{E}_{\pm 1} = \mathbb{TR}$. This equality can be equivalently deduced from a more recent result on tree eigenvalues due to Salez [30, Theorem 1]. As far as I know, the sets $\mathcal{I}_1$ and $\mathcal{I}_{\pm 1}$ still wait for manageable characterizations. For instance, the Perron-Frobenius theorem yields $\mathcal{I}_1 \subseteq \mathbb{TR} \cap \mathbb{AP}$, yet it is still dubious whether this inclusion can be reversed.

Results in this article allow one to determine $\mathcal{E}_{\mathbb{T}}$ and $\mathcal{I}_{\mathbb{T}}$. It turns out that $\mathcal{E}_{\mathbb{T}} = \mathbb{R}$. This equality is a consequence of Theorem 1.3 or Corollary 3.8. In contrast, the equality $\mathcal{I}_{\mathbb{T}} = \{0\} \cup [1, +\infty)$ immediately comes from Theorem 1.1, which is the first main result of this article. In its statement, the symbol $\lceil a \rceil$ denotes the smallest integer not less than the real number a.

Theorem 1.1. *Let C denote the class of all connected $\mathbb{T}$-gain graphs, and let a be a real number.*
(*i*) *The set $\Omega(a) := \{\Gamma \in C \,|\, \lambda_1(\Gamma) = a\}$ is nonempty if and only if a belongs to $\{0\} \cup [1, +\infty)$.*
(*ii*) *For $a \geqslant 1$, the set $\Omega(a)$ contains infinitely many $\mathbb{T}$-gain graphs.*
(*iii*) *For $a \in \{0\} \cup [1, +\infty)$, a graph of minimal order in $\Omega(a)$ has $\lceil a \rceil + 1$ vertices.*

A simple graph G is said to be a-regular if each $v \in V_G$ has vertex degree a. Let now $\Sigma = (G, \sigma)$ be a signed graph. The net-degree $d^{\pm}_{\Sigma}(u)$ of a vertex u is given by the difference $d^{+}_{\Sigma}(u) - d^{-}_{\Sigma}(u)$, where $d^{+}_{\Sigma}(u)$ (resp. $d^{-}_{\Sigma}(u)$) is the cardinality of the positive (resp. negative) edges incident on $u \in V_G$. The signed graph Σ is said to be a-net regular if $d^{\pm}(u) = a$ for all $u \in V_G$. Net-regularity for signed graphs has been studied in connections with several different problems (see, for instance, [7,18,29,32,33]). The following notion of a-$\mathbb{T}$-regularity appropriately extends net-regularity to $\mathbb{T}$-gain graphs.

Let u be a vertex of a $\mathbb{T}$-gain graph $\Gamma = (G, \gamma)$. The numbers

$$d^{\rightarrow}_{\Gamma}(u) = \sum_{v \sim u} \gamma(uv) \quad \text{and} \quad d^{\leftarrow}_{\Gamma}(u) = \sum_{v \sim u} \gamma(vu)$$

are, respectively, called the $\mathbb{T}$-*ingain* and the $\mathbb{T}$-*outgain* of the vertex u. If u is an isolated vertex, then $d^{\rightarrow}_{\Gamma}(u)$ and $d^{\leftarrow}_{\Gamma}(u)$ are assumed to be 0. In all cases, $d^{\leftarrow}_{\Gamma}(u)$ is the complex conjugate of $d^{\rightarrow}_{\Gamma}(u)$, with γ being a gain function.

Definition 1.2. An a-$\mathbb{T}$-regular graph is a $\mathbb{T}$-gain graph $\Gamma = (G, \gamma)$ such that $d^{\rightarrow}_{\Gamma}(u) = a$ for all $u \in V_G$.

It is straightforward to check that a-regularity, a-net regularity, and a-$\mathbb{T}$-regularity all have the same spectral characterization: they occur if and only if a is an eigenvalue for the adjacency matrix of the simple, signed, or $\mathbb{T}$-gain graph (of order n) under consideration, and the corresponding eigenspace contains the all-ones vector $\mathbf{j}_n$. By definition, a-regularity can possibly occur only if a is a nonnegative integer; a-net regularity is possible only if a is an integer; finally, a must be a real number in order to possibly get a-$\mathbb{T}$-regularity, since a is an eigenvalue of a Hermitian matrix. It should be pointed out that a-$\mathbb{T}$-regularity for $\mathbb{T}$-gain graphs, like net-regularity for signed graphs, is a purely combinatorial invariant; for instance, $\mathbb{T}$-gain graphs that are switching equivalent to a-$\mathbb{T}$-regular graph are not in general a-$\mathbb{T}$-regular. In any case, suitable constructions that preserve a-$\mathbb{T}$-regularity do exist. Using them, the following result will be proved.

Theorem 1.3. *For every fixed real number a, there exists infinitely many a-$\mathbb{T}$-regular graphs.*

Let $A(\Gamma)$ be the adjacency matrix of a $\mathbb{T}$-gain graph $\Gamma = (G, \gamma)$. It is immediately seen from the definitions that $d_\Gamma^{\rightarrow}(u_j)$ (resp. $d_\Gamma^{\leftarrow}(u_j)$) gives the sum of the elements in the j-th row (resp. column) of $A(\Gamma)$. This means that a $\mathbb{T}$-gain graph $\Gamma = (G, \gamma)$ is a-$\mathbb{T}$-regular if and only if the sum of elements on each fixed row or fixed column of the matrix $A(\Gamma)$ is always equal to a. For this reason, Theorem 1.3 could be restated in purely matrix-theoretical terms: it says that for each $a \in \mathbb{R}$ there exist infinitely many Hermitian matrices with trace zero, entries in $\mathbb{T} \cup \{0\}$, and constant row sum equal to a.

The remainder of the article is structured as follows. Section 2 contains some preliminaries on complex unit gain graphs, their non-complete extended p-sums, and two different lexicographic products. In that section, the set $\{\mathcal{K}_3'(z) \,|\, z \in \mathbb{T}\}$ of $\mathbb{T}$-gain triangles (Figure 1) that plays an important role in the proofs of Theorems 1.1 and 1.3 is also introduced. Section 3 is devoted to the proof of Theorem 1.1. Its main ingredient is the detected spectrum of the matrices (3.2), whose eigenvalues are somehow related to Dirichlet kernels and could be of independent interest. In Section 4, it is shown that non-complete extended p-sums (NEPS) and the HG-lexicographic product behave well with respect to a-regularity. This fact, together with the properties of a suitably defined join of $\mathbb{T}$-gain graphs, allows one to prove Theorem 1.3.

2 Preliminaries

2.1 Complex unit gain graphs

For the figures of this article, we adopt the following drawing convention: close to each depicted arc uv the value $\gamma(uv)$ is specified. Obviously, the gain assigned to the non-depicted arc vu is $\gamma(uv)^*$. For instance, the arcs u_1u_2, u_1u_3, and u_2u_3 of the gain triangle $\mathcal{K}_3(z)$ depicted in Figure 1 all have the same gain $z \in \mathbb{T}$, whereas $\gamma(u_2u_1) = \gamma(u_3u_1) = \gamma(u_3u_2) = z^*$.

Let $\Gamma = (G, \gamma)$ be a $\mathbb{T}$-gain graph. The spectrum of Γ, i.e., the multiset of the eigenvalues of $A(\Gamma)$, will be denoted by $\mathrm{Sp}(\Gamma)$. The *negation* of a $\mathbb{T}$-gain graph Γ is $-\Gamma \coloneqq (\Gamma, -\gamma)$. Clearly, $A(-\Gamma) = -A(\Gamma)$ and $\lambda_i(-\Gamma) = -\lambda_{n-i+1}(\Gamma)$. A walk $W = e_{i_1i_2}e_{i_2i_3} \cdots e_{i_{l-1}i_l}$ is said to be *neutral*, *negative*, or *imaginary*, depending if its *gain* $\gamma(W) \coloneqq \gamma(e_{i_1i_2})\gamma(e_{i_2i_3}) \cdots \gamma(e_{i_{l-1}i_l})$ is 1, −1, or imaginary. We write $(\Gamma, 1)$ for the $\mathbb{T}$-gain graph with all neutral arcs. The following result is due to Reff.

Proposition 2.1. [27, Theorem 5.1] *Let (C_n, γ) be a (directed) $\mathbb{T}$-gain cycle whose gain is $e^{i\alpha}$. Then,*

$$\mathrm{Sp}(C_n, \gamma) = \left\{2\cos\left(\frac{\alpha + 2\pi j}{n}\right) \,\middle|\, 0 \leqslant j \leqslant n-1\right\}. \tag{2.1}$$

Example 2.2. Let $\mathcal{K}_3(z) = (K_3, \gamma_{3,z})$ and $\mathcal{K}_3'(z) = (K_3, \gamma_{3,z}')$ be the $\mathbb{T}$-gain graphs depicted in Figure 1. The gains of the directed cycles $u_1u_2u_3$ and $v_1v_2v_3$ in Figure 1 are

$$\gamma_{3,z}(u_1u_2)\gamma_{3,z}(u_2u_3)\gamma_{3,z}(u_3u_1) = z^2 \cdot z^* = z \quad \text{and} \quad \gamma_{3,z}'(v_1v_2)\gamma_{3,z}'(v_2v_3)\gamma_{3,z}'(v_3v_1) = z^3$$

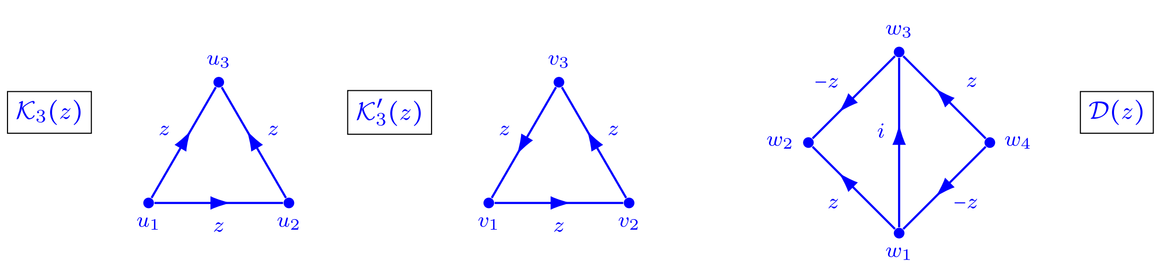

Figure 1: The $\mathbb{T}$-gain graphs $\mathcal{K}_3(z)$, $\mathcal{K}_3'(z)$, and $\mathcal{D}(z)$.

respectively. If $z = e^{i\theta}$, by Proposition 2.1, we obtain

$$\mathrm{Sp}(\mathcal{K}_3(z)) = \left\{2\cos\left(\frac{\theta}{3}\right),\ 2\cos\left(\frac{\theta+2\pi i}{3}\right),\ 2\cos\left(\frac{\theta+4\pi i}{3}\right)\right\} \tag{2.2}$$

and

$$\mathrm{Sp}(\mathcal{K}_3'(z)) = \left\{2\cos\theta,\ 2\cos\left(\theta+\frac{2\pi i}{3}\right),\ 2\cos\left(\theta+\frac{4\pi i}{3}\right)\right\}. \tag{2.3}$$

The spectra (2.2) and (2.3) can also be obtained by directly computing the characteristic polynomials of the matrices

$$A(\mathcal{K}_3(z)) = \begin{pmatrix} 0 & z & z \\ z^* & 0 & z \\ z^* & z^* & 0 \end{pmatrix} \quad \text{and} \quad A(\mathcal{K}_3'(z)) = \begin{pmatrix} 0 & z & z^* \\ z^* & 0 & z \\ z & z^* & 0, \end{pmatrix}. \tag{2.4}$$

It turns out that

$$p_{\mathcal{K}_3(z)}(\lambda) = \lambda^3 - 3\lambda - 2\cos\theta \quad \text{and} \quad p_{\mathcal{K}_3'(z)}(\lambda) = \lambda^3 - 3\lambda - 2\cos(3\theta).$$

It is easily seen that the gain diamond $\mathcal{D}(z)$ in Figure 1 has two imaginary triangles and a neutral quadrangle. More precisely, the directed cycles $C_3' = w_1w_2w_3$, $C_3'' = w_1w_3w_4$ have both gain $e^{i\frac{\pi}{2}} = i$. It is somehow instructive to see that

$$A(\mathcal{D}(z)) = \begin{pmatrix} 0 & z & i & -z^* \\ z^* & 0 & -z^* & 0 \\ -i & -z & 0 & z^* \\ -z & 0 & z & 0 \end{pmatrix} \quad \text{and} \quad p_{\mathcal{D}(z)}(\lambda) = \lambda^4 - 5\lambda^2 \quad \forall z \in \mathbb{T}. \tag{2.5}$$

Thus, $\mathrm{Sp}(\mathcal{D}(z)) = \{-\sqrt{5}, 0^{(2)}, \sqrt{5}\}$, where the exponent of 0 in parentheses stands for its multiplicity.

If $\Lambda = (H, \zeta)$ is an induced ($\mathbb{T}$-gain) subgraph of Γ, i.e., if H is an induced subgraph of G and $\zeta = \gamma_{|\vec{E}_H}$, then $A(\Lambda)$ is the principal matrix of $A(\Gamma)$ correspondent to the rows indexed by vertices in V_H. Therefore, [22, Corollary 4.3.37] yields the well-known interlacing phenomenon recalled in the following proposition.

Proposition 2.3. *Let $\Lambda = (H, \zeta)$ be an induced subgraph of $\Gamma = (G, \gamma)$ with $|V_G| = n$ and $|V_H| = m$. Denoted by $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ and $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_m$ the eigenvalues of Γ and Λ, respectively, the following inequalities hold: $\lambda_{n-m+i} \leqslant \mu_i \leqslant \lambda_i$ for $1 \leqslant i \leqslant m$.*

Corollary 2.4. *If $\Gamma = (G, \gamma)$ is a nonempty $\mathbb{T}$-gain graph, then $\lambda_1(\Gamma) \geqslant 1$.*

Proof. Let u and v be two adjacent vertices in G, and let $H \coloneqq G[u, v]$ be the subgraph of G induced by u and v. The index of $\Lambda \coloneqq (H, \gamma_{|\vec{E}_H})$ is equal to 1; in fact,

$$A(\Lambda) = \begin{pmatrix} 0 & \gamma(uv) \\ \gamma(uv)^* & 0 \end{pmatrix} \quad \text{and} \quad p_\Lambda(\lambda) = \lambda^2 - 1. \tag{2.6}$$

Proposition 2.3 now yields $\lambda_1(\Gamma) \geqslant \lambda_1(\Lambda) = 1$. □

Expert scholars know very well that, whatever gain function is chosen on the arcs of a tree, one always retrieves the spectrum of the unsigned tree. This phenomenon occurs whenever the $\mathbb{T}$-gain graph $\Gamma = (G, \gamma)$ is balanced, since in this case Γ is switching equivalent to $(G, 1)$. Similarly, the lack of dependence on z for the coefficients of $p_{\mathcal{D}(z)}$ in (2.5) is not surprising, as the various graphs in the set $\{\mathcal{D}(z) | z \in \mathbb{T}\}$ are all switching equivalent.

For the notions of balance and switching equivalence, the reader is referred to [10,12,27]. In spite of their significance in the theory of gain graphs, these notions will never be mentioned again in the remainder of this article.

2.2 NEPS and lexicographic products of $\mathbb{T}$-gain graphs

In [9], Belardo et al. introduced NEPS of complex unit gain graphs. In order to keep the article reasonably self-contained, the definition of NEPS will be recalled here.

Let $\mathfrak{B}$ be a nonempty subset of $\mathfrak{F}_h \coloneqq \{0,1\}^h \backslash \{(0,\ldots,0)\}$, the set of $\{0,1\}$-h-tuples with at least one among their components. Cvektović defined $G \coloneqq \mathrm{NEPS}(G_1,\ldots,G_h;\ \mathfrak{B})$, the non-complete extended p-sums (or simply NEPS) of the simple graphs $G_1,\ldots,G_h$ with basis $\mathfrak{B}$ (see, for instance, [14, p. 66]): the vertex set V_G is the Cartesian product $V_{G_1}\times\cdots\times V_{G_h}$, and the vertices $u \coloneqq (u_1,\ldots,u_h)$ and $v \coloneqq (v_1,\ldots,v_h)$ are adjacent if and only if there exists a (unique) h-tuple $\mathbf{b} = (b_1,\ldots,b_h)$ in $\mathfrak{B}$ such that $u_i = v_i$ whenever $b_i = 0$, and $u_i \sim v_i$ in G_i if $b_i = 1$. Note that

$$\vec{E}_G = \bigsqcup_{\mathbf{b}\in\mathfrak{B}} \vec{E}_{\mathrm{NEPS}(G_1,\ldots,G_h;\ \mathbf{b})}, \tag{2.7}$$

where the symbol $\sqcup$ denotes the disjoint union.

Definition 2.5. Let $\Gamma_1 = (G_1,\gamma_1),\ldots,\Gamma_h = (G_h,\gamma_h)$ be $h\mathbb{T}$-gain graphs. The NEPS (or *Cvektović product*) of $\Gamma_1,\ldots,\Gamma_h$ with basis $\mathfrak{B}$ are the $\mathbb{T}$-gain graph $\Gamma = (G,\gamma)$ defined as follows:
- the underlying graph G is $\mathrm{NEPS}(G_1,\ldots,G_h;\ \mathfrak{B})$,
- for each pair of adjacent vertices $u \coloneqq (u_1,\ldots,u_h)$ and $v \coloneqq (v_1,\ldots,v_h)$ in G,

$$\gamma(uv) \coloneqq \prod_{j=1}^{h} \gamma_j(u_jv_j), \tag{2.8}$$

where $\gamma_j(u_jv_j)$ is understood to be 1 whenever $u_j = v_j$.

The $\mathbb{T}$-gain graph $\Gamma = (G,\gamma)$ will be denoted by $\mathrm{NEPS}(\Gamma_1,\ldots,\Gamma_h;\ \mathfrak{B})$.

The map γ in (2.8) is indeed a gain function. In fact,

$$\gamma(vu) = \prod_{j=1}^{h}\gamma_j(v_ju_j) = \prod_{j=1}^{h}\gamma_j(u_jv_j)^* = \gamma(uv)^*.$$

For $1 \leqslant p \leqslant h$, let $\mathfrak{B}_{h,p}$ be the subset of $\mathfrak{F}_h$ of all h-tuples containing precisely p 1's, and let $\mathbf{j}_h \in \mathfrak{F}_h$ be the all-ones h-tuple $(1,\ldots,1)$. Clearly, $\mathfrak{B}_{h,h} = \{\mathbf{j}_h\}$. The $\mathbb{T}$-gain graph $\mathrm{NEPS}(\Gamma_1,\ldots,\Gamma_h;\ \mathfrak{B}_{h,p})$ is called the *(complete) p-sum* of $\Gamma_1,\ldots,\Gamma_h$, and

$$\square_{i=1}^{h}\Gamma_i \coloneqq \mathrm{NEPS}(\Gamma_1,\ldots,\Gamma_h;\ \mathfrak{B}_{h,1}),\quad \times_{i=1}^{h}\Gamma_i \coloneqq \mathrm{NEPS}(\Gamma_1,\ldots,\Gamma_h;\{\mathbf{j}_h\}),\quad \boxtimes_{i=1}^{h}\Gamma_i \coloneqq \mathrm{NEPS}(\Gamma_1,\ldots,\Gamma_h;\ \mathfrak{F}_h)$$

are the *Cartesian* product, the *direct* or *tensor* product, and the *strong* product, respectively.

In order to describe how the adjacency matrix of an NEPS is related to those of its factors, we need to recall that, given two matrices $B = (b_{ij})$ and $C = (c_{ij})$ of type $k \times m$ and $l \times n$, respectively, the Kronecker product $B \otimes C$ is the following $kl \times mn$ matrix:

$$B \otimes C \coloneqq \begin{pmatrix} b_{11}C & b_{12}C & \cdots & \cdots & b_{1m}C \\ b_{21}C & b_{22}C & \cdots & \cdots & b_{2m}C \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ b_{k1}C & b_{k2}C & \cdots & \cdots & b_{km}C \end{pmatrix}.$$

Let A be an $n \times n$ matrix. In the following statement, A^0 stands for the $n \times n$ identity matrix I_n.

Proposition 2.6. [9, Proposition 3.3] *For* $1 \leqslant i \leqslant h$, *let* Γ_i *be a* $\mathbb{T}$*-gain graph with* n_i *vertices. The adjacency matrix of* $\Gamma = \mathrm{NEPS}(\Gamma_1, \ldots, \Gamma_h; \mathfrak{B})$ *is given by*

$$A(\Gamma) = \sum_{(b_1,\ldots,b_h)\in\mathfrak{B}} A(\Gamma_1)^{b_1} \otimes \cdots \otimes A(\Gamma_h)^{b_h}. \tag{2.9}$$

Moreover, if $\lambda_{i1} \geqslant \lambda_{i2} \geqslant \cdots \geqslant \lambda_{in_i}$ *are the eigenvalues of* $A(\Gamma_i)$, *then* $\mathrm{Sp}(\Gamma) = \{\lambda_{k_1,\ldots,k_h} | 1 \leqslant k_i \leqslant n_i\}$, *where*

$$\lambda_{k_1,\ldots,k_h} \coloneqq \sum_{(b_1,\ldots,b_h)\in\mathfrak{B}} \lambda_{1k_1}^{b_1} \cdots \lambda_{hk_h}^{b_h}. \tag{2.10}$$

The following corollary is extracted from [9, Equation (3.6) and Corollary 3.4].

Corollary 2.7. *Let* n_i *be the order of the* $\mathbb{T}$*-gain graph* Γ_i *for* $1 \leqslant i \leqslant h$. *The adjacency matrix of the Cartesian product* $\square_{i=1}^h \Gamma_i$ *is*

$$A(\square_{i=1}^h \Gamma_i) = A(\Gamma_1) \otimes I_{n_2} \otimes \cdots \otimes I_{n_h} + I_{n_1} \otimes A(\Gamma_2) \otimes \cdots \otimes I_{n_h} + \cdots + I_{n_1} \otimes I_{n_2} \otimes \cdots \otimes A(\Gamma_h). \tag{2.11}$$

The spectrum of $\square_{i=1}^h \Gamma_i$ *is*

$$\mathrm{Sp}(\square_{i=1}^h \Gamma_i) = \{\lambda_{1k_1} + \cdots + \lambda_{hk_h} | 1 \leqslant k_i \leqslant n_i\},$$

where $\lambda_{i1} \geqslant \lambda_{i2} \geqslant \cdots \geqslant \lambda_{in_i}$ *are the eigenvalues of* $A(\Gamma_i)$.

In [19], Harary introduced the notion of *composition* of (simple) graphs later known also as *lexicographic product* [17, ch. I,4]. In the literature, there are two different extensions to signed graphs of Harary's composition. The first attempt was made by Hameed and Germina [18], the second is due to Brunetti et al. [11]. In [3], they are respectively called the HG- and the BCD-lexicographic product. We now generalize these two products to $\mathbb{T}$-gain graphs, employing a notation tendentially consistent with [3].

Let G and H be two (unsigned) graphs. The *composition* or *lexicographic product* $G[H]$ is a graph whose set of vertices is $V(G) \times V(H)$, with $(u, v) \sim (u', v')$ whenever $u \sim u'$ or $u = u'$ and $v \sim v'$.

Definition 2.8. Let $\Gamma = (G, \gamma)$ and $\Lambda = (H, \zeta)$ be two $\mathbb{T}$-gain graphs. The maps

$$\gamma_{\mathrm{HG}} : ((u, v)(u', v')) \in \vec{E}_{G[H]} \mapsto \begin{cases} \gamma(uu') & \text{if } u \sim u', \\ \zeta(vv') & \text{if } u = u' \text{ and } v \sim v', \end{cases}$$

and

$$\gamma_{\mathrm{BCD}} : ((u, v)(u', v')) \in \vec{E}_{G[H]} \mapsto \begin{cases} \gamma(u, u') & \text{if } u \sim u' \text{ and } v \nsim v', \\ \gamma(uu')\zeta(vv') & \text{if } u \sim u' \text{ and } v \sim v', \\ \zeta(vv') & \text{if } u = u' \text{ and } v \sim v', \end{cases}$$

give rise to the following $\mathbb{T}$-gain graphs with underlying graph $G[H]$: the HG-lexicographic product $\Gamma[\Lambda] = (G[H], \gamma_{\mathrm{HG}})$ and the BCD-lexicographic product $\Gamma * \Lambda = (G[H], \gamma_{\mathrm{BCD}})$.

The reader will easily realize that Definition 2.8 is well-posed: the equalities

$$\gamma_{\mathrm{HG}}((u, v)(u', v')) = \gamma_{\mathrm{HG}}((u', v')(u, v))^* \quad \text{and} \quad \gamma_{\mathrm{BCD}}((u, v)(u', v')) = \gamma_{\mathrm{BCD}}((u', v')(u, v))^*$$

are elementary to check; they show that γ_{HG} and γ_{BCD} are both gain functions. The HG-lexicographic product and the BCD-lexicographic product can be iterated. By inductively defining $\Gamma^k[\Lambda] \coloneqq \Gamma[\Gamma^{k-1}[H]]$ and $(*^k\Gamma) * \Lambda \coloneqq (*^{k-1}\Gamma) * (\Gamma * \Lambda)$ for all $k \geqslant 2$, one obtains $\mathbb{T}$-gain graphs whose underlying graph is $G^k[H]$, the iterated lexicographic product of simple graphs spectrally investigated by Abreu et al. [1].

The following proposition, whose proof just comes from the definition of the HG-lexicographic product, is the $\mathbb{T}$-gain analog of [18, Theorem 8]. In its statement, J_m denotes the $m \times m$ all-ones matrix.

Proposition 2.9. *Let $\Gamma = (G, \gamma)$ and $\Lambda = (H, \zeta)$ be two $\mathbb{T}$-gain graphs with $V_G = \{u_1, \dots, u_n\}$ and $V_H = \{v_1, \dots, v_m\}$. Chosen for $V_{G\times H}$ the lexicographic ordering, the following equality holds*:

$$A(\Gamma[\Lambda]) = A(\Gamma) \otimes J_m + I_n \otimes A(\Lambda). \tag{2.12}$$

3 Spectrum of a useful matrix

Let $n \geqslant 1$ and let $u_1, u_2, \dots, u_n$ be the vertices of the complete simple graph K_n. In this section, for every $z \in \mathbb{T}$, we compute the characteristic polynomial and spectrum of the $\mathbb{T}$-gain graph $\mathcal{K}_n(z) = (K_n, \gamma_{n,z})$, where $\gamma_{1,z}$ is the empty gain function and, for $n \geqslant 2$,

$$\gamma_{n,z}(u_i u_j) = z \quad \text{whenever } \ i < j.$$

To lighten the notation, we set $A_{n,z} \coloneqq A(\mathcal{K}_n(z))$ and $p_{n,z}(\lambda) \coloneqq p_{\mathcal{K}_n(z)}(\lambda) = \det(\lambda I - A_{n,z})$. By definition,

$$A_{n,z} = \begin{pmatrix} 0 & z & z & z & \dots & z \\ z^* & 0 & z & z & \dots & z \\ z^* & z^* & 0 & z & \dots & z \\ z^* & z^* & z^* & 0 & \dots & z \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ z^* & z^* & z^* & z^* & \dots & 0 \end{pmatrix}. \tag{3.1}$$

Remark 3.1. The matrix $A(\mathcal{K}_n(1))$ is the adjacency matrix of the complex graph K_n, whose spectrum is $\{(-1)^{(n-1)}, n-1\}$ (see, for instance, [14, p. 72]). Since $A(\mathcal{K}_n(-1)) = -A(\mathcal{K}_n(1))$, we immediately obtain $\mathrm{Sp}(\mathcal{K}_n(-1)) = \{1^{(n-1)}, -(n-1)\}$. The determinant of an adjacency matrix is the product of its eigenvalues; therefore, $\det(A_{n,1}) = (-1)^{n-1}(n-1)$ and $\det(A_{n,-1}) = -(n-1)$.

Lemma 3.2. *For $n \geqslant 1$ and $z = \mathrm{e}^{i\theta}$, the following equality holds*:

$$\det(A_{n,z}) = \begin{cases} (-1)^{n-1}(n-1) & \text{for } \theta = 0 \ (\text{i.e. } z = 1), \\ -(n-1) & \text{for } \theta = \pi \ (\text{i.e. } z = -1), \\ (-1)^{n-1}\dfrac{\sin((n-1)\theta)}{\sin\theta} & \text{for } 0 < \theta < \pi. \end{cases} \tag{3.2}$$

Proof. The cases $\theta \in \{0, \pi\}$ are dealt in Remark 3.1. From now on, we assume $0 < \theta < \pi$. Clearly,

$$\det(A_{1,z}) = 0 \quad \text{and} \quad \det(A_{2,z}) = -1 \quad \text{for all } \ z \in \mathbb{T}. \tag{3.3}$$

Thus, (3.2) holds for $n = 2$. Let now $n \geqslant 3$. By subtracting the second row to the first one, and the second column to the first one afterwards, one obtains

$$\det(A_{n,z}) = \det\begin{pmatrix} -2\cos\theta & z & 0 & 0 & \dots & 0 \\ z^* & 0 & z & z & \dots & z \\ 0 & z^* & 0 & z & \dots & z \\ 0 & z^* & z^* & 0 & \dots & z \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & z^* & z^* & z^* & \dots & 0 \end{pmatrix}.$$

In fact, $z + z^* = 2\cos\theta$. Cofactor expansion along the first row gives

$$\det(A_{n,z}) = -2\cos\theta\det(A_{n-1,z}) - z\det\begin{pmatrix} z^* & z & z & \dots & z \\ 0 & 0 & z & \dots & z \\ 0 & z^* & 0 & \dots & z \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & z^* & z^* & \dots & 0 \end{pmatrix} = -2\cos\theta\det(A_{n-1.z}) - \det(A_{n-2.z})$$

or equivalently

$$\det(A_{n,z}) = -(z + z^*)\det(A_{n-1,z}) - \det(A_{n-2,z}). \tag{3.4}$$

For $0 < \theta < \pi$, the complex number $z - z^*$ is nonzero. The recursive formula (3.4) can be solved by taking into account (3.3), arriving at the formula

$$\det(A_{n,z}) = (-1)^{n-1}\frac{z^{n-1} - \bar{z}^{\,n-1}}{z - \bar{z}} = (-1)^{n-1}\frac{\sin((n-1)\theta)}{\sin\theta}, \tag{3.5}$$

which, *a posteriori*, can also be proved from (3.3) and (3.4) by an inductive argument. □

Proposition 3.3. *Let $n \geqslant 2$ and $z = \mathrm{e}^{i\theta}$. The characteristic polynomial $p_{n,z}(\lambda) = \det(\lambda I - A_{n,z})$ has the following form*:

$$p_{n,z}(\lambda) = \begin{cases} (\lambda - n + 1)(\lambda + 1)^{n-1} & \text{for } \theta = 0 \text{ (i.e. } z = 1), \\ (\lambda + n - 1)(\lambda - 1)^{n-1} & \text{for } \theta = \pi \text{ (i.e. } z = -1), \\ \lambda^n - \displaystyle\sum_{k=1}^{n}\binom{n}{k}\frac{\sin((k-1)\theta)}{\sin\theta}\lambda^{n-k} & \text{for } 0 < \theta < \pi. \end{cases} \tag{3.6}$$

Proof. When θ belongs to $\{0, \pi\}$, the expression of $p_{n,z}(\lambda)$ in (3.6) is immediately retrieved from the known spectra $\mathrm{Sp}(K_n) = \{(-1)^{(n-1)}, n-1\}$ and $\mathrm{Sp}(-(K_n, 1)) = \{1^{(n-1)}, -(n-1)\}$. Let now $0 < \theta < \pi$. Note that $p_{1,z}(\lambda) = \lambda$ and $p_{2,z}(\lambda) = \lambda^2 - 1$, consistently with (3.6).

Let now $n \geqslant 3$. The coefficient of λ^{n-k} in $p_{n,z}(\lambda)$ is given by $(-1)^k$ times the sum of all principal $k \times k$ minors of the matrix $A_{n,z}$. They are all equal to $\det(A_{k,z})$ and there are $\binom{n}{k}$ of them. Now, equation (3.5) yields

$$\begin{aligned} p_{n,z}(\lambda) &= \lambda^n + \sum_{k=1}^{n}\binom{n}{k}(-1)^k\det(A_{k,z})\lambda^{n-k} \\ &= \lambda^n + \sum_{k=1}^{n}\binom{n}{k}(-1)^k(-1)^{k-1}\frac{\sin((k-1)\theta)}{\sin\theta}\lambda^{n-k} \\ &= \lambda^n - \sum_{k=1}^{n}\binom{n}{k}\frac{\sin((k-1)\theta)}{\sin\theta}\lambda^{n-k} \end{aligned} \tag{3.7}$$

as wanted. □

The proof of the next result employs the prosthaphaeresis formulas:

$$\cos a - \cos b = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right) \tag{3.8}$$

and

$$\sin a \sin b = \frac{\cos(a-b) - \cos(a+b)}{2}. \tag{3.9}$$

Proposition 3.4. *Let $z = \mathrm{e}^{i\theta}$ with $0 < \theta < \pi$. The roots of $p_{n,z}(\lambda)$, i.e., the eigenvalues of the matrix* (3.2), *are given by the numbers*

$$f_{n,k}(\theta) = \frac{\sin\left(\frac{(n-1)\theta - k\pi}{n}\right)}{\sin\left(\frac{\theta + k\pi}{n}\right)} \quad \text{for } 0 \leqslant k \leqslant n-1. \tag{3.10}$$

Moreover,

$$f_{n,0}(\theta) > f_{n,1}(\theta) > \cdots > f_{n,n-1}(\theta). \tag{3.11}$$

Proof. Let $\mathbb{T}_n = \{e^{\frac{2h\pi i}{n}} | 0 \leqslant h \leqslant n-1\}$ denote the set of the n-th roots of unity. In our hypotheses, $z - z^*$ is nonzero and, from (3.5) and (3.6), it follows that

$$p_{n,z}(\lambda) = \lambda^n - \sum_{k=1}^{n} \binom{n}{k} \frac{z^{k-1} - \bar{z}^{\,k-1}}{z - \bar{z}} \lambda^{n-k} = \frac{z(\lambda + z^*)^n - z^*(\lambda + z)^n}{z - z^*}.$$

Therefore, $p_{n,z}(\lambda) = 0$ if and only if

$$z^*(\lambda + z)^n = z(\lambda + z^*)^n. \tag{3.12}$$

By extracting the n-th root, (3.12) is equivalent to

$$\xi^*(\lambda + z) = q\ \xi(\lambda + z^*) \quad \text{for } \ \xi \coloneqq e^{\frac{i\theta}{n}} \ \text{and some } \ q \in \mathbb{T}_n. \tag{3.13}$$

Observe that the complex number $q\xi - \xi^2$ is nonzero for all $q \in \mathbb{T}_n$; in fact, by setting $q = e^{\frac{2k\pi i}{n}}$ and $\alpha \coloneqq (\theta + k\pi)/n$,

$$\begin{aligned} |q\xi - \xi^2|^2 &= (q\ \xi - \xi^2)(q^*\xi^* - (\xi^*)^2) \\ &= 2(1 - \mathrm{Re}(q\xi^2)) \\ &= 2(1 - \cos(2\alpha)) \\ &= 4\sin^2\alpha \end{aligned}, \tag{3.14}$$

which is nonzero, since

$$0 < \frac{\theta}{n} \leqslant \alpha \leqslant \frac{\theta + (n-1)\pi}{n} < \pi.$$

Hence, all the roots of (3.12) can be obtained by collecting the values of λ satisfying (3.13); namely,

$$\frac{\xi^{n-1} - q(\xi^*)^{n-1}}{q\xi - \xi^*} = \frac{\xi^{n-1} - q(\xi^*)^{n-1}}{|q\xi - \xi^*|^2} \cdot (q^*\xi^* - \xi) = 2\frac{\mathrm{Re}(q^*\xi^{n-2}) - \mathrm{Re}(\xi^n)}{|q\xi - \xi^*|^2} \quad \text{for } \ q \in \mathbb{T}_n.$$

From (3.14) and the equalities

$$\mathrm{Re}(q^*\xi^{n-2}) = \cos\left(\frac{(n-2)\theta - 2k\pi}{n}\right) \quad \text{and} \quad \mathrm{Re}(\xi^n) = \mathrm{Re}(z) = \cos(\theta),$$

one sees that the roots of $p_{n,z}(\lambda)$ are

$$f_{n,k}(\theta) = \frac{\cos\left(\frac{(n-2)\theta - 2k\pi}{n}\right) - \cos(\theta)}{2\sin^2\left(\frac{\theta + k\pi}{n}\right)} \quad \text{for } \ 0 \leqslant k \leqslant n-1, \tag{3.15}$$

and we arrive at (3.10) by replacing the numerator of (3.15) according to formula (3.8).

The remainder of the proof consists in a suitable sequence of "if and only if" steps. Let $f_{n,k_1}(\theta)$ and $f_{n,k_2}(\theta)$ be two eigenvalues of $A_{n,z}$. The first, the second, and the fourth of the following steps are, respectively, due to the positivity of all the denominators in (3.10), (3.9), and (3.8).

$$\begin{aligned} f_{n,k_1}(\theta) > f_{n,k_2}(\theta) &\Leftrightarrow \sin\left(\frac{(n-1)\theta - k_1\pi}{n}\right)\sin\left(\frac{\theta + k_2\pi}{n}\right) > \sin\left(\frac{(n-1)\theta - k_2\pi}{n}\right)\sin\left(\frac{\theta + k_1\pi}{n}\right) \\ &\Leftrightarrow \cos\left(\frac{(n-2)\theta - (k_1 + k_2)\pi}{n}\right) - \cos\left(\theta + \frac{k_2 - k_1}{n}\right) > \cos\left(\frac{(n-2)\theta - (k_1 + k_2)\pi}{n}\right) \\ &\quad - \cos\left(\theta + \frac{(k_1 - k_2)\pi}{n}\right), \\ &\Leftrightarrow \cos\left(\theta - \frac{(k_2 - k_1)\pi}{n}\right) - \cos\left(\theta + \frac{(k_2 - k_1)\pi}{n}\right) > 0, \\ &\Leftrightarrow 2\sin\theta\sin\left(\frac{(k_2 - k_1)\pi}{n}\right) > 0. \end{aligned}$$

The latter is true if and only if $k_2 > k_1$, proving the sequence of inequalities (3.11) and ending the proof. □

The trigonometric identity recalled in the following lemma concerns the collection of real functions $\{D_m(x)\}_{m\geqslant 0}$ known as the Dirichlet kernel (one of the existing proofs can be found, for instance, in [4, p. 175]).

Lemma 3.5. *Let m be any nonnegative integer. The following equality holds*:

$$D_m(x) := 1 + 2\sum_{h=1}^{m} \cos(hx) = \frac{\sin\left(\left(m+\frac{1}{2}\right)x\right)}{\sin\left(\frac{x}{2}\right)},$$

for every $x \notin \{2k\pi | k \in \mathbb{Z}\}$.

Proposition 3.6. *For all* $n \geqslant 3$, *the map* $\Psi_n : \theta \in [0,\pi] \longmapsto \lambda_1(\mathcal{K}_n(e^{i\theta})) \in \mathbb{R}$ *is continuous and strictly decreasing. The image of* Ψ_n *is the interval* $[1, n-1]$.

Proof. The maps Ψ_1 and Ψ_2 are constant. In fact, $\Psi_1(\theta) = 0$ and $\Psi_2(\theta) = 1$ for every $\theta \in [0,1]$. Let now $n \geqslant 3$ and let $f_{n,0} : (0,\pi) \longrightarrow \mathbb{R}$ be one of the functions considered in (3.10). Since

$$\lim_{\theta\to 0} f_{n,0}(\theta) = n-1 \quad \text{and} \quad \lim_{\theta\to\pi} f_{n,0}(\theta) = 1,$$

the map

$$\tilde{f}_{n,0} : \theta \in [0,\pi] \longmapsto \begin{cases} n-1 & \text{if } \theta = 0, \\ f_{n,0}(\theta) & \text{if } 0 < \theta < \pi, \\ 1 & \text{if } \theta = \pi, \end{cases}$$

is continuous in its entire domain and, by Remark 3.1 and Proposition 3.4, it turns out that $\Psi_n(\theta) = \tilde{f}_{n,0}(\theta)$ for all $\theta \in [0,1]$. In order to see that Ψ_n is strictly decreasing, a different argument is needed according to the parity of n.

Case 1: n is even. Let $n = 2k$ with $k \geqslant 2$. The number $\Psi_n(\theta) = \tilde{f}_{n,0}(\theta)$ can be expressed in terms of a suitable Dirichlet kernel (see Lemma 3.5). More precisely,

$$\Psi_n(\theta) = \tilde{f}_{n,0}(\theta) = D_{k-1}(2\theta/n) = 1 + 2\sum_{h=1}^{k-1} \cos\left(\frac{h\theta}{k}\right). \tag{3.16}$$

Thus, $\Psi_{2k}(\theta)$ is strictly decreasing since it is sum of functions which are strictly decreasing in the interval $[0,\pi]$; namely,

$$1 + 2\cos\left(\frac{\theta}{k}\right), \quad 2\cos\left(\frac{2\theta}{k}\right), \ \dots\ , \quad 2\cos\left(\frac{(k-1)\theta}{k}\right).$$

Case 2: n is odd. Let $n = 2k+1$ with $k \geqslant 1$. The addition formula for the sine gives

$$f_{2k+1,0}(\theta) = \frac{\sin\left(\frac{(n-2)\theta}{n} + \frac{\theta}{n}\right)}{\sin\left(\frac{\theta}{n}\right)} = f_{n-1}(\theta)\cos\left(\frac{\theta}{n}\right) + \cos\left(\frac{(n-2)\theta}{n}\right) \quad \text{for } 0 < \theta < \pi. \tag{3.17}$$

Since $f_{n-1}(\theta) = f_{2k}(\theta)$ is constant for $k = 1$ and it is strictly decreasing in $(0,\pi)$ for $k > 1$ by Case 1, the function $f_{2k+1,0}(\theta)$ is also strictly decreasing in $(0,\pi)$, being the sum of two strictly decreasing functions; namely, $f_{n-1,0}(\theta)\cos(\theta/n)$ and $\cos((n-2)\theta/n)$. The already proved continuity of Ψ_n in the closed interval $[0,\pi]$ ends the proof. □

Proof of Theorem 1.1. Let $\Omega(a)$ be the class of all connected $\mathbb{T}$-gain graphs Γ such that $\lambda_1(\Gamma) = a$. The set $\Omega(0)$ is a singleton; in fact, it only contains K_1 equipped with the empty gain function. Moreover, $\Omega(a) = \varnothing$ for all

$a \in (0,1)$ by Corollary 2.4. Along the proof of that corollary it has also been observed that the $\mathbb{T}$-gain graphs $\mathcal{K}_2(z)$ are in $\Omega(1)$ for all $z \in \mathbb{T}$.

Let now $a > 1$. Since the number $\lceil a \rceil + 1$ is at least 3, Proposition 3.6 ensures that an inverse function $\Psi_n^{-1} : [1, n-1] \to [0, \pi]$ of Ψ_n exists for $n \geqslant \lceil a \rceil + 1 \geqslant 3$. Let $\theta_n \coloneqq \Psi_n^{-1}(a)$. The inclusion

$$\{\mathcal{K}_n(\mathrm{e}^{i\theta_n}) | n \geqslant \lceil a \rceil + 1\} \subseteq \Omega(a). \tag{3.18}$$

proves both Parts (i) and (ii) of Theorem 1.1.

Part (iii) also comes from (3.18) and the fact that $\Omega(a)$ does not contain $\mathbb{T}$-gain graphs with less than $\lceil a \rceil + 1$ vertices. In order to see this, let $\varDelta_G$ denote the maximum vertex degree of a graph G with $|V_G| \leqslant \lceil a \rceil$. Clearly $\varDelta_G \leqslant |V_G| - 1 \leqslant \lceil a \rceil - 1$ and, as a consequence of [27, Theorem 4.3], the index of $\Gamma = (G, \gamma)$ cannot be larger than $\varDelta_G \leqslant \lceil a \rceil - 1 < a$. □

Remark 3.7. Theorem 1.1 admits a shorter but more conceptual proof, for which Propositions 3.3, 3.4, and 3.6 are not really needed: since the entries of the Hermitian matrix $A_{n,\mathrm{e}^{i\theta}}$ continuously depend on θ, the eigenvalues

$$\lambda_1(A_{n,\mathrm{e}^{i\theta}}) \geqslant \cdots \geqslant \lambda_n(A_{n,\mathrm{e}^{i\theta}})$$

are continuous functions with respect to θ. This is quite a well-known fact in perturbation theory sometimes referred as “the Rellich’s theorem.” In fact, a proof can be found in [28, p. 39]. It follows that the image of the map

$$\Psi_n : \theta \in [0, \pi] \mapsto \lambda_1(A_{n,\mathrm{e}^{i\theta}}) \in \mathbb{R}$$

is a connected subset of $\mathbb{R}$. Since $1 = \lambda_1(A_{n,-1})$ and $n - 1 = \lambda_1(A_{n,1})$ are both in Θ, then $[1, n-1] \subseteq \mathrm{Im}(\Psi_n)$. In spite of its elegance and shortness, this approach does not allow one to detect, for any fixed $a \in \mathbb{R}$, which (and how many) z’s make $\mathcal{K}_n(z)$ belonging to $\Omega(a)$.

Corollary 3.8. *For every real number $c \in \mathbb{R}$, there exists a $\mathbb{T}$-gain graph $\Gamma = (G, \gamma)$ such that c belongs to* $\mathrm{Sp}(\Gamma)$.

Proof. If c belongs to $\mathrm{Sp}(\Gamma)$, then $-c$ belongs to $\mathrm{Sp}(-\Gamma)$. Thus, it is not restrictive to assume $c \geqslant 0$. For $c \in \{0\} \cup [1, +\infty)$, the result immediately follows from Theorem 1.1. If instead $c \in (0,1)$, then $c \in \mathrm{Sp}(\mathcal{K}_3'(\mathrm{e}^{i\theta}))$ for $\theta = \arccos(c/2)$ by (2.3) (the graph $\mathcal{K}_3'(\mathrm{e}^{i\theta})$ is depicted in Figure 1). □

4 a-$\mathbb{T}$-regularity and products

In Section 1, a-$\mathbb{T}$-regular graphs have been defined, and it has also been observed that a-$\mathbb{T}$-regular graphs with n vertices are precisely the $\mathbb{T}$-gain graphs having the all-ones vector $\mathbf{j}_n$ in their spectrum. However, it should be noted that, contrarily to what happens for simple graphs as a consequence of the Perron-Froebenius theorem, the number a is not necessarily the largest eigenvalue of a-$\mathbb{T}$-regular graphs. For instance, the graph $\mathcal{K}_3'(\mathrm{e}^{i\theta})$ considered in Example 2.2 is $2\cos\theta$-regular; yet, $2\cos\theta$ is not the largest eigenvalue for $\pi/3 < \theta \leqslant \pi$. The next proposition shows how a-$\mathbb{T}$-regularity behaves with respect to the several products recalled in Section 2.

Proposition 4.1. *For $1 \leqslant i \leqslant h$, let Γ_i be an a_i-$\mathbb{T}$-regular graph with n_i vertices, and let $\mathfrak{B}$ be a nonempty subset of $\mathfrak{F}_h \coloneqq \{0,1\}^h \backslash \{(0, \ldots, 0)\}$.*

(1) *The $\mathbb{T}$-gain graph $\Gamma = \mathrm{NEPS}(\Gamma_1, \ldots, \Gamma_h; \mathfrak{B})$ is r-$\mathbb{T}$-regular for*

$$r = \sum_{(b_1, \ldots, b_h) \in \mathfrak{B}} a_1^{b_1} \cdots a_h^{b_h}.$$

(2) *The HG-lexicographic product $\Gamma_1[\Gamma_2]$ is s-$\mathbb{T}$-regular for $s = a_1 n_2 + a_2$.*

Proof. The $\mathbb{T}$-gain graph Γ has $n \coloneqq \prod_{i=1}^{h} n_i$ vertices. Using Proposition 2.6 and the properties of the tensor product of matrices,

$$\begin{aligned} A(\Gamma)\mathbf{j}_n &= \left(\sum_{(b_1,\dots,b_h)\in\mathfrak{B}} A(\Gamma_1)^{b_1} \otimes \cdots \otimes A(\Gamma_h)^{b_h} \right) \otimes_{i=1}^{h} \mathbf{j}_{n_i} \\ &= \sum_{(b_1,\dots,b_h)\in\mathfrak{B}} (A(\Gamma_1)^{b_1}\mathbf{j}_{n_1}) \otimes \cdots \otimes (A(\Gamma_1)^{b_h}\mathbf{j}_{n_h}) \\ &= \sum_{(b_1,\dots,b_h)\in\mathfrak{B}} (a_1^{b_1}\mathbf{j}_{n_1}) \otimes \cdots \otimes (a_h^{b_h}\mathbf{j}_{n_h}) \\ &= \left(\sum_{(b_1,\dots,b_h)\in\mathfrak{B}} a_1^{b_1} \cdots a_h^{b_h} \right) \mathbf{j}_{n_1} \otimes \cdots \otimes \mathbf{j}_{n_h} \\ &= r\mathbf{j}_n. \end{aligned}$$

This proves Part (i). The proof of Part (ii) relies on (2.9):

$$\begin{aligned} A(\Gamma[\Lambda])\mathbf{j}_{n_1 n_2} &= (A(\Gamma) \otimes J_{n_2})(\mathbf{j}_{n_1} \otimes \mathbf{j}_{n_2}) + (I_{n_1} \otimes A(\Lambda))(\mathbf{j}_{n_1} \otimes \mathbf{j}_{n_2}) \\ &= (A(\Gamma)\mathbf{j}_{n_1}) \otimes (J_{n_2}\,\mathbf{j}_{n_2}) + \mathbf{j}_{n_1} \otimes A(\Lambda)\,\mathbf{j}_{n_2} \\ &= (a_1\,\mathbf{j}_{n_1}) \otimes (n_2\,\mathbf{j}_{n_2}) + \mathbf{j}_{n_1} \otimes (a_2\,\mathbf{j}_{n_2}) \\ &= (a_1 n_2 + a_2)\mathbf{j}_{n_1} \otimes \mathbf{j}_{n_2} = s\,\mathbf{j}_{n_1 n_2}. \end{aligned}$$

□

The reader could ask whether the BCD-lexicographic product preserves $\mathbb{T}$-regularity. The answer is negative in general. In Figure 2, the graph $\Gamma \coloneqq (K_2, 1)$ is 1-$\mathbb{T}$-regular, whereas the signed hourglass $\Lambda \coloneqq \mathcal{H} = (H, \zeta)$ is 0-$\mathbb{T}$-regular (the gain function ζ is defined as follows: $\zeta(v_1v_2) = \zeta(v_2v_3) = \zeta(v_4v_5) = 1$ and $\zeta(v_1v_3) = \zeta(v_2v_4) = \zeta(v_2v_5) = -1$). The submatrix of $A(\Gamma * \Lambda)$ consisting of the rows indexed by (u_1, v_1) and (u_1, v_2) is

$$B = \begin{pmatrix} 0 & 1 & -1 & 0 & 0 & 1 & 1 & -1 & 1 & 1 \\ 1 & 0 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 \end{pmatrix}.$$

By looking at B, one realizes that the $\mathbb{T}$-outgains of the vertices (u_1, v_1) and (u_1, v_2) in $\Gamma * \Lambda$ are different. More precisely,

$$d^{\vec{}}_{\Gamma*\Lambda}(u_1, v_1) = 3 \neq 1 = d^{\vec{}}_{\Gamma*\Lambda}(u_1, v_2).$$

Proposition 4.1 and (2.11) lead without difficulty to the following result.

Corollary 4.2. *For $1 \leqslant i \leqslant h$, let Γ_i be an a_i-$\mathbb{T}$-regular graph. The Cartesian product $\square_{i=1}^{h}\Gamma_i$ is an a-$\mathbb{T}$-regular graph with $a = a_1 + \cdots + a_h$.*

Proof of Theorem 1.3. For $a \in \mathbb{R}$, let $\mathcal{R}(a)$ be the set of connected a-$\mathbb{T}$-regular graphs, and let v be a vertex of a $\mathbb{T}$-gain graph Γ. Since $d^{\vec{}}_{-\Gamma}(v) = -d^{\vec{}}_{\Gamma}(v)$, it will be suffice to show that $\mathcal{R}(a)$ contains infinitely many $\mathbb{T}$-gain graphs for $a \geqslant 0$. In Example 2.2, the 2Re(z)-regular triangle $\mathcal{K}_3'(z)$ has been introduced. Corollary 4.2 implies

$$\widetilde{\Gamma}_h \coloneqq \underbrace{\mathcal{K}_3'(i)\square\cdots\square\mathcal{K}_3'(i)}_{h \text{ times}} \in \mathcal{R}(0) \quad \text{for all} \quad h \geqslant 1,$$

proving the statement for $a = 0$. For $a > 0$, three cases will be considered. In all of them, Corollary 4.2 will be used.

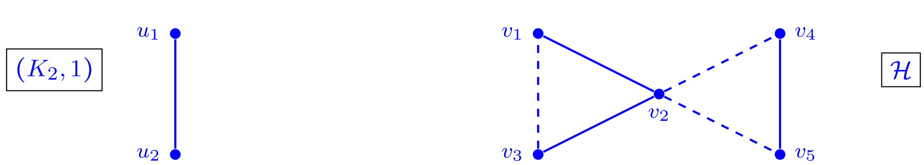

Figure 2: $(K_2, 1)$ and the signed hourglass $\mathcal{H}$. Dashed lines represent negative edges.

Case 1: a is a positive integer. Since $(K_{a+1}, 1)$ is a-$\mathbb{T}$-regular,

$$\{(K_{a+1}, 1)\} \cup \{(K_{a+1}, 1) \,\Box\, \widetilde{\Gamma}_h | h \geqslant 1\} \subseteq \mathcal{R}(a).$$

Case 2: $0 < a < 2$. If $\theta_a \coloneqq \arccos(a/2)$, $\{\mathcal{K}_3'(e^{i\theta_a})\} \cup \{\mathcal{K}_3'(e^{i\theta_a}) \,\Box\, \widetilde{\Gamma}_h | h \geqslant 1\} \subseteq \mathcal{R}(a)$.

Case 3: $a > 2$. The number $p \coloneqq (a - \lfloor a \rfloor + 1)/2$ belongs to $[1/2, 1)$. For $\omega_a \coloneqq \arccos p$, the $\mathbb{T}$-gain graph $\Lambda_a \coloneqq (K_{\lfloor a \rfloor}, 1) \,\Box\, \mathcal{K}_3'(e^{i\omega_a})$ is a-$\mathbb{T}$-regular; in fact, $\lfloor a \rfloor - 1 + 2\cos\omega_a = \lfloor a \rfloor - 1 + 2p = a$. Therefore,

$$\{\Lambda_a\} \cup \{\Lambda_a \Box \widetilde{\Gamma}_h | h \geqslant 1\} \subseteq \mathcal{R}(a),$$

and the proof is completed. □

In the proof of Theorem 1.3, Case 1 could be absorbed in Cases 2 and 3: if a is a positive integer $(K_{a+1}, 1)$ and $\Lambda_a \coloneqq (K_a, 1) \Box \mathcal{K}_3'(e^{\frac{1}{2}i})$ are both a-$\mathbb{T}$-regular, yet the former has a smaller number of vertices. In fact, for $a \geqslant 0$, $\mathcal{R}(a)$ surely contains: a graph of order $a + 1$ if a is an integer, and a graph of order $\max\{3, 3\lfloor a \rfloor\}$ if a is not an integer. In any case, a suitable join-like construction, when a is not an integer, will lead to the detections of items in $\mathcal{R}(a)$ with no more than $\lfloor a \rfloor + 4$ vertices.

Definition 4.3. Let $\Gamma = (G, \gamma)$ and $\Gamma' = (G', \gamma')$ be two nonempty $\mathbb{T}$-gain graphs with $V_G = \{u_1, \ldots, u_n\}$ and $V_{G'} = \{u_1', \ldots, u_m'\}$, and let $G \vee G'$ denote the usual join between G and G'. For $z \in \mathbb{T}$, we define a gain function γ_z on the set $\vec{E}_{G \vee G'}$ as follows:

$$\gamma_z(u_i u_j) \coloneqq \gamma(u_i u_j), \quad \gamma_z(u_i' u_j') \coloneqq \gamma'(u_i' u_j') \quad \text{and} \quad \gamma_z(u_i u_j') = \gamma_z(u_j' u_i)^* = \begin{cases} z & \text{if } i + j \text{ is even;} \\ z^* & \text{if } i + j \text{ is odd.} \end{cases}$$

The z-join $\Gamma \vee_z \Gamma'$ is, by definition, the $\mathbb{T}$-gain graph $(G \vee G', \gamma_z)$.

Chosen for $V_{G \vee G'}$ the ordering $u_1, \ldots, u_n, u_1', \ldots, u_m'$, the adjacency matrix of $\Gamma \vee_z \Gamma'$ reads as follows:

$$A(\Gamma \vee_z \Gamma') = \left(\begin{array}{c|ccc} A(\Gamma) & \begin{matrix} z & z^* & \ldots \\ z^* & z & \ldots \\ \vdots & \vdots & \ddots \end{matrix} \\ \hline \begin{matrix} z^* & z & \ldots \\ z & z^* & \ldots \\ \vdots & \vdots & \ddots \end{matrix} & A(\Gamma') \end{array}\right).$$

The following lemma is straightforward.

Lemma 4.4. *If Γ and Γ' are a-$\mathbb{T}$-regular graphs having the same even number of vertices n, then $\Gamma \vee_z \Gamma'$ is a b-$\mathbb{T}$-regular graph, with $b = a + n\mathrm{Re}(z)$.*

The following Theorem 4.5 is the last result of this article.

Theorem 4.5. *Let a be any real number. Then, $\mathcal{R}(a)$ contains at least one $\mathbb{T}$-gain graph whose order is not larger than $|a| + 5$.*

Proof. If Γ belongs to $\mathcal{R}(a)$, then $-\Gamma$ belongs to $\mathcal{R}(-a)$. Thus, it is not restrictive to assume $a \geqslant 0$. If $0 \leqslant a \leqslant 2$, the statement surely holds since (as seen in Case 2 of the previous proof) $\mathcal{R}(a)$ contains a $\mathbb{T}$-gain triangle. Furthermore, the statement is trivially verified when a is a positive integer, since $(K_{a+1}, 1) \in \mathcal{R}(a)$. From now on, a is assumed to be a real number larger than 2 which is not an integer, and $h \coloneqq \lfloor a/4 \rfloor$.

Case 1: $\lfloor a \rfloor = 4h + \varepsilon$ for $\varepsilon \in \{0, 1, 2\}$. Let $\tilde{z}_1$ be a complex unit such that

$$\mathrm{Re}(\tilde{z}_1) = \frac{2h + \varepsilon - 1 + a - \lfloor a \rfloor}{2h + 2}.$$

From Lemma 4.4, the $\mathbb{T}$-gain graph $(K_{2h+2}, 1) \vee_{\tilde{z}_1} (K_{2h+2}, 1)$, whose order is $\lfloor a \rfloor + 4 - \varepsilon < a + 5$, belongs to $\mathcal{R}(a)$.

Case 2: $\lfloor a \rfloor = 4h + 3$. Let $\tilde{z}_2$ be a complex unit such that

$$\mathrm{Re}(\tilde{z}_2) = \frac{2h + a - \lfloor a \rfloor}{2h + 4}.$$

From Lemma 4.4, the $\mathbb{T}$-gain graph $(K_{2h+4}, 1) \vee_{\tilde{z}_2} (K_{2h+4}, 1)$ is a-$\mathbb{T}$-regular. Its order is $\lfloor a \rfloor + 5 < a + 5$. □

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