Research Article
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New constructions of nonregular cospectral graphs

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Abstract: We consider two types of joins of graphs G_1 and G_2, $G_1 \veebar G_2$ – the neighbors splitting join and $G_1 \underset{=}{\vee} G_2$ – the nonneighbors splitting join, and compute the adjacency characteristic polynomial, the Laplacian characteristic polynomial, and the signless Laplacian characteristic polynomial of these joins. When G_1 and G_2 are regular, we compute the adjacency spectrum, the Laplacian spectrum, the signless Laplacian spectrum of $G_1 \underset{=}{\vee} G_2$, and the normalized Laplacian spectrum of $G_1 \veebar G_2$ and $G_1 \underset{=}{\vee} G_2$. We use these results to construct nonregular, nonisomorphic graphs that are cospectral with respect to the four matrices: adjacency, Laplacian, signless Laplacian and normalized Laplacian.

Keywords: spectra of graphs, nonisomorphic cospectral graphs, NS and NNS joins, coronal and Schur complement

MSC 2020: 05C50

1 Introduction

Spectral graph theory is the study of graphs via the spectrum of matrices associated with them [3,6,8,22,27]. The graphs in this article are undirected and simple. There are several matrices associated with a graph, four of which are considered here: the adjacency matrix, the Laplacian matrix, the signless Laplacian matrix, and the normalized Laplacian matrix.

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$.

Definition 1.1. The adjacency matrix of G, $A(G)$, is defined as follows:

$$(A(G))_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent;} \\ 0, & \text{otherwise.} \end{cases}$$

Let $d_i = d_G(v_i)$ be the degree of vertex v_i in G, and let $D(G)$ be the diagonal matrix with diagonal entries $d_1, d_2, \ldots, d_n$.

Dedicated to Prof. Frank J. Hall, in celebration of his many contributions to matrix theory.

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Definition 1.2. The Laplacian matrix, $L(G)$, and the signless Laplacian matrix, $Q(G)$, of G are defined as $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$, respectively.

Definition 1.3. [6] The normalized Laplacian matrix, $\mathcal{L}(G)$, is defined to be $\mathcal{L}(G) = I_n - D(G)^{-\frac{1}{2}}A(G)D(G)^{-\frac{1}{2}}$ (with the convention that if the degree of vertex v_i in G is 0, then $(d_i)^{-\frac{1}{2}} = 0$). In other words,

$$(\mathcal{L}(G))_{ij} = \begin{cases} 1, & \text{if } i = j \text{ and } d_i \neq 0; \\ -\frac{1}{\sqrt{d_i d_j}}, & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j; \\ 0, & \text{otherwise}. \end{cases}$$

Notation 1.4. For an $n \times n$ matrix M, we denote the characteristic polynomial $\det(xI_n - M)$ of M by $f_M(x)$, where I_n is the identity matrix of order n. In particular, for a graph G, $f_{X(G)}(x)$ is the X-characteristic polynomial of G, for $X \in \{A, L, Q, \mathcal{L}\}$. The roots of the X-characteristic polynomial of G are the X-eigenvalues of G and the collection of the X-eigenvalues, including multiplicities, is called the X-spectrum of G.

Notation 1.5. The multiplicity of an eigenvalue λ is denoted by a superscript above λ.

Example 1.6. The A-spectrum of the complete graph K_n is $\{n-1, (-1)^{[n-1]}\}$.

Remark 1.7. If

$$\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G),$$
$$\mu_1(G) \leq \mu_2(G) \leq \cdots \leq \mu_n(G),$$
$$\nu_1(G) \geq \nu_2(G) \geq \cdots \geq \nu_n(G),$$
$$\delta_1(G) \leq \delta_2(G) \leq \cdots \leq \delta_n(G)$$

are the eigenvalues of $A(G)$, $L(G)$, $Q(G)$, and $\mathcal{L}(G)$, respectively. Then, $\sum_{i=1}^{n} \lambda_i = 0$, $\mu_1(G) = 0$, $\nu_n(G) \geq 0$, and $\delta_1(G) = 0$, $\delta_n(G) \leq 2$ (equality iff G is bipartite).

Remark 1.8. If G is an r-regular graph, then $\mu_i(G) = r - \lambda_i(G)$, $\nu_i(G) = r + \lambda_i(G)$, and $\delta_i(G) = 1 - \frac{1}{r}\lambda(G)$.

Definition 1.9. Two graphs G and H are X-cospectral if they have the same X-spectrum. If X-cospectral graphs are not isomorphic, we say that they are XNICS.

Definition 1.10. Let S be a subset of $\{A, L, Q, \mathcal{L}\}$. The graphs G and H are SNICS if they are XNICS for all $X \in S$.

Definition 1.11. A graph G is determined by its X-spectrum if every graph H that is X-cospectral with G is isomorphic to G.

A basic problem in spectral graph theory, [28, 29], is determining which graphs are determined by their spectrum or finding nonisomorphic X-cospectral graphs.

Theorem 1.12. [28] *If G is regular, then the following are equivalent*;
- *G is determined by its A-spectrum,*
- *G is determined by its L-spectrum,*
- *G is determined by its Q-spectrum,*
- *G is determined by its $\mathcal{L}$-spectrum.*

Thus, for regular graphs G and H, we say that G and H are cospectral if they are X-cospectral with respect to any $X \in \{A, L, Q, \mathcal{L}\}$.

Proposition 1.13. [28] *Every regular graph with less than ten vertices is determined by its spectrum.*

Example 1.14. Graphs in Figure 1 are regular and cospectral. They are nonisomorphic since in G, there is an edge that lies in three triangles but there is no such edge in H.

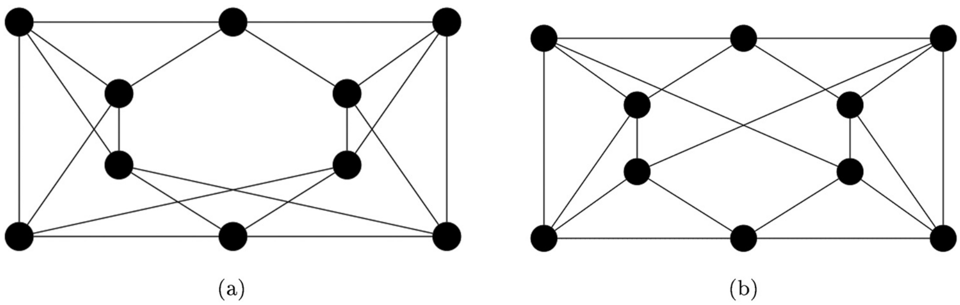


Figure 1: Two regular nonisomorphic cospectral graphs. (a) (G) and (b) (H).

In recent years, several researchers studied the spectral properties of graphs which are constructed by graph operations. These operations include disjoint union, the Cartesian product, the Kronecker product, the strong product, the lexicographic product, the rooted product, the corona, the edge corona, the neighborhood corona, etc. We refer the reader to [1,2,8,9,12,13,16,21,23–26] and the references therein for more graph operations and the results on the spectra of these graphs.

Many operations are based on the join of graphs.

Definition 1.15. [14] The join of two graphs is their disjoint union together with all the edges that connect all the vertices of the first graph with all the vertices of the second graph.

Recently, many researchers provided several variants of join operations of graphs and investigated their spectral properties. Some examples are Cardoso [5], Indulal [17], Liu and Zhang [18], Varghese and Susha [30], and Das and Panigrahi [11].

Butler [4] constructed nonregular bipartite graphs, which are cospectral with respect to both the adjacency and the normalized Laplacian matrices. He asked for examples of nonregular $\{A, L, \mathcal{L}\}$NICS graphs. A slightly more general question is

Question 1.16. Construct nonregular $\{A, L, Q, \mathcal{L}\}$NICS graphs.

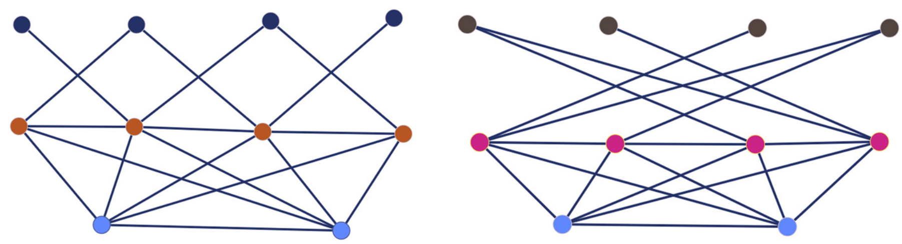
Such examples can be constructed using special join operation defined by Lu et al. [19] and a variant of this operation, suggested in this article.

Definition 1.17. [19] Let G_1 and G_2 be two vertex disjoint graphs with $V(G_1) = \{u_1, u_2, \ldots, u_n\}$. The splitting V-vertex join of G_1 and G_2, denoted by $G_1 \veebar G$, is obtained by adding new vertices $u_1', u_2', \ldots, u_n'$ to $G_1 \vee G_2$ and connecting u_i' to u_j if and only if $(u_i, u_j) \in E(G_1)$.

We refer to the splitting V-vertex join as neighbors splitting (NS) join and define a new type of join, nonneighbors splitting (NNS) join.

Definition 1.18. Let G_1 and G_2 be two vertex disjoint graphs with $V(G_1) = \{u_1, u_2, \ldots, u_n\}$. The NNS join of G_1 and G_2, denoted by $G_1 \underset{=}{\vee} G_2$, is obtained by adding new vertices $u_1', u_2', \ldots, u_n'$ to $G_1 \vee G_2$ and connecting u_i' to u_j iff $(u_i, u_j) \notin E(G_1)$.

Example 1.19. Let G_1 and G_2 be the path P_4 and the path P_2, respectively. The graphs $P_4 \underset{=}{\vee} P_2$ and $P_4 \veebar P_2$ are given in Figure 2.



(a) (b)

Figure 2: The graphs $P_4 \veebar P_2$ and $P_4 \underset{=}{\vee} P_2$. (a) $P_4 \veebar P_2$ and (b) $P_4 \underset{=}{\vee} P_2$.

The structure of the article is as follows; after preliminaries, we compute the adjacency characteristic polynomial, the Laplacian characteristic polynomial, and the signless Laplacian characteristic polynomial of $G_1 \underset{=}{\vee} G_2$ and $G_1 \veebar G_2$, and use the results to construct $\{A, L, Q\}$NICS graphs, and finally, under regularity assumptions, we compute the A-spectrum, the L-spectrum, the Q-spectrum, and the $\mathcal{L}$-spectrum of NS and NNS joins and use the results to construct $\{A, L, Q, \mathcal{L}\}$NICS graphs.

2 Preliminaries

Notation 2.1.
- $\mathbf{1}_n$ *denotes* $n \times 1$ *column whose all entries are* 1,
- $J_{s\times t} = \mathbf{1}_s\mathbf{1}_t^T$, $J_s = J_{s\times s}$,
- $O_{s\times t}$ *denotes the zero matrix of order* $s \times t$,
- $\operatorname{adj}(A)$ *denotes the adjugate of* A.
- $\overline{G}$ *denotes the complement of graph* G.

Definition 2.2. [7,20] The coronal $\Gamma_M(x)$ of an $n \times n$ matrix M is the sum of the entries of the inverse of the characteristic matrix of M, i.e.,

$$\Gamma_M(x) = \mathbf{1}_n^T(xI_n - M)^{-1}\mathbf{1}_n. \tag{2.1}$$

Lemma 2.3. [7,20] *Let M be an $n \times n$ matrix with all row sums equal to r (e.g., the adjacency matrix of a r-regular graph). Then,*

$$\Gamma_M(x) = \frac{n}{x - r}.$$

Definition 2.4. Let M be a block matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

such that its blocks A and D are square. If A is invertible, the Schur complement of A in M is

$$M/A = D - CA^{-1}B,$$

and if D is invertible, the Schur complement of D in M is

$$M/D = A - BD^{-1}C.$$

Issai Schur proved the following lemma.

Lemma 2.5. [15] *If D is invertible, then*

$$\det M = \det(M/D)\det D,$$

and if A is invertible, then

$$\det M = \det(M/A)\det A.$$

Lemma 2.6. *Let M be a block matrix*

$$M = \begin{pmatrix} A & B & J_{n_1\times n_2} \\ B & C & O_{n_1\times n_2} \\ J_{n_2\times n_1} & O_{n_2\times n_1} & D \end{pmatrix},$$

where A, B, and C are square matrices of order n_1 and D is a square matrix of order n_2. Then, the Schur complement of $xI_{n_2} - D$ in the characteristic matrix of M is

$$\begin{pmatrix} xI_{n_1} - A - \Gamma_D(x)J_{n_1} & -B \\ -B & xI_{n_1} - C \end{pmatrix}.$$

Proof. The characteristic matrix of M is

$$xI_{2n_1+n_2} - M = \begin{pmatrix} xI_{n_1} - A & -B & -J_{n_1\times n_2} \\ -B & xI_{n_1} - C & O_{n_1\times n_2} \\ -J_{n_2\times n_1} & O_{n_2\times n_1} & xI_{n_2} - D \end{pmatrix}.$$

The Schur complement of $(xI_{n_2} - D)$ is

$$\begin{aligned}(xI_{2n_1+n_2} - M)/(xI_{n_2} - D) &= \begin{pmatrix} xI_{n_1} - A & -B \\ -B & xI_{n_1} - C \end{pmatrix} - \begin{pmatrix} -J_{n_1\times n_2} \\ O_{n_1\times n_2} \end{pmatrix}(xI_{n_2} - D)^{-1}\begin{pmatrix} -J_{n_2\times n_1} & O_{n_2\times n_1} \end{pmatrix} \\ &= \begin{pmatrix} xI_{n_1} - A & -B \\ -B & xI_{n_1} - C \end{pmatrix} - \begin{pmatrix} \mathbf{1}_{n_1}\mathbf{1}_{n_2}^T \\ O_{n_1\times n_1} \end{pmatrix}((xI_{n_2} - D)^{-1})\begin{pmatrix} \mathbf{1}_{n_2}\mathbf{1}_{n_1^T} & O_{n_1\times n_1} \end{pmatrix} \\ &= \begin{pmatrix} xI_{n_1} - A - \Gamma_D(x)J_{n_1} & -B \\ -B & xI_{n_1} - C \end{pmatrix}.\end{aligned}$$

□

Lemma 2.7. [8] *If A is an $n \times n$ real matrix and a is an real number, then*

$$\det(A + \alpha J_n) = \det(A) + \alpha \mathbf{1}_n^T \mathrm{adj}(A)\mathbf{1}_n. \tag{2.2}$$

3 The characteristic polynomials of the NNS and NS joins

Lu et al. [19] computed the adjacency, Laplacian, and signless Laplacian characteristic polynomials of $G_1 \veebar G_2$, where G_1 and G_2 are regular.

Here, we compute the characteristic polynomials of $G_1 \underset{=}{\vee} G_2$ and $G_1 \veebar G_2$, where G_1 and G_2 are arbitrary graphs. The proofs for the two joins (NS and NNS) are quite similar and use Lemma 2.5 (twice) and Lemmas 2.6 and 2.7. The results are used to construct nonregular $\{A, L, Q\}$NICS graphs.

3.1 Adjacency characteristic polynomial

Theorem 3.1. *Let G_i be a graph on n_i vertices for $i = 1, 2$. Then,*

(a) $f_{A(G_1 \underset{=}{\vee} G_2)}(x) = x^{n_1} f_{A(G_2)}(x) \det\left(xI_{n_1} - A(G_1) - \frac{1}{x}A^2(\overline{G}_1)\right)\left[1 - \Gamma_{A(G_2)}(x)\Gamma_{A(G_1)+\frac{1}{x}A^2(\overline{G}_1)}(x)\right]$.

(b) $f_{A(G_1 \veebar G_2)}(x) = x^{n_1} f_{A(G_2)}(x) \det\left(xI_{n_1} - A(G_1) - \frac{1}{x}A^2(G_1)\right)\left[1 - \Gamma_{A(G_2)}(x)\Gamma_{A(G_1)+\frac{1}{x}A^2(G_1)}(x)\right]$.

Proof. We prove (a). The proof of (b) is similar.

With a suitable ordering of the vertices of $G_1 \underset{=}{\vee} G_2$, we obtain

$$A(G_1 \underset{=}{\vee} G_2) = \begin{pmatrix} A(G_1) & A(\overline{G}_1) & J_{n_1 \times n_2} \\ A(\overline{G}_1) & O_{n_1 \times n_1} & O_{n_1 \times n_2} \\ J_{n_2 \times n_1} & O_{n_2 \times n_1} & A(G_2) \end{pmatrix}.$$

Thus,

$$\begin{aligned} f_{A(G_1 \underset{=}{\vee} G_2)}(x) &= \det(xI_{2n_1+n_2} - A(G_1 \underset{=}{\vee} G_2)) \\ &= \det\left(\begin{array}{cc|c} xI_{n_1} - A(G_1) & -A(\overline{G}_1) & -J_{n_1 \times n_2} \\ -A(\overline{G}_1) & xI_{n_1} & O_{n_1 \times n_2} \\ \hline -J_{n_2 \times n_1} & O_{n_2 \times n_1} & xI_{n_2} - A(G_2) \end{array}\right) \\ &= \det(xI_{n_2} - A(G_2)) \det((xI_{2n_1+n_2} - A(G_1 \underset{=}{\vee} G_2))/(xI_{n_2} - A(G_2))) \end{aligned}$$

by the Lemma of Schur (Lemma 2.5).

By Lemma 2.6,

$$(xI_{2n_1+n_2} - A(G_1 \underset{=}{\vee} G_2))/(xI_{n_2} - A(G_2)) = \begin{pmatrix} xI_{n_1} - A(G_1) - \Gamma_{A(G_2)}(x)J_{n_1 \times n_1} & -A(\overline{G}_1) \\ -A(\overline{G}_1) & xI_{n_1} \end{pmatrix}.$$

Using again Lemma 2.5, we obtain

$$\det((xI_{2n_1+n_2} - A(G_1 \underset{=}{\vee} G_2))/(xI_{n_2} - A(G_2))) = \det(xI_{n_1})\det\left(xI_{n_1} - A(G_1) - \Gamma_{A(G_2)}(x)J_{n_1 \times n_1} - \frac{1}{x}A^2(\overline{G}_1)\right).$$

By Lemma 2.7, we obtain

$$\begin{aligned} &\det((xI_{2n_1+n_2} - A(G_1 \underset{=}{\vee} G_2))/(xI_{n_2} - A(G_2))) \\ &= x^{n_1}\left(\det(xI_{n_1} - A(G_1) - \frac{1}{x}A^2(\overline{G}_1)) - \Gamma_{A(G_2)}(x)1_{n_1}^T \operatorname{adj}\left(xI_{n_1} - A(G_1) - \frac{1}{x}A^2(\overline{G}_1)\right)1_{n_1}\right) \\ &= x^{n_1}\det\left(xI_{n_1} - A(G_1) - \frac{1}{x}A^2(\overline{G}_1)\right)\left[1 - \Gamma_{A(G_2)}(x)1_{n_1}^T(xI_{n_1} - A(G_1) - \frac{1}{x}A^2(\overline{G}_1))^{-1}1_{n_1}\right] \\ &= x^{n_1}\det\left(xI_{n_1} - A(G_1) - \frac{1}{x}A^2(\overline{G}_1)\right)\left[1 - \Gamma_{A(G_2)}(x)\Gamma_{A(G_1)+\frac{1}{x}A^2(\overline{G}_1)}(x)\right]. \end{aligned}$$

Thus,

$$f_{A(G_1 \underset{=}{\vee} G_2)}(x) = x^{n_1} f_{A(G_2)}(x) \det\left(xI_{n_1} - A(G_1) - \frac{1}{x}A^2(\overline{G}_1)\right)\left[1 - \Gamma_{A(G_2)}(x)\Gamma_{A(G_1)+\frac{1}{x}A^2(\overline{G}_1)}(x)\right].$$ □

3.2 Laplacian characteristic polynomial

In this section, we derive the Laplacian characteristic polynomials of $G_1 \veebar G_2$ and $G_1 \underset{=}{\vee} G_2$ when $G_1$and G_2 are arbitrary graphs.

Theorem 3.2. *Let G_i be a graph on n_i vertices for $i = 1, 2$. Then,*
(a)

$$
\begin{aligned}
f_{L(G_1 \underset{=}{\vee} G_2)}(x) = {} & \det((x - n_1)I_{n_2} - L(G_2))\det((x - n_1 + 1)I_{n_1} + D(G_1)) \\
& \det((x - n_1 - n_2 + 1)I_{n_1} - L(G_1) + D(G_1) - A(\overline{G}_1)((x - n_1 + 1)I_{n_1} + D(G_1))^{-1}A(\overline{G}_1)) \\
& [1 - \Gamma_{L(G_2)}(x - n_1)\Gamma_{L(G_1)-D(G_1)+A(\overline{G}_1)((x-n_1+1)I_{n_1}+D(G_1))^{-1}A(\overline{G}_1)}(x - n_1 - n_2 + 1)].
\end{aligned}
$$

(b)

$$
\begin{aligned}
f_{L(G_1 \veebar G_2)} = {} & \det((x - n_1)I_{n_2} - L(G_2))\det(xI_{n_1} - D(G_1)) \\
& \det((x - n_2)I_{n_1} - L(G_1) - D(G_1) - A(G_1)(xI_{n_1} - D(G_1))^{-1}A(G_1)) \\
& [1 - \Gamma_{L(G_2)}(x - n_1)\Gamma_{L(G_1)+D(G_1)+A(G_1)(xI_{n_1}-D(G_1))^{-1}A(G_1)}(x - n_2)].
\end{aligned}
$$

Proof. (a) With a suitable ordering of the vertices of $G_1 \underset{=}{\vee} G_2$, we obtain

$$
L(G_1 \underset{=}{\vee} G_2) = \begin{pmatrix} (n_1 + n_2 - 1)I_{n_1} + L(G_1) - D(G_1) & -A(\overline{G}_1) & -J_{n_1 \times n_2} \\ -A(\overline{G}_1) & (n_1 - 1)I_{n_1} - D(G_1) & O_{n_1 \times n_2} \\ -J_{n_2 \times n_1} & O_{n_2 \times n_1} & n_1 I_{n_2} + L(G_2) \end{pmatrix}.
$$

The Laplacian characteristic polynomial is

$$
\begin{aligned}
f_{L(G_1 \underset{=}{\vee} G_2)}(x) &= \det(xI_{2n_1+n_2} - L(G_1 \underset{=}{\vee} G_2)) \\
&= \det((x - n_1)I_{n_2} - L(G_2))\det((xI_{2n_1+n_2} - L(G_1 \underset{=}{\vee} G_2))/((x - n_1)I_{n_2} - L(G_2)))
\end{aligned}
$$

by the Lemma of Schur (Lemma 2.5).

By Lemma 2.6,

$$
\begin{aligned}
& (xI_{2n_1+n_2} - L(G_1 \underset{=}{\vee} G_2))/((x - n_1)I_{n_2} - L(G_2)) \\
& = \begin{pmatrix} (x - n_1 - n_2 + 1)I_{n_1} - L(G_1) + D(G_1) - \Gamma_{L(G_2)}(x - n_1)J_{n_1 \times n_1} & A(\overline{G}_1) \\ A(\overline{G}_1) & (x - n_1 + 1)I_{n_1} + D(G_1) \end{pmatrix}.
\end{aligned}
$$

Using again Lemma 2.5, we obtain

$$
\det((xI_{2n_1+n_2} - L(G_1 \underset{=}{\vee} G_2))/((x - n_1)I_{n_2} - L(G_2))) = \det((x - n_1 + 1)I_{n_1} + D(G_1))\det(B - \Gamma_{L(G_2)}(x - n_1)J_{n_1 \times n_1}),
$$

where

$$
B = (x - n_1 - n_2 + 1)I_{n_1} - L(G_1) + D(G_1) - A(\overline{G}_1)((x - n_1 + 1)I_{n_1} + D(G_1))^{-1}A(\overline{G}_1).
$$

By Lemma 2.7, we obtain

$$
\begin{aligned}
& \det((xI_{2n_1+n_2} - L(G_1 \underset{=}{\vee} G_2))/((x - n_1)I_{n_2} - L(G_2))) \\
& = \det((x - n_1 + 1)I_{n_1} + D(G_1))(\det(B) - \Gamma_{L(G_2)}(x - n_1)\mathbf{1}_{n_1}^T \operatorname{adj}(B)\mathbf{1}_{n_1}) \\
& = \det((x - n_1 + 1)I_{n_1} + D(G_1))\det(B)[1 - \Gamma_{L(G_2)}(x - n_1)\mathbf{1}_{n_1}^T B^{-1}\mathbf{1}_{n_1}] \\
& = \det((x - n_1 + 1)I_{n_1} + D(G_1)) \\
& \quad \det((x - n_1 - n_2 + 1)I_{n_1} - L(G_1) + D(G_1) - A(\overline{G}_1)((x - n_1 + 1)I_{n_1} + D(G_1))^{-1}A(\overline{G}_1)) \\
& \quad [1 - \Gamma_{L(G_2)}(x - n_1)\Gamma_{L(G_1)-D(G_1)+A(\overline{G}_1)((x-n_1+1)I_{n_1}+D(G_1))^{-1}A(\overline{G}_1)}(x - n_1 - n_2 + 1)].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
f_{L(G_1 \underset{=}{\vee} G_2)}(x) = {} & \det((x - n_1)I_{n_2} - L(G_2))\det((x - n_1 + 1)I_{n_1} + D(G_1)) \\
& \det((x - n_1 - n_2 + 1)I_{n_1} - L(G_1) + D(G_1) - A(\overline{G}_1)((x - n_1 + 1)I_{n_1} + D(G_1))^{-1}A(\overline{G}_1)) \\
& [1 - \Gamma_{L(G_2)}(x - n_1)\Gamma_{L(G_1)-D(G_1)+A(\overline{G}_1)((x-n_1+1)I_{n_1}+D(G_1))^{-1}A(\overline{G}_1)}(x - n_1 - n_2 + 1)].
\end{aligned}
$$

The proof of (b) is similar. □

3.3 Signless Laplacian characteristic polynomial

Theorem 3.3. *Let G_i be a graph on n_i vertices for $i = 1, 2$. Then,*
(*a*)

$$\begin{aligned} f_{Q(G_1 \underset{=}{\vee} G_2)}(x) = {} & \det((x - n_1)I_{n_2} - Q(G_2))\det((x - n_1 + 1)I_{n_1} + D(G_1)) \\ & \det((x - n_1 - n_2 + 1)I_{n_1} - Q(G_1) + D(G_1) - A(\overline{G}_1)((x - n_1 + 1)I_{n_1} + D(G_1))^{-1}A(\overline{G}_1)) \\ & [1 - \Gamma_{Q(G_2)}(x - n_1)\Gamma_{Q(G_1) - D(G_1) + A(\overline{G}_1)((x - n_1 + 1)I_{n_1} + D(G_1))^{-1}A(\overline{G}_1)}(x - n_1 - n_2 + 1)]. \end{aligned}$$

(*b*)

$$\begin{aligned} f_{Q(G_1 \veebar G_2)} = {} & \det((x - n_1)I_{n_2} - Q(G_2))\det(xI_{n_1} - D(G_1)) \\ & \det((x - n_2)I_{n_1} - Q(G_1) - D(G_1) - A(G_1)(xI_{n_1} - D(G_1))^{-1}A(G_1)) \\ & [1 - \Gamma_{Q(G_2)}(x - n_1)\Gamma_{Q(G_1) + D(G_1) + A(G_1)(xI_{n_1} - D(G_1))^{-1}A(G_1)}(x - n_2)]. \end{aligned}$$

Proof. The proof is similar to the proof of Theorem 3.2. ☐

Corollary 3.4. *Let F and H be r-regular nonisomorphic cospectral graphs. Then, for every G,*
(*a*) *$G \veebar F$ and $G \veebar H$ are $\{A, Q, L\}$ NICS.*
(*b*) *$G \underset{=}{\vee} F$ and $G \underset{=}{\vee} H$ are $\{A, Q, L\}$ NICS.*

Proof. (a) $G \veebar F$ and $G \veebar H$ are nonisomorphic since F and H are nonisomorphic. By Theorems 3.1, 3.2, and 3.3, $f_{A(G \veebar F)}(x) = f_{A(G \veebar H)}(x)$, $f_{L(G \veebar F)}(x) = f_{L(G \veebar H)}(x)$, and $f_{Q(G \veebar F)}(x) = f_{Q(G \veebar H)}(x)$ since the matrices $A(F)$ and $A(H)$ have the same coronal (Lemma 2.3) and the same characteristic polynomial. This completes the proof of (a).

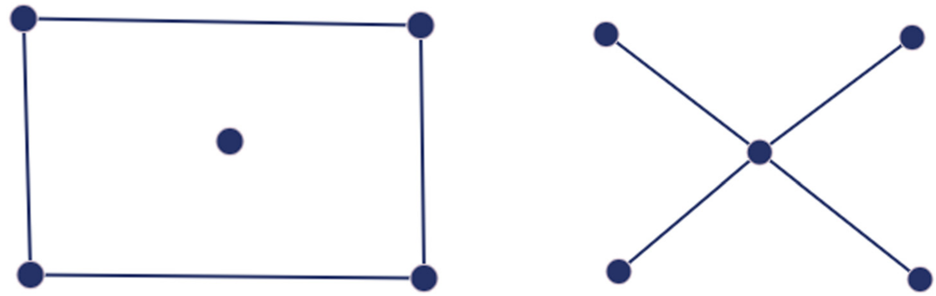
The proof of (b) is similar. ☐

Corollary 3.5. *Let F and H be r-regular nonisomorphic cospectral graphs. Then, for every G,*
(*a*) *$F \veebar G$ and $H \veebar G$ are $\{A, Q, L\}$ NICS.*
(*b*) *$F \underset{=}{\vee} G$ and $H \underset{=}{\vee} G$ are $\{A, Q, L\}$ NICS.*

Proof. The proof is similar to the proof of Corollary 3.4. ☐

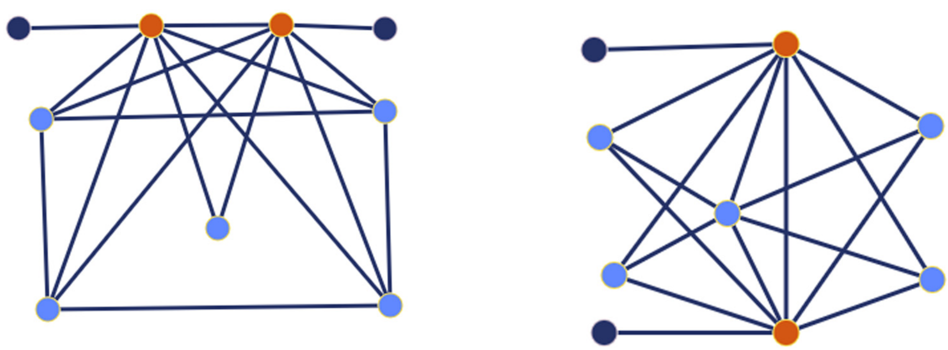
Remark 3.6. The following examples demonstrate the importance of the regularity of the graphs F and H.

Example 3.7. The graphs F and H in Figure 3 are nonregular and A-cospectral [10]. The joins $K_2 \veebar F$ and $K_2 \veebar H$ in Figure 4 are not A-cospectral since the A-spectrum of $K_2 \veebar F$ is $\{-2.2332, -2, -1.618, 0^{[3]}, 0.577, 0.618, 4.6562\}$ and the A-spectrum of $K_2 \veebar H$ is $\{-2.7039, -1.618, -1.2467, 0^{[3]}, 0.2526, 0.618, 4.698\}$. The joins $K_2 \underset{=}{\vee} F$ and $K_2 \underset{=}{\vee} H$ in Figure 5 are not A-cospectral since the A-spectrum of $K_2 \underset{=}{\vee} F$ is $\{(-2)^{[2]}, -1, 0^{[4]}, 0.4384, 4.5616\}$ and the A-spectrum of $K_2 \underset{=}{\vee} H$ is $\{-2.6056, (-1)^{[2[}, 0^{[5]}, 4.6056\}$. The joins $F \veebar K_2$ and $H \veebar K_2$ in Figure 6 are not A-cospectral since the A-spectrum of $F \veebar K_2$ is $\{-3.2361, -2.5205, -1, -0.5812, 0^{[5]}, 1.0895, 1.2361, 5.0122\}$ and the A-spectrum of $H \veebar K_2$ is $\{-3.5337, -2.1915, -1, 0^{[6]}, 0.3034, 1.3403, 5.0815\}$, and the joins $F \underset{=}{\vee} K_2$ and $H \underset{=}{\vee} K_2$ in Figure 7 are not A-cospectral since the A-spectrum of $F \underset{=}{\vee} K_2$ is $\{-3.1903, -2.4142, -1.2946, (-1)^{[3]}, 0.4046, 0.4142, 1^{[2]}, 1.8201, 5.2602\}$ and the A-spectrum of $H \underset{=}{\vee} K_2$ is $\{-4.1337, (-1)^{[5]}, 0, 0.8194, 1^{[3]}, 5.3143\}$.



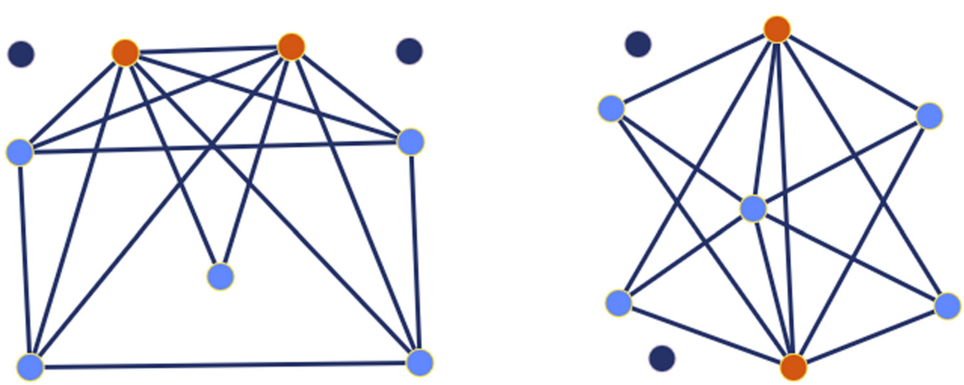
(a) (b)

Figure 3: Two nonregular A-cospectral graphs F and H. (a) F and (b) H.



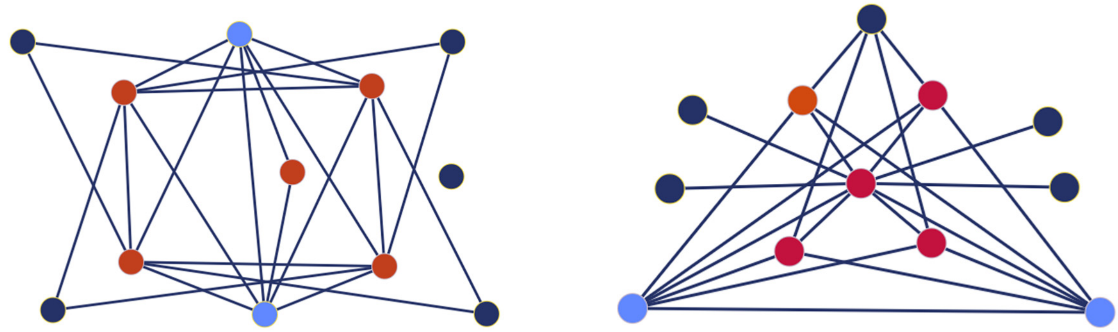
(a) (b)

Figure 4: Two non-A-cospectral graphs $K_2 \veebar F$ and $K_2 \veebar H$. (a) $K_2 \veebar F$ and (b) $K_2 \veebar H$.



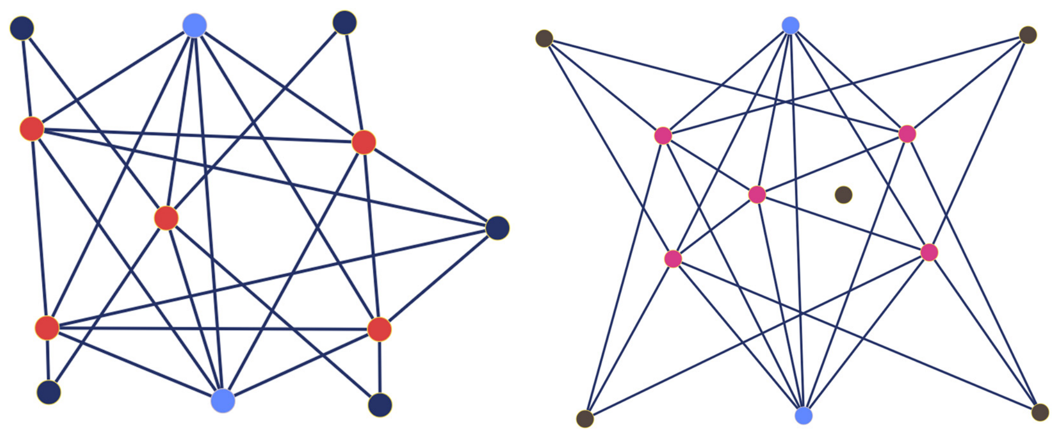
(a) (b)

Figure 5: Two non-A-cospectral graphs $K_2 \underset{=}{\vee} F$ and $K_2 \underset{=}{\vee} H$. (a) $K_2 \underset{=}{\vee} F$ and (b) $K_2 \underset{=}{\vee} H$.



(a) (b)

Figure 6: Two non-A-cospectral graphs $F \veebar K_2$ and $H \veebar K_2$. (a) $F \veebar K_2$ and (b) $H \veebar K_2$.



(a) (b)

Figure 7: Two non-A-cospectral graphs $F \underset{=}{\vee} K_2$ and $H \underset{=}{\vee} K_2$. (a) $F \underset{=}{\vee} K_2$ and (b) $H \underset{=}{\vee} K_2$.

Remark 3.8. Numerical computations suggest that in the Corollaries 3.4 and 3.5, $\{A, Q, L\}$ can be replaced by $\{A, Q, L, \mathcal{L}\}$.

Conjecture 3.9.

(a) *Let H_1 and H_2 be regular nonisomorphic cospectral graphs. Then, for every G, $G \veebar H_1$ and $G \veebar H_2$ are $\{A, Q, L, \mathcal{L}\}$NICS and $G \underset{=}{\vee} H_1$ and $G \underset{=}{\vee} H_2$ are $\{A, Q, L, \mathcal{L}\}$NICS.*

(b) *Let G_1 and G_2 be regular nonisomorphic cospectral graphs. Then, for every H, $G_1 \veebar H$, and $G_2 \veebar H$ are $\{A, Q, L, \mathcal{L}\}$ NICS and $G_1 \underset{=}{\vee} H$ and $G_2 \underset{=}{\vee} H$ are $\{A, Q, L, \mathcal{L}\}$NICS.*

4 $\mathcal{L}$-Spectra of NS joins

Let G_i and H_i be r_i-regular graphs, $i = 1, 2$. Lu et al. showed that if G_1 and H_1 are cospectral and G_2 and H_2 are cospectral and nonisomorphic, then $G_1 \veebar G_2$ and $H_1 \veebar H_2$ are $\{A, L, Q\}$NICS. In this section, we extend this result by showing that $G_1 \veebar G_2$ and $H_1 \veebar H_2$ are $\{A, L, Q, \mathcal{L}\}$NICS. To do it, we determine the spectrum of the normalized Laplacian of the graph $G_1 \veebar G_2$.

Theorem 4.1. *Let G_1 be an r_1-regular graph with n_1 vertices and G_2 be an r_2-regular graph with n_2 vertices. Then, the normalized Laplacian spectrum of $G_1 \veebar G_2$ consists of:*
- $1 + \frac{r_2(\delta_i(G_2)-1)}{n_1+r_2}$ *for* $i = 2, 3, \ldots, n_2$;
- $1 + \frac{(\delta_i(G_1)-1)(\sqrt{9r_1^2+4r_1n_2}+r_1)}{2(2r_1+n_2)}$ *for* $i = 2, 3, \ldots, n_1$;
- $1 + \frac{(1-\delta_i(G_1))(\sqrt{9r_1^2+4r_1n_2}-r_1)}{2(2r_1+n_2)}$ *for* $i = 2, 3, \ldots, n_1$;
- *the three roots of the equation*

$$(2r_1r_2 + 2r_1n_1 + n_2r_2 + n_1n_2)x^3 - (3r_1r_2 + 5r_1n_1 + 2r_2n_2 + 3n_1n_2)x^2 + (3r_1n_1 + n_2r_2 + 2n_1n_2)x = 0.$$

Proof. Let $u_1, u_2, \ldots, u_{n_1}$be the vertices of G_1, $u_1', u_2', \ldots, u_{n_1}'$ be the vertices added by the splitting, and $v_1, v_2, \ldots, v_{n_2}$ be the vertices of G_2. Under this vertex partitioning, the adjacency matrix of $G_1 \veebar G_2$ is

$$A(G_1 \veebar G_2) = \begin{pmatrix} A(G_1) & A(G_1) & J_{n_1\times n_2} \\ A(G_1) & O_{n_1\times n_1} & O_{n_1\times n_2} \\ J_{n_2\times n_1} & O_{n_2\times n_1} & A(G_2) \end{pmatrix}$$

The corresponding degrees matrix of $G_1 \veebar G_2$ is,

$$D(G_1 \veebar G_2) = \begin{pmatrix} (2r_1+n_2)I_{n_1} & O_{n_1\times n_1} & O_{n_1\times n_2} \\ O_{n_1\times n_1} & r_1I_{n_1} & O_{n_1\times n_2} \\ O_{n_2\times n_1} & O_{n_2\times n_1} & (r_2+n_1)I_{n_2} \end{pmatrix}.$$

By simple calculation, we obtain

$$\mathcal{L}(G_1 \veebar G_2) = \begin{pmatrix} I_{n_1} - \dfrac{A(G_1)}{2r_1+n_2} & \dfrac{-A(G_1)}{\sqrt{r_1(2r_1+n_2)}} & \dfrac{-J_{n_1\times n_2}}{\sqrt{(2r_1+n_2)(r_2+n_1)}} \\ \dfrac{-A(G_1)}{\sqrt{r_1(2r_1+n_2)}} & I_{n_1} & O_{n_1\times n_2} \\ \dfrac{-J_{n_2\times n_1}}{\sqrt{(2r_1+n_2)(r_2+n_1)}} & O_{n_2\times n_1} & I_{n_2} - \dfrac{A(G_2)}{r_2+n_1} \end{pmatrix}.$$

We prove the theorem by constructing an orthogonal basis of eigenvectors of $\mathcal{L}(G_1 \veebar G_2)$. Since G_2 is r_2-regular, the vector $\mathbf{1}_{n_2}$ is an eigenvector of $A(G_2)$ that corresponds to $\lambda_1(G_2) = r_2$. For $i = 2, 3, \ldots, n_2$, let Z_i be an eigenvector of $A(G_2)$ that corresponds to $\lambda_i(G_2)$. Then, $\mathbf{1}_{n_2}^T Z_i = 0$ and $(0_{1\times n_1}, 0_{1\times n_1}, Z_i^T)^T$ is an eigenvector of $\mathcal{L}(G_1 \veebar G_2)$ corresponding to the eigenvalue $1 - \frac{\lambda_i(G_2)}{r_2+n_1}$.

By Remark 1.8, $1 + \frac{r_2(\delta_i(G_2)-1)}{n_1+r_2}$ are eigenvalues of $\mathcal{L}(G_1 \veebar G_2)$ for $i = 2, \ldots, n_2$.

For $i = 2, \ldots, n_1$, let X_i be an eigenvector of $A(G_1)$ corresponding to the eigenvalue $\lambda_i(G_1)$. We now look for a nonzero real number α such that $(X_i^T \quad \alpha X_i^T \quad 0_{1\times n_2})^T$ is an eigenvector of $\mathcal{L}(G_1 \veebar G_2)$.

$$\mathcal{L}\begin{pmatrix} X_i \\ \alpha X_i \\ 0_{n_2\times 1} \end{pmatrix} = \begin{pmatrix} X_i - \dfrac{\lambda_i(G_1)}{2r_1+n_2}X_i - \dfrac{\lambda_i(G_1)\alpha}{\sqrt{r_1(2r_1+n_2)}}X_i \\ -\dfrac{\lambda_i(G_1)}{\sqrt{r_1(2r_1+n_2)}}X_i + \alpha X_i \\ 0_{n_2\times 1} \end{pmatrix} = \begin{pmatrix} 1 - \dfrac{\lambda_i(G_1)}{2r_1+n_2} - \dfrac{\lambda_i(G_1)\alpha}{\sqrt{r_1(2r_1+n_2)}} \\ -\dfrac{\lambda_i(G_1)}{\alpha\sqrt{r_1(2r_1+n_2)}} + 1 \\ 0_{n_2\times 1} \end{pmatrix} \begin{pmatrix} X_i \\ \alpha X_i \\ 0_{n_2\times 1} \end{pmatrix},$$

then, α must be a root of the equation

$$1 - \frac{\lambda_i(G_1)}{2r_1+n_2} - \frac{\lambda_i(G_1)\alpha}{\sqrt{r_1(2r_1+n_2)}} = -\frac{\lambda_i(G_1)}{\alpha\sqrt{r_1(2r_1+n_2)}} + 1, \tag{4.1}$$

$$\sqrt{2r_1+n_2}\,\alpha^2+\sqrt{r_1}\,\alpha-\sqrt{2r_1+n_2}=0.$$

Thus,

$$\alpha=\frac{2\sqrt{2r_1+n_2}}{\sqrt{9r_1+4n_2}+\sqrt{r_1}}\ \text{or}\ \alpha=\frac{-2\sqrt{2r_1+n_2}}{\sqrt{9r_1+4n_2}-\sqrt{r_1}}.$$

Substituting the values of α in the right side of equation (4.1), we obtain, by Remark 1.8, that

$$1+\frac{(\delta_i(G_1)-1)(\sqrt{9r_1^2+4r_1n_2}+r_1)}{n_1+r_2},1+\frac{(1-\delta_i(G_1))(\sqrt{9r_1^2+4r_1n_2}-r_1)}{2(2r_1+n_2)}$$

are eigenvalues of $\mathcal{L}(G_1 \veebar G_2)$ for $i = 2, 3, \ldots, n_1$.

So far, we obtained $n_2 - 1 + 2(n_1 - 1) = 2n_1 + n_2 - 3$ eigenvalues of $\mathcal{L}(G_1 \veebar G_2)$. Their eigenvectors are orthogonal to $(\mathbf{1}_{n_1}^T, 0_{1\times n_1}, 0_{1\times n_2})^T$, $(0_{1\times n_1}, 1_{n_1}^T, 0_{1\times n_2})^T$, and $(0_{1\times n_1}, 0_{1\times n_1}, 1_{n_2}^T)^T$.

To find three additional eigenvalues, we look for eigenvectors of $\mathcal{L}(G_1 \veebar G_2)$ of the form $Y = (\alpha\mathbf{1}_{n_1}^T, \beta\mathbf{1}_{n_1}^T, \gamma\mathbf{1}_{n_2}^T)^T$ for $(\alpha, \beta, \gamma) \neq (0, 0, 0)$. Let x be an eigenvalue of $\mathcal{L}(G \veebar G_2)$ corresponding to the eigenvector Y. From $\mathcal{L}Y = xY$, we obtain

$$\begin{cases}\alpha-\dfrac{r_1}{2r_1+n_2}\alpha-\dfrac{r_1}{\sqrt{r_1(2r_1+n_2)}}\beta-\dfrac{n_2}{\sqrt{(2r_1+n_2)(r_2+n_1)}}\gamma=\alpha x\\ \dfrac{-r_1}{\sqrt{r_1(2r_1+n_2)}}\alpha+\beta=\beta x\\ \dfrac{-n_1}{\sqrt{(2r_1+n_2)(r_2+n_1)}}\alpha+\gamma-\dfrac{r_2}{r_2+n_1}\gamma=\gamma x.\end{cases}$$

Thus,

$$\alpha-\frac{r_1}{2r_1+n_2}\alpha+\frac{r_1^2\alpha}{r_1(2r_1+n_2)(x-1)}+\frac{n_1n_2(r_2+n_1)\alpha}{(2r_1+n_2)(r_2+n_1)((x-1)(r_2+n_1)+r_2)}=\alpha x.$$

Note that $\alpha \neq 0$, since if $\alpha = 0$, then $\alpha = \beta = \gamma = 0$, and also $x \neq 1$, since $x = 1$ implies that $\alpha = 0$.

Dividing by α, we obtain the following cubic equation

$$(2r_1r_2+2r_1n_1+n_2r_2+n_1n_2)x^3-(3r_1r_2+5r_1n_1+2r_2n_2+3n_1n_2)x^2+(3r_1n_1+n_2r_2+2n_1n_2)x=0,$$

and this completes the proof. □

Now we can answer Question 1.16 by constructing pairs of nonregular $\{A, L, Q, \mathcal{L}\}$NICS graphs.

Corollary 4.2. *Let G_i and H_i be r_i-regular graphs, $i = 1, 2$. If, G_1 and H_1 are cospectral and G_2 and H_2 are cospectral and nonisomorphic then $G_1 \veebar G_2$ and $H_1 \veebar H_2$ are $\{A, L, Q, \mathcal{L}\}$ NICS.*

Proof. $G_1 \veebar G_2$ and $H_1 \veebar H_2$ are nonisomorphic since G_2 and H_2 are nonisomorphic. By Theorem 4.1 and Theorems 3.1–3.3 in [19], the graphs $H_1 \veebar H_2$ are $\{A, L, Q, \mathcal{L}\}$NICS. □

Example 4.3. Let $G_1 = H_1 = C_4$, and if we choose $G_2 = G$ and $H_2 = H$, where G and H are graphs in Figure 1, then the graphs in Figure 8 are $\{A, L, Q, \mathcal{L}\}$NICS.

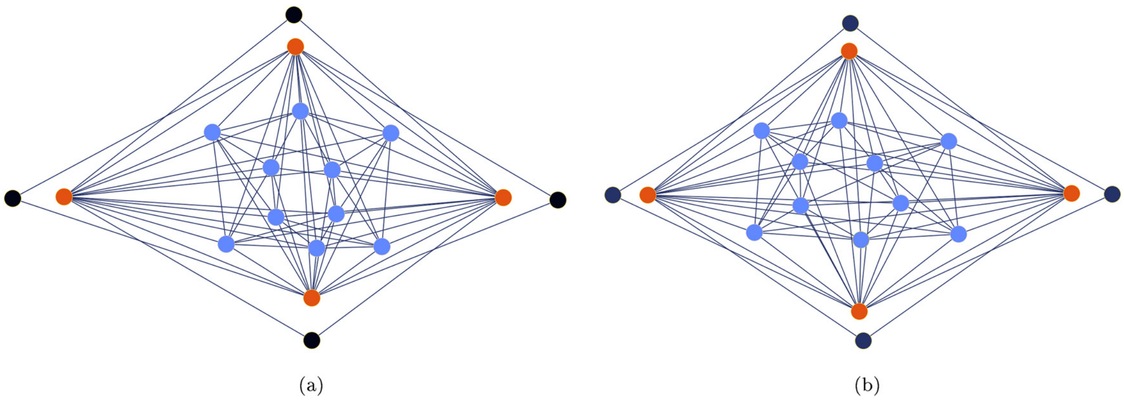


Figure 8: Nonregular $\{A, L, Q, \mathcal{L}\}$NICS graphs. (a) $C_4 \veebar G_2$ and (b) $C_4 \veebar H_2$.

5 Spectra of NNS joins

In this section, we compute the A-spectrum, L-spectrum, Q-spectrum, and $\mathcal{L}$-spectrum of $G_1 \underset{=}{\vee} G_2$, where G_1 and G_2 are regular.

We use it to answer Question 1.16 by constructing pairs of nonregular $\{A, L, Q, \mathcal{L}\}$ NICS graphs.

5.1 A-spectra of NNS join

The adjacency matrix of $G_1 \underset{=}{\vee} G_2$ can be written in a block form

$$A(G_1 \underset{=}{\vee} G_2) = \begin{pmatrix} A(G_1) & A(\overline{G}_1) & J_{n_1\times n_2} \\ A(\overline{G}_1) & O_{n_1\times n_1} & O_{n_1\times n_2} \\ J_{n_2\times n_1} & O_{n_2\times n_1} & A(G_2) \end{pmatrix}. \tag{5.1}$$

Theorem 5.1. *Let G_1 be an r_1-regular graph with n_1 vertices and G_2 be an r_2-regular graph with n_2 vertices. Then, the adjacency spectrum of $G_1 \underset{=}{\vee} G_2$ consists of:*

(*i*) *$\lambda_j(G_2)$ for each $j = 2, 3, \ldots, n_2$;*

(*ii*) *two roots of the equation*

$$x^2 - (\lambda_i(G_1))x - (\lambda_i^2(G_1) + 2\lambda_i(G_1) + 1) = 0$$

for each $i = 2, 3, \ldots, n_1$;

(*iii*) *the three roots of the equation*

$$x^3 - (r_1 + r_2)x^2 + (r_1r_2 - (n_1 - r_1 - 1)^2 - n_1n_2)x + r_2(n_1 - r_1 - 1)^2 = 0$$

Proof. By Theorem 3.1, the adjacency characteristic polynomial of $G_1 \underset{=}{\vee} G_2$ is

$$\begin{aligned} f_{A(G_1 \underset{=}{\vee} G_2)}(x) &= x^{n_1} f_{A(G_2)}(x) \det\left(xI_{n_1} - A(G_1) - \frac{1}{x}A^2(\overline{G}_1)\right)\left[1 - \Gamma_{A(G_2)}(x)\Gamma_{A(G_1)+\frac{1}{x}A^2(\overline{G}_1)}(x)\right] \\ &= x^{n_1}\prod_{j=1}^{n_2}(x - \lambda_j(G_2)) \det\left(xI_{n_1} - A(G_1) - \frac{1}{x}A^2(\overline{G}_1)\right)\left[1 - \Gamma_{A(G_2)}(x)\Gamma_{A(G_1)+\frac{1}{x}A^2(\overline{G}_1)}(x)\right]. \end{aligned}$$

Since G_1 and G_2 are regular, we can use Lemma 2.3 to obtain

$$f_{A(G_1 \underset{=}{\vee} G_2)}(x) = x^{n_1}\prod_{j=1}^{n_2}(x-\lambda_j(G_2))\det\left(xI_{n_1} - A(G_1) - \frac{1}{x}A^2(\overline{G}_1)\right)\left[1 - \frac{n_2}{x-r_2}\,\frac{n_1}{x-r_1-\frac{1}{x}(n_1-r_1-1)^2}\right]$$

$$= x^{n_1}\prod_{j=1}^{n_2}(x-\lambda_j(G_2))\det\left(xI_{n_1} - A(G_1) - \frac{1}{x}(J-I-A(G_1))^2\right)\left[1 - \frac{n_1n_2}{(x-r_2)(x-r_1-\frac{1}{x}(n_1-r_1-1)^2)}\right]$$

$$= x^{n_1}\prod_{j=1}^{n_2}(x-\lambda_j(G_2))\det\left(xI_{n_1} - A(G_1) - \frac{1}{x}(J^2 - 2J + I - 2r_1J + 2A(G_1) + A^2(G_1))\right)$$
$$\left[1 - \frac{n_1n_2}{(x-r_2)(x-r_1-\frac{1}{x}(n_1-r_1-1)^2)}\right]$$

$$= x^{n_1}\prod_{j=1}^{n_2}(x-\lambda_j(G_2))\left[\det(B) - \frac{1}{x}(n_1-2-2r_1)1_{n_1}^T\operatorname{adj}(B)1_{n_1}\right]\left[1 - \frac{n_1n_2}{(x-r_2)(x-r_1-\frac{1}{x}(n_1-r_1-1)^2)}\right],$$

where $B = (x-\frac{1}{x})I_{n_1} - (1+\frac{2}{x})A(G_1) - \frac{1}{x}A^2(G_1)$.

Thus, based on Definition 2.2 and Lemma 2.3, we have

$$f_{A(G_1 \underset{=}{\vee} G_2)}(x) = x^{n_1}\prod_{j=1}^{n_2}(x-\lambda_j(G_2))\prod_{i=1}^{n_1}\left[\left(x-\frac{1}{x}\right) - \left(1+\frac{2}{x}\right)\lambda_i(G_1) - \frac{1}{x}\lambda_i^2(G_1)\right]$$
$$\left[1 - \frac{1}{x}(n_1-2-2r_1)\Gamma_{\frac{1}{x}I_{n_1}+(1+\frac{2}{x})A(G_1)+\frac{1}{x}A^2(G_1)}(x)\right]\left[1 - \frac{n_1n_2}{(x-r_2)(x-r_1-\frac{1}{x}(n_1-r_1-1)^2)}\right]$$

$$= x^{n_1}\prod_{j=1}^{n_2}(x-\lambda_j(G_2))\prod_{i=1}^{n_1}\left[\left(x-\frac{1}{x}\right) - \left(1+\frac{2}{x}\right)\lambda_i(G_1) - \frac{1}{x}\lambda_i^2(G_1)\right]\left[1 - \frac{n_1-2-2r_1}{x}\cdot\frac{n_1}{x-\frac{1}{x}-r_1-\frac{2r_1}{x}-\frac{r_1^2}{x}}\right]$$
$$\cdot\left[1 - \frac{n_1n_2}{(x-r_2)(x-r_1-\frac{1}{x}(n_1-r_1-1)^2)}\right]$$

$$= \prod_{j=1}^{n_2}(x-\lambda_j(G_2))x^{n_1}\prod_{i=1}^{n_1}\left(\left(x-\frac{1}{x}\right) - \left(1+\frac{2}{x}\right)\lambda_i(G_1) - \frac{1}{x}\lambda_i^2(G_1)\right)$$
$$\left(1 - \frac{n_1(n_1-2-2r_1)}{x^2-r_1x-r_1^2-2r_1-1}\right)\left(\frac{(x-r_2)(x(x-r_1)-(n_1-r_1-1)^2)-n_1n_2x}{(x-r_2)(x(x-r_1)-(n_1-r_1-1)^2)}\right)$$

$$= \prod_{j=2}^{n_2}(x-\lambda_j(G_2))\prod_{i=2}^{n_1}(x^2-1-(x+2)\lambda_i(G_1)-\lambda_i^2(G_1))$$
$$(x(x-r_1)-(n_1-r_1-1)^2)\left(\frac{(x-r_2)(x(x-r_1)-(n_1-r_1-1)^2)-n_1n_2x}{(x-r_2)(x(x-r_1)-(n_1-r_1-1)^2)-n_1n_2x}\right)$$

$$= \prod_{j=2}^{n_2}(x-\lambda_j(G_2))\prod_{i=2}^{n_1}(x^2-1-(x+2)\lambda_i(G_1)-\lambda_i^2(G_1))$$
$$((x-r_2)(x(x-r_1)-(n_1-r_1-1)^2)-n_1n_2x)$$

$$= \prod_{j=2}^{n_2}(x-\lambda_j(G_2))\prod_{i=2}^{n_1}(x^2-(\lambda_i(G_1))x-(\lambda_i^2(G_1)+2\lambda_i(G_1)+1))$$
$$(x^3-(r_1+r_2)x^2+(r_1r_2-(n_1-r_1-1)^2-n_1n_2)x+r_2(n_1-r_1-1)^2).\square$$

5.2 L-spectra of NNS join

The degrees of the vertices of $G_1 \underline{\vee} G_2$ are as follows:

$$d_{G_1 \underline{\vee} G_2}(u_i) = n_1 + n_2 - 1, \quad i = 1, \ldots, n_1,$$

$$d_{G_1 \underline{\vee} G_2}(u_i') = n_1 - r_1 - 1, \quad i = 1, \ldots, n_1,$$

$$d_{G_1 \underline{\vee} G_2}(v_j) = r_2 + n_1, \quad j = 1, \ldots, n_2,$$

so the degrees matrix of $G_1 \underline{\vee} G_2$ that corresponds to equation (5.1) is

$$D(G_1 \underline{\vee} G_2) = \begin{pmatrix} (n_1 + n_2 - 1)I_{n_1} & O_{n_1 \times n_1} & O_{n_1 \times n_2} \\ O_{n_1 \times n_1} & (n_1 - r_1 - 1)I_{n_1} & O \\ O_{n_2 \times n_1} & O_{n_2 \times n_1} & (r_2 + n_1)I_{n_2} \end{pmatrix} \tag{5.2}$$

Theorem 5.2. *Let G_1 be an r_1-regular graph with n_1 vertices and G_2 be an r_2-regular graph with n_2 vertices. Then, the Laplacian spectrum of $G_1 \underline{\vee} G_2$ consists of the following*:

- *$n_1 + \mu_j(G_2)$ for each $j = 2, 3, \ldots, n_2$;*
- *two roots of the equation*

$$x^2 + (2r_1 - 2n_1 - n_2 - \mu_i(G_1) + 2)x + n_1^2 - 2r_1n_1 - 2n_1 + n_1n_2 - r_1n_2 - n_2 + \mu_i(G_1)(n_1 + r_1 + 1 - \mu_i(G_1)) = 0$$

  for each $i = 2, 3, \ldots, n_1$;
- *the three roots of the equation*

$$x^3 + (2r_1 - 3n_1 - n_2 + 2)x^2 + (n_1n_2 - n_2r_1 - n_2 + 2n_1^2 - 2r_1n_1 - 2n_1)x = 0.$$

Proof. By substituting $D(G_1) = r_1 I_{n_1}$ in Theorem 3.2, the Laplacian characteristic polynomial of $G_1 \underline{\vee} G_2$ is

$$\begin{aligned} f_{L(G_1 \underline{\vee} G_2)}(x) = &\det((x - n_1)I_{n_2} - L(G_2))\det(x - n_1 + 1 + r_1)I_{n_1} \\ &\det\left((x - n_1 - n_2 + r_1 + 1)I_{n_1} - L(G_1) - \frac{1}{x - n_1 + r_1 + 1}A^2(\overline{G}_1)\right) \\ &\left[1 - \Gamma_{L(G_2)}(x - n_1)\Gamma_{L(G_1) + \frac{1}{x - n_1 + 1 + r_1}A^2(\overline{G}_1)}(x - n_1 - n_2 + 1 + r_1)\right]. \end{aligned}$$

Using Lemma 2.3, we obtain

$$\begin{aligned}
f_{L(G_1 \underline{\vee} G_2)}(x) &= \det((x-n_1)I_{n_2} - L(G_2))(x-n_1+r_1+1)^{n_1} \\
&\cdot \det\left((x-n_1-n_2+r_1+1)I_{n_1} - L(G_1) - \frac{1}{x-n_1+r_1+1}A^2(\overline{G}_1)\right) \\
&\cdot \left[1 - \frac{n_2 n_1}{(x-n_1)\left(x-n_1-n_2+1+r_1-\frac{(n_1-r_1-1)^2}{x-n_1+r_1+1}\right)}\right] \\
&= \prod_{j=1}^{n_2}(x-n_1-\mu_j(G_2))(x-n_1+r_1+1)^{n_1}\left((x-n_1-n_2+1+r_1) - \frac{(n_1-r_1-1)^2}{x-n_1+r_1+1}\right) \\
&\cdot \prod_{i=2}^{n_1}\left(x-n_1-n_2+1+r_1-\mu_i(G_1) - \frac{(\mu_i(G_1)-r_1-1)^2}{x-n_1+r_1+1}\right) \\
&\cdot \left[1 - \frac{n_2 n_1}{(x-n_1)\left(x-n_1-n_2+1+r_1-\frac{(n_1-r_1-1)^2}{x-n_1+r_1+1}\right)}\right] \\
&= \prod_{j=2}^{n_2}(x-n_1-\mu_j(G_2))\prod_{i=2}^{n_1}(x^2+(2r_1-2n_1-n_2-\mu_i(G_1)+2)x+n_1^2-2r_1n_1 \\
&\quad -2n_1+n_1n_2-r_1n_2-n_2+\mu_i(G_1)(n_1+r_1+1-\mu_i(G_1))) \\
&\cdot [x^3+(2r_1-3n_1-n_2+2)x^2+(n_1n_2-n_2r_1-n_2+2n_1^2-2r_1n_1-2n_1)x].
\end{aligned}$$

This completes the proof. □

5.3 Q-spectra of NNS join

Theorem 5.3. *Let G_1 be an r_1-regular graph with n_1 vertices and G_2 be an r_2-regular graph with n_2 vertices. Then, the signless Laplacian spectrum of $G_1 \underline{\vee} G_2$ consists of the following:*

- $n_1 + \nu_j(G_2)$ *for each* $j = 2, 3, \dots, n_2$;
- *two roots of the equation* $x^2 + (2r_1 - 2n_1 - n_2 - \nu_i(G_1) + 2)x + n_1^2 - 2r_1n_1 - 2n_1 + n_1n_2 - r_1n_2 - n_2 + 4r_1 + \nu_i(G_1)(n_1 + r_1 - 3 - \nu_i(G_1)) = 0$ *for each* $i = 2, 3, \dots, n_1$;
- *the three roots of the equation*

$$\begin{aligned}
&x^3 + (2-3n_1-n_2-2r_2)x^2 + (n_1n_2 - n_2r_1 - n_2 + 2n_1^2 + 2r_1n_1 - 2n_1 - 2r_1 - 2r_1^2 + 4r_2n_1 - 4r_2 + 2r_2n_2)x \\
&\quad + 2n_1r_1^2 + 2r_1n_1 - 2r_1n_1^2 - 2r_2n_1n_2 + 2r_1r_2n_2 + 2r_2n_2 + 4r_2r_1^2 + 4r_1r_2 - 4r_1r_2n_1 = 0
\end{aligned}$$

Proof. The proof is similar to the proof of Theorem 5.2. □

5.4 $\mathcal{L}$-spectra of NNS join

Let G_1 be an r_1-regular graph on order n_1. Let S be a subset of $\{2, 3, \dots, n_1\}$ such that $\delta_i(G_1) = 1 + \frac{1}{r_1}$ for $i \in S$ and denote the cardinality of S by $n(S)$. Let G_2 be an r_2-regular graph on order n_2. In the following theorem, we determine the normalized Laplacian spectrum of $G_1 \underline{\vee} G_2$ in terms of the normalized Laplacian eigenvalues of G_1 and G_2. The proof is slightly more complicated than the proof of Theorem 4.1, and we consider three cases.

Theorem 5.4.

(*a*) *If $S = \Phi$, then the normalized Laplacian spectrum of $G_1 \underset{=}{\vee} G_2$ consists of the following*:

(*i*) $1 + \frac{r_2(\delta_i(G_2) - 1)}{n_1 + r_2}$ *for each* $i = 2, 3, \ldots, n_2$;

(*ii*) $1 + \frac{2(1 + r_1 - r_1\delta_i(G_1))^2}{r_1(1 - \delta_i(G_1))(n_1 - r_1 - 1) \mp \sqrt{(n_1 - r_1 - 1)[r_1^2(1 - \delta_i(G_1))^2(n_1 - r_1 - 1) + 4(1 + r_1 - r_1\delta_i(G_1))^2(n_1 + n_2 - 1)]}}$ for each $i = 2, 3, \ldots, n_1$;

(*iii*) the three roots of the equation

$$(n_1^2 + n_1n_2 - n_1 + r_2n_1 + r_2n_2 - r_2)x^3 - (3n_1^2 + 3n_1n_2 - 3n_1 - r_1n_1 + 2r_2n_1 + 2r_2n_2 - 2r_2 - r_1r_2)x^2 + (2n_1^2 + 2n_1n_2 - 2n_1 - r_1n_1 + r_2n_2)x = 0.$$

(*b*) *If $S = \{2, 3, \ldots, n_1\}$, then the normalized Laplacian spectrum of $G_1 \underset{=}{\vee} G_2$ consists of the following*:

(*i*) $1 + \frac{r_2(\delta_i(G_2) - 1)}{n_1 + r_2}$ for each $i = 2, 3, \ldots, n_2$;

(*ii*) $(1 + \frac{1}{n_1 + n_2 - 1})^{[n_1 - 1]}$, $0^{[n_1 + 1]}$, and $\frac{n_1^2 + 2n_1n_2 + r_2n_2 - n_1}{(r_2 + n_1)(n_1 + n_2 - 1)}$.

(*c*) *If $S \neq \Phi$ and $S \neq \{2, 3, \ldots, n_1\}$, then the normalized Laplacian spectrum of $G_1 \underset{=}{\vee} G_2$ consists of the following*:

(*i*) $1 + \frac{r_2(\delta_i(G_2) - 1)}{n_1 + r_2}$ for each $i = 2, 3, \ldots, n_2$;

(*ii*) $1 + \frac{2(1 + r_1 - r_1\delta_i(G_1))^2}{r_1(1 - \delta_i(G_1))(n_1 - r_1 - 1) \mp \sqrt{(n_1 - r_1 - 1)[r_1^2(1 - \delta_i(G_1))^2(n_1 - r_1 - 1) + 4(1 + r_1 - r_1\delta_i(G_1))^2(n_1 + n_2 - 1)]}}$ for each $i \in \{2, 3, \ldots, n_1\} \backslash S$;

(*iii*) $1^{[n(S)]}$ *and* $(1 + \frac{1}{n_1 + n_2 - 1})^{[n(S)]}$.

(*iv*) *the three roots of the equation*

$$(n_1^2 + n_1n_2 - n_1 + r_2n_1 + r_2n_2 - r_2)x^3 - (3n_1^2 + 3n_1n_2 - 3n_1 - r_1n_1 + 2r_2n_1 + 2r_2n_2 - 2r_2 - r_1r_2)x^2 + (2n_1^2 + 2n_1n_2 - 2n_1 - r_1n_1 + r_2n_2)x = 0.$$

Proof. (a) If $S = \Phi$, then $\delta_i(G_1) \neq 1 + \frac{1}{r_1}$ for each $i = 2, 3, \ldots, n_1$, so $\lambda_i(G_1) \neq -1$ for each $i = 2, 3, \ldots, n_1$. The normalized Laplacian matrix of $G_1 \underset{=}{\vee} G_2$ is:

$$\mathcal{L}(G_1 \underset{=}{\vee} G_2) = \begin{pmatrix} I_{n_1} - \dfrac{A(G_1)}{n_1 + n_2 - 1} & \dfrac{-A(\overline{G}_1)}{\sqrt{(n_1 + n_2 - 1)(n_1 - r_1 - 1)}} & \dfrac{-J_{n_1 \times n_2}}{\sqrt{(n_1 + n_2 - 1)(r_2 + n_1)}} \\ \dfrac{-A(\overline{G}_1)}{\sqrt{(n_1 + n_2 - 1)(n_1 - r_1 - 1)}} & I_{n_1} & O_{n_1 \times n_2} \\ \dfrac{-J_{n_2 \times n_1}}{\sqrt{(n_1 + n_2 - 1)(r_2 + n_1)}} & O_{n_2 \times n_1} & I_{n_2} - \dfrac{A(G_2)}{r_2 + n_1} \end{pmatrix}.$$

Since G_2 is r_2-regular, the vector $\mathbf{1}_{n_2}$is an eigenvector of $A(G_2)$ that corresponds to $\lambda_1(G_2) = r_2$. For $i = 2, 3, \ldots, n_2$ let Z_i be an eigenvector of $A(G_2)$ that corresponds to $\lambda_i(G_2)$. Then, $\mathbf{1}_{n_2}^T Z_i = 0$ and $(0_{1 \times n_1}, 0_{1 \times n_1}, Z_i^T)^T$ is an eigenvector of $\mathcal{L}(G_1 \underset{=}{\vee} G_2)$ corresponding to the eigenvalue $1 - \frac{\lambda_i(G_2)}{r_2 + n_1}$. By Remark 1.8, $1 + \frac{r_2(\delta_i(G_2) - 1)}{n_1 + r_2}$ are an eigenvalues of $L(G_1 \underset{=}{\vee} G_2)$ for $i = 2, \ldots, n_2$.

For $i = 2, \ldots, n_1$, let X_i be an eigenvector of $A(G_1)$ corresponding to the eigenvalue $\lambda_i(G_1)$. We now look for a nonzero real number α such that $(X_i^T \quad \alpha X_i^T \quad 0_{1 \times n_2})^T$ is an eigenvector of $\mathcal{L}(G_1 \underset{=}{\vee} G_2)$. Note that $\alpha \neq 0$, since if $\alpha = 0$, then $\lambda_i(G_1) = -1$.

$$\mathcal{L}\begin{pmatrix} X_i \\ \alpha X_i \\ 0_{n_1\times 1}\end{pmatrix} = \begin{pmatrix} X_i - \dfrac{\lambda_i(G_1)}{n_1+n_2-1}X_i + \dfrac{1+\lambda_i(G_1)}{\sqrt{(n_1+n_2-1)(n_1-r_1-1)}}\alpha X_i \\ \dfrac{1+\lambda_i(G_1)}{\sqrt{(n_1+n_2-1)(n_1-r_1-1)}}X_i + \alpha X_i \\ 0_{n_1\times 1}\end{pmatrix}$$

$$= \begin{pmatrix} 1 - \dfrac{\lambda_i(G_1)}{n_1+n_2-1} + \dfrac{1+\lambda_i(G_1)}{\sqrt{(n_1+n_2-1)(n_1-r_1-1)}}\alpha \\ \dfrac{1+\lambda_i(G_1)}{\alpha\sqrt{(n_1+n_2-1)(n_1-r_1-1)}} + 1 \\ 0_{n_1\times 1}\end{pmatrix}\begin{pmatrix} X_i \\ \alpha X_i \\ 0_{n_1\times 1}\end{pmatrix}.$$

Thus,

$$1 - \frac{\lambda_i(G_1)}{n_1+n_2-1} + \frac{1+\lambda_i(G_1)}{\sqrt{(n_1+n_2-1)(n_1-r_1-1)}}\alpha = \frac{1+\lambda_i(G_1)}{\alpha\sqrt{(n_1+n_2-1)(n_1-r_1-1)}} + 1 \tag{5.3}$$

$$-\frac{\lambda_i(G_1)}{n_1+n_2-1} + \frac{1+\lambda_i(G_1)}{\sqrt{(n_1+n_2-1)(n_1-r_1-1)}}\alpha = \frac{1+\lambda_i(G_1)}{\alpha\sqrt{(n_1+n_2-1)(n_1-r_1-1)}}$$

$$\frac{\alpha^2(1+\lambda_i(G_1)) - (1+\lambda_i(G_1))}{\alpha\sqrt{(n_1+n_2-1)(n_1-r_1-1)}} = \frac{\lambda_i(G_1)}{n_1+n_2-1}$$

$$(1+\lambda_i(G_1))\sqrt{n_1+n_2-1}\,\alpha^2 - \lambda_i(G_1)\sqrt{n_1-r_1-1}\,\alpha - (1+\lambda_i(G_1))\sqrt{n_1+n_2-1} = 0,$$

so

$$\alpha_{1,2} = \frac{\lambda_i(G_1)\sqrt{n_1-r_1-1}}{2(1+\lambda_i(G_1))\sqrt{n_1+n_2-1}} \mp \sqrt{\frac{\lambda_i(G_1)^2(n_1-r_1-1)}{4(1+\lambda_i(G_1))^2(n_1+n_2-1)} + 1}.$$

Substituting the values of α in the right side of equation (5.3), we obtain by Remark 1.8 that

$$1 + \frac{2(1+r_1-r_1\delta_i(G_1))^2}{r_1(1-\delta_i(G_1))(n_1-r_1-1) \mp \sqrt{(n_1-r_1-1)[r_1^2(1-\delta_i(G_1))^2(n_1-r_1-1) + 4(1+r_1-r_1\delta_i(G_1))^2(n_1+n_2-1)]}}$$

are eigenvalues of $\mathcal{L}(G_1 \underline{\vee} G_2)$ for each $i = 2, 3, \dots, n_1$.

So far, we have obtained $n_2 - 1 + 2(n_1 - 1) = 2n_1 + n_2 - 3$ eigenvalues of $\mathcal{L}(G_1 \underline{\vee} G_2)$. The corresponding eigenvectors are orthogonal to $(\mathbf{1}_{n_i}^T, 0_{1\times n_1}, 0_{1\times n_2})^T$, $(0_{1\times n_1}, \mathbf{1}_{n_i}^T, 0_{1\times n_2})^T$ and $(0_{1\times n_1}, 0_{1\times n_1}, \mathbf{1}_{n_2}^T)^T$. To find three additional eigenvalues, we look for eigenvectors of $\mathcal{L}(G_1 \underline{\vee} G_2)$ of the form $Y = (\alpha\mathbf{1}_{n_1}^T, \beta\mathbf{1}_{n_1}^T, \gamma\mathbf{1}_{n_2}^T)^T$ for $(\alpha, \beta, \gamma) \neq (0, 0, 0)$. Let x be an eigenvalue of $\mathcal{L}(G \underline{\vee} G_2)$ corresponding to the eigenvector Y. From $\mathcal{L}Y = xY$, we obtain

$$\alpha - \frac{\alpha r_1}{n_1+n_2-1} + \frac{(1+r_1-n_1)\beta}{\sqrt{n_1+n_2-1}\sqrt{n_1-r_1-1}} - \frac{n_2\gamma}{\sqrt{(n_1+n_2-1)(r_2+n_1)}} = \alpha x \tag{5.4}$$

$$\frac{(1+r_1-n_1)\alpha}{\sqrt{n_1-r_1-1}\sqrt{n_1+n_2-1}} + \beta = \beta x \tag{5.5}$$

$$\frac{-n_1\alpha}{\sqrt{r_2+n_1}\sqrt{n_1+n_2-1}} + \gamma - \frac{r_2\gamma}{r_2+n_1}\gamma x \tag{5.6}$$

Thus,

$$\alpha - \frac{\alpha r_1}{n_1+n_2-1} + \frac{\alpha(n_1-1-r_1)}{(n_1+n_2-1)(x-1)} + \frac{\alpha(n_1n_2)}{(n_1+n_2-1)(xr_2+n_1(x-1))} = \alpha x.$$

Note that $\alpha \neq 0$, since if $\alpha = 0$, then $\alpha = \beta = \gamma = 0$, and also $x \neq 1$, since $x = 1$ implies that $\alpha = 0$.

Dividing by α, we obtain

$$1 - \frac{r_1}{n_1 + n_2 - 1} + \frac{n_1 - 1 - r_1}{(n_1 + n_2 - 1)(x - 1)} + \frac{n_1 n_2}{(n_1 + n_2 - 1)(x r_2 + n_1(x - 1))} = x.$$

Then,

$$(n_1^2 + n_1 n_2 - n_1)(x - 1)^3 + (r_2 n_1 x + r_2 n_2 x - r_2 x + r_1 n_1)(x - 1)^2 + (r_1 r_2 x - n_1^2 + n_1 + r_1 n_1 - n_1 n_2)(x - 1) \\ - r_2(n_1 - 1 - r_1)x = 0,$$

then, by simple calculation, we see that x is a root of the cubic equation

$$(n_1^2 + n_1 n_2 - n_1 + r_2 n_1 + r_2 n_2 - r_2)x^3 - (3n_1^2 + 3n_1 n_2 - 3n_1 - r_1 n_1 + 2r_2 n_1 + 2r_2 n_2 - 2r_2 - r_1 r_2)x^2 \\ + (2n_1^2 + 2n_1 n_2 - 2n_1 - r_1 n_1 + r_2 n_2)x = 0,$$

and this completes the proof of (a).

(b) The proof of (i) is similar to the proof of (i) in (a). Now we prove (ii). If $S = \{2, 3, \ldots, n_1\}$, then $\delta_j(G_1) = 1 + \frac{1}{r_1}$ for each $i = 2, 3, \ldots, n_1$, so $\lambda_i(G_1) = -1$ for each $i = 2, 3, \ldots, n_1$, i.e., $G_1 = K_{n_1}$ and $r_1 = n_1 - 1$. So the normalized Laplacian matrix of $G_1 \underline{\vee} G_1$ is as follows:

$$\mathcal{L}(G_1 \underline{\vee} G_2) = \begin{pmatrix} I_{n_1} - \dfrac{A(G_1)}{n_1 + n_2 - 1} & O_{n_1 \times n_1} & \dfrac{-J_{n_1 \times n_2}}{\sqrt{(n_1 + n_2 - 1)(r_2 + n_1)}} \\ O_{n_1 \times n_1} & O_{n_1 \times n_1} & O_{n_1 \times n_2} \\ \dfrac{-J_{n_2 \times n_1}}{\sqrt{(n_1 + n_2 - 1)(r_2 + n_1)}} & O_{n_2 \times n_1} & I_{n_2} - \dfrac{A(G_2)}{r_2 + n_1} \end{pmatrix}.$$

For $i = 2, \ldots, n_1$, let X_i be an eigenvector of $A(G_1)$ corresponding to the eigenvalue $\lambda_i(G_1) = -1$. So $\begin{pmatrix} X_i^T & 0_{1 \times n_1} & 0_{1 \times n_2} \end{pmatrix}^T$ is an eigenvector of $\mathcal{L}(G_1 \underline{\vee} G_2)$ corresponding to the eigenvalue $1 + \frac{1}{n_1 + n_2 - 1}$ and $\begin{pmatrix} 0_{1 \times n_1} & X_i^T & 0_{1 \times n_2} \end{pmatrix}^T$ is an eigenvector of $\mathcal{L}(G_1 \underline{\vee} G_2)$ corresponding to the eigenvalue 0 because,

$$\mathcal{L}\begin{pmatrix} X_i \\ 0_{n_1 \times 1} \\ 0_{n_2 \times 1} \end{pmatrix} = \begin{pmatrix} X_i - \dfrac{\lambda_i(G_1) X_i}{n_1 + n_2 - 1} \\ 0_{n_1 \times 1} \\ 0_{n_2 \times 1} \end{pmatrix} = \left(1 + \frac{1}{n_1 + n_2 - 1}\right)\begin{pmatrix} X_i \\ 0_{n_1 \times 1} \\ 0_{n_2 \times 1} \end{pmatrix} \quad \text{and} \quad \mathcal{L}\begin{pmatrix} 0_{n_1 \times 1} \\ X_i \\ 0_{n_2 \times 1} \end{pmatrix} = \begin{pmatrix} 0_{n_1 \times 1} \\ 0_{n_1 \times 1} \\ 0_{n_2 \times 1} \end{pmatrix}.$$

Therefore, $(1 + \frac{1}{n_1 + n_2 - 1})^{[n_1 - 1]}$ and $0^{[n_1 - 1]}$ are eigenvalues of $\mathcal{L}(G_1 \underline{\vee} G_2)$.

So far, we have obtained $n_2 - 1 + 2(n_1 - 1) = 2n_1 + n_2 - 3$ eigenvalues of $\mathcal{L}(G_1 \underline{\vee} G_2)$. Their eigenvectors are orthogonal to $(\mathbf{1}_{n_1}^T, 0_{1 \times n_1}, 0_{1 \times n_2})^T$, $(0_{1 \times n_1}, 1_{n_1}^T, 0_{1 \times n_2})^T$, and $(0_{1 \times n_1}, 0_{1 \times n_1}, 1_{n_2}^T)^T$. To find three additional eigenvalues, we look for eigenvectors of $\mathcal{L}(G \underline{\vee} G_2)$ of the form $Y = (\alpha \mathbf{1}_{n_1}^T, \beta \mathbf{1}_{n_1}^T, \gamma \mathbf{1}_{n_2}^T)^T$ for $(\alpha, \beta, \gamma) \neq (0, 0, 0)$. Let x be an eigenvalue of $\mathcal{L}(G \underline{\vee} G_2)$ corresponding to the eigenvector Y. Then, from $\mathcal{L}Y = xY$, we obtain

$$\alpha - \frac{r_1 \alpha}{n_1 + n_2 - 1} - \frac{\gamma n_2}{\sqrt{(n_1 + n_2 - 1)(r_2 + n_1)}} = \alpha x \tag{5.7}$$

$$\beta x = 0 \tag{5.8}$$

$$\frac{-\alpha n_1}{\sqrt{(n_1 + n_2 - 1)(r_2 + n_1)}} + \gamma - \frac{r_2 \gamma}{r_2 + n_1} = \gamma x \tag{5.9}$$

If $\beta \neq 0$, then $(0, \beta, 0)$ is a one of the solutions of the above three equations, so $(0_{1 \times n_1}, \beta \mathbf{1}_{n_1}^T, 0_{1 \times n_2})^T$ is an eigenvector corresponding to the eigenvalue 0. On the other hand, if $\beta = 0$, we obtain

$$\alpha\left(1 - x - \frac{r_1}{n_1 + n_2 - 1}\right) = \frac{\gamma n_2}{\sqrt{(n_1 + n_2 - 1)(r_2 + n_1)}} \tag{5.10}$$

$$\gamma\left(1 - x - \frac{r_2}{r_2 + n_1}\right) = \frac{an_1}{\sqrt{(n_1 + n_2 - 1)(r_2 + n_1)}}. \tag{5.11}$$

By solving the above two equations, we obtain the following equation:

$$(r_2 + n_1)(n_1 + n_2 - 1)x^2 + (n_1 - r_2 n_2 - 2n_1 n_2 - n_1^2)x = 0,$$

whose roots are 0 and $\frac{n_1^2 + 2n_1n_2 + r_2n_2 - n_1}{(r_2 + n_1)(n_1 + n_2 - 1)}$.

This completes the proof of (b).

(c) The proofs of (i), (ii), and (iv) are similar to the proofs of (i), (ii) and (iii) of (a), respectively. Now we prove (iii), Let $S \neq \Phi$ and $S \neq \{2, 3, \dots, n_1\}$. If $i \in S$, then $1 + \frac{1}{n_1 + n_2 - 1}$ and 1 are eigenvalues of $\mathcal{L}(G \underset{=}{\vee} G_2)$ because if X_i is an eigenvector corresponding to the eigenvalue $\delta_i(G_1)$, then

$$\mathcal{L}\begin{pmatrix} X_i \\ 0_{n_1\times 1} \\ 0_{n_2\times 1}\end{pmatrix} = \begin{pmatrix} X_i + \dfrac{X_i}{n_1 + n_2 - 1} \\ 0_{n_1\times 1} \\ 0_{n_2\times 1}\end{pmatrix} = \left(1 + \frac{1}{n_1 + n_2 - 1}\right)\begin{pmatrix} X_i \\ 0_{n_1\times 1} \\ 0_{n_2\times 1}\end{pmatrix} \quad \text{and} \quad \mathcal{L}\begin{pmatrix} 0_{n_1\times 1} \\ X_i \\ 0_{n_2\times 1}\end{pmatrix} = \begin{pmatrix} 0_{n_1\times 1} \\ X_i \\ 0_{n_2\times 1}\end{pmatrix}.$$

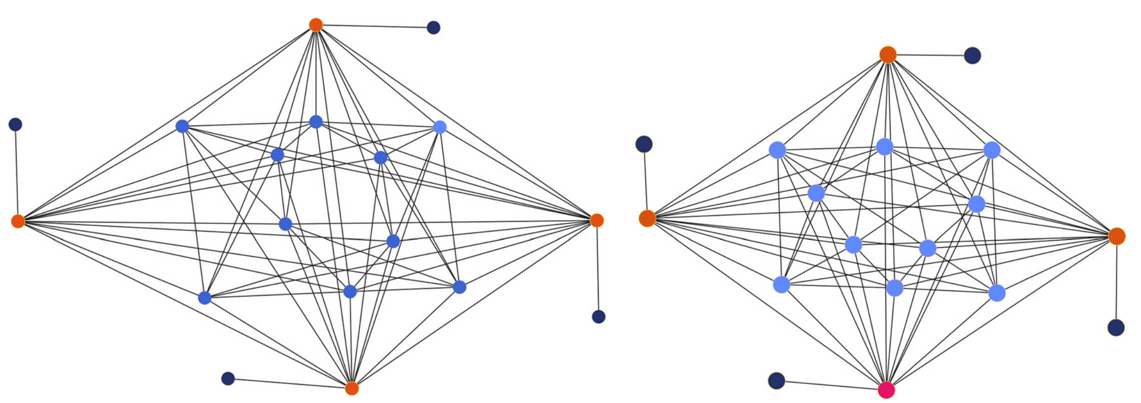
So, $1^{[n(S)]}$ and $(1 + \frac{1}{n_1 + n_2 - 1})^{[n(S)]}$ are eigenvalues of $\mathcal{L}(G \underset{=}{\vee} G_2)$, and this completes the proof of (c). □

Now we can give another answer to Question 1.16 by constructing several pairs of nonregular $\{A, L, Q, \mathcal{L}\}$NICS graphs.

Corollary 5.5. *Let G_1 and H_1 be cospectral regular graphs and G_2 and H_2 be nonisomorphic, regular, and cospectral graphs. Then, $G_1 \underset{=}{\vee} G_2$ and $H_1 \underset{=}{\vee} H_2$ are nonregular* $\{A, L, Q, \mathcal{L}\}$NICS.

Proof. $G_1 \underset{=}{\vee} G_2$ and $H_1 \underset{=}{\vee} H_2$ are nonisomorphic since G_2 and H_2 are nonisomorphic. By Theorems 5.1–5.4, we obtain that $G_1 \underset{=}{\vee} G_2$ and $H_1 \underset{=}{\vee} H_2$ are nonregular $\{A, L, Q, \mathcal{L}\}$ NICS. □

Example 5.6. Let $G_1 = H_1 = C_4$, and if we choose $G_2 = G$ and $H_2 = H$, where G and H are graphs in Figure 1, then the graphs in Figure 9 are $\{A, L, Q, \mathcal{L}\}$NICS.



(a) (b)

Figure 9: Non regular $\{A, L, Q, \mathcal{L}\}$ NICS graphs. (a) $C_4 \underset{=}{\vee} G_2$ and (b) $C_4 \underset{=}{\vee} H_2$.

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