Research Article

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The effect of removing a 2-downer edge or a cut 2-downer edge triangle for an eigenvalue

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Abstract: Edges in the graph associated with a square matrix over a field may be classified as to how their removal affects the multiplicity of an identified eigenvalue. There are five possibilities: +2 (2-Parter); +1 (Parter); no change (neutral); −1 (downer); and −2 (2-downer). Especially, it is known that 2-downer edges for an eigenvalue comprise cycles in the graph. We investigate the effect for the statuses of other edges or vertices by removing a 2-downer edge. Then, we investigate the change in the multiplicity of an eigenvalue by removing a cut 2-downer edge triangle.

Keywords: eigenvalue, graph of a matrix, 2-downer edge cycle

MSC 2020: 15A18, 05C50

1 Introduction

Throughout, G denotes a simple, undirected graph on n vertices, and we denote by $\mathcal{S}(G)$ the set of all n-by-n real symmetric matrices, $\mathcal{H}(G)$ the set of all n-by-n Hermitian matrices, the graph whose off-diagonal entries has an edge $\{i, j\}$ iff $a_{ij}a_{ji} \neq 0$ for $A = [a_{ij}]$; no restriction is placed on the diagonal entries, other than that they be real. For a given matrix A, we denote the multiplicity of an eigenvalue λ of A by $m_A(\lambda)$.

When a graph G is a tree T, there are many papers relating the structure of T to the multiplicities of the eigenvalues of the matrices in $\mathcal{S}(T)$.

In this article, we consider a general simple graph G. There are papers that relate the structure of G to the multiplicities of the eigenvalues of the matrices in $\mathcal{S}(G)$, [1,2,4,6,8–12, etc.]. When a vertex v is removed from G, the remaining graph is denoted by $G(v)$, and we denote by $A(v)$ the $(n-1)$-by-$(n-1)$ principal submatrix of $A \in \mathcal{S}(G)$, resulting from deletion of the row and column corresponding to v. When an edge e_{ij} is removed from G, we denote the remaining graph by $G(e_{ij})$; then $A(e_{ij}) \in \mathcal{S}(G(e_{ij}))$ denotes the matrix obtained from A by changing the entries corresponding to e_{ij} to zero. Further, when a vertex v and an edge e_{ij} are removed from G, we denote the remaining graph by $G(v, e_{ij})$ and the corresponding submatrix by $A(v, e_{ij})$. For an identified $A \in \mathcal{S}(G)$, we often speak interchangeably about the graph and the matrix, and we identify vertices in a graph with indices of a matrix, for convenience.

From the interlacing inequalities for a symmetric matrix, the multiplicity of an eigenvalue may change by at most 1, when a particular vertex is deleted. A vertex v of a graph G is called *Parter* (respectively *neutral*, *downer*) in G for an eigenvalue λ of $A \in \mathcal{S}(G)$, if

$$m_{A(u)}(\lambda) = m_A(\lambda) + 1(\text{resp. } m_A(\lambda), m_A(\lambda) - 1).$$

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We call these the *status* (or *classifications*) of the vertex v for the eigenvalue λ relative to A. We denote Parter, neutral, and downer by P, N, and D, respectively.

The change in the multiplicity of an eigenvalue when an edge is removed was investigated in [3].

Lemma 1. [3] *Let G be a graph, and $A \in \mathcal{H}(G)$. For an edge e_{ij} in G, and $\lambda \in \sigma(A)$,*

$$m_A(\lambda) - 2 \le m_{A(e_{ij})(\lambda)} \le m_A(\lambda) + 2.$$

We can define the status of an edge. An edge e_{ij} is called *2-Parter* (resp. *Parter, neutral, downer, 2-downer*) for λ relative to A, if for $A \in \mathcal{H}(G)$ and $A(e_{ij}) \in \mathcal{H}(G(e_{ij}))$,

$$m_{A(e_{ij})}(\lambda) - m_A(\lambda) = 2(\text{resp. } 1, 0, -1, -2).$$

If the status of a vertex or an edge is neutral, Parter, or 2-Parter, then the status of it is called *at least neutral*.

The classification number of a vertex or an edge is described in [6]. Given $A \in \mathcal{S}(G)$, and $\lambda \in \sigma(A)$, the classification number of a vertex is defined in a natural numerical way, Parter is 1, neutral 0, and downer −1. In particular, we may numerically classify a vertex i in A relative to the identified eigenvalue λ, $S_A(i)$, as follows:

$$S_A(i) = m_{A(i)}(\lambda) - m_A(\lambda).$$

An edge's classification number is defined as 2, 1, 0, −1, and −2, depending on whether the edge is 2-Parter, Parter, neutral, downer, or 2-downer, respectively. We may numerically classify an edge e_{ij}, $S_A(e_{ij})$, as follows:

$$S_A(e_{ij}) = m_{A(e_{ij})}(\lambda) - m_A(\lambda).$$

There is a relationship between the classification number of an edge and the classification number of the incident vertices,

$$S_A(e_{ij}) = S_A(i) - S_{A(e_{ij})}(i) = S_A(j) - S_{A(e_{ij})}(j). \tag{1}$$

If vertex j is a neighbor of vertex i in G and j is a downer vertex for λ in $A(i)$, then we call j a *downer neighbor* of i for $A \in \mathcal{S}(G)$ and λ [5].

Definition 2. Let G be a graph, $A \in \mathcal{S}(G)$, and $\lambda \in \sigma(A)$ with $m_A(\lambda) \ge 2$. If there is a cycle Γ in G whose edges on Γ are all 2-downer edges for λ relative to A, then we call Γ a *2-downer edge cycle for λ relative to A.*

As a simple example of a 2-downer edge cycle, there is a cycle C_n with n vertices whose eigenvalues are $2\cos\frac{2\pi}{n}j$ $(j = 1, \ldots, n)$, and every edge of C_n can be a 2-downer edge of a certain double eigenvalue.

Definition 3. Let G be a graph, $A \in \mathcal{S}(G)$, and $\lambda \in \sigma(A)$ with $m_A(\lambda) \ge 2$. If Γ is a 2-downer edge cycle for λ relative to A and it is not connected to other 2-downer edge cycles with 2-downer edges for λ, then we call Γ a *primitive 2-downer edge cycle for λ relative to A.*

As an example of a primitive 2-downer edge cycle, we refer to Example 1 in Section 4, in which the triangle 1, 2, 3, and the triangle 8, 9, 10 are primitive 2-downer edge cycles for the eigenvalue 1.

We note that if Γ is a primitive 2-downer edge cycle for λ, it is separated from other 2-downer edge cycles, then edges incident to vertices on Γ that are not on Γ are edges other than 2-downer edges.

We are interested in a 2-downer edge cycle in this article, and in Section 2, we investigate the effect of removing a 2-downer edge from the cycle, the change of the statuses of other edges or vertices. Then we investigate the change in the multiplicity of an eigenvalue by removing a cut 2-downer edge triangle in Section 3.

2 Removing a 2-downer edge from a cycle

In this article, we are particular about in a 2-downer edge cycle and focus on the effect of removing a 2-downer edge in a cycle or a 2-downer edge triangle. In [11], the possible classification for an edge e_{ij} for λ relative to $A \in \mathcal{H}(G)$ is given as in Table 1, when the classifications of adjacent vertices i and j are known. Here, we refer to the two theorems we require later.

Theorem 4. [11, Theorem 6] *Let G be a connected graph, $A \in \mathcal{H}(G)$, and $\lambda \in \sigma(A)$ with $m_A(\lambda) \geq 2$. An edge e_{ij} is 2-downer for λ in G if and only if the status of i is downer for λ in G, and j is a downer neighbor of i in G. Here, i and j are interchangeable.*

Theorem 5. [11, Theorem 7] *Let G be a graph, $A \in \mathcal{H}(G)$, and $\lambda \in \sigma(A)$. If an edge e_{ij} is 2-Parter for λ in G, then each of i and j are Parter for λ in G, and each is a downer neighbor for the other in G.*

We need a necessary and sufficient condition for a Parter vertex.

If a graph is a tree T, a Parter vertex v for λ relative to $A \in \mathcal{S}(T)$ is characterized by the existence of a downer branch at v [5]. However, when G is a general graph, a necessary and sufficient condition for a vertex to be a Parter vertex is given in [12]. We give the proof of it to be self-contained here.

Theorem 6. [12, Theorem 3] *A vertex i is Parter for λ in G relative to $A \in \mathcal{S}(G)$ if and only if there is a downer neighbor j at i and the edge e_{ij} is at least neutral for λ in A (i.e., $m_{A(e_{ij})}(\lambda) \geq m_A(\lambda)$).*

Proof. Wlog, we may assume that the index of i is 1, $\lambda = 0$ and that A has the following form:

$$A = \begin{bmatrix} \alpha & \mathrm{x}^T \\ \mathrm{x} & B \end{bmatrix}, \tag{2}$$

in which the x is a nonzero column vector, α is a scalar, and the B is a square matrix.

If the index 1 is Parter for λ in A, there has to be at least one downer neighbor j adjacent to 1 in B. Because if there is no downer neighbor adjacent to 1 in B, then all adjacent vertices to 1 are neutral or Parter for λ in B. Then, every column (resp. row) relative to the adjacent vertex in B is not a linear combination of the remaining columns (resp. rows). Let RS(B) (resp. CS(B)) denote the row space (resp. column space)of B. Then, $e_k^T \in \mathrm{RS}(B)$, (resp. $e_k \in \mathrm{CS}(B)$), in which e_k is a normal unit vector and k corresponds to some indices to which index 1 is adjacent in B. Since x^T is a linear combination of some e_k^Ts $(k \geq 1)$, $\mathrm{x}^T \in \mathrm{RS}(B)$, a contradiction because the index 1 is Parter in A. Furthermore, e_{1j} cannot be a downer or a 2-downer edge in A, since 1 is Parter. Therefore, e_{1j} is at least neutral for λ in A.

Next, we give a proof for sufficiency. Suppose that there is a downer neighbor j of 1 in B and e_{1j} is at least neutral.

To reach a contradiction, suppose that the index 1 is not Parter for λ in A satisfying the aforementioned conditions. If the index 1 is neutral for λ in A, then $\mathrm{x}^T \in \mathrm{RS}(B)$. When e_{1j} is removed from G,

Table 1: Possible classification for edges in $\mathcal{H}(G)$

i	j	**Possible classifications for edge e_{ij}**
P	P	2-Parter or neutral
P	N	Parter or neutral
P	D	Neutral
N	N	Parter or neutral
N	D	Downer
D	D	2-downer, downer, or neutral

P: Parter vertex, N: neutral vertex, D: downer vertex.

$x^T - a_{1j}e_j^T \notin \mathrm{RS}(B)$, since $e_j^T \notin \mathrm{RS}(B)$, because j is a downer neighbor of 1 in B. Then index 1 becomes Parter in $A(e_{1j})$ and $S_A(e_{1j}) = S_A(1) - S_{A(e_{1j})}(1) = -1$. This means the edge e_{1j} is a downer edge in A, a contradiction to the assumption.

Next, if index 1 is downer for λ in A satisfying the aforementioned conditions, then $x^T \in \mathrm{RS}(B)$. When e_{1j} is removed from G, $x^T - a_{1j}e_j^T \notin \mathrm{RS}(B)$, since $e_j^T \notin \mathrm{RS}(B)$ because j is a downer neighbor at 1 in B. Then index 1 becomes Parter in $A(e_{1j})$. Therefore, $S_A(e_{1j}) = S_A(1) - S_{A(e_{1j})}(1) = -2$, which means the edge e_{1j} is a 2-downer edge in A, a contradiction to the assumption.

Thus, when the conditions are satisfied, index 1 must be Parter in A. □

In [12], it was observed that if there is a 2-downer edge in a graph G for λ relative to $A \in \mathcal{S}(G)$, then there is a 2-downer edge cycle for λ in the graph. To be self-contained in this article, we give the proof here.

Theorem 7. [12] *Suppose G is a graph, $A \in \mathcal{H}(G)$ and $\lambda \in \sigma(A)$ with $m_A(\lambda) \geq 2$. Then each 2-downer edge for λ is contained in a 2-downer edge cycle of G or it is on a path connecting 2-downer edge cycles.*

Proof. Let e_{ij} be a 2-downer edge for λ in $A \in \mathcal{H}(G)$. Then, i and j are downer for λ in A and downer neighbors for each other. When the edge e_{ij} is removed from G, the status of i changes to Parter . Then, there has to be a downer neighbor k distinct from j in $A(e_{ij})$ by Theorem 6. Then, we note that k is also a downer neighbor of i in A, and i is originally downer in A. Thus, e_{ik} is a 2-downer edge in A by Theorem 4.

From a similar argument, there must be another 2-downer edge incident to e_{ij} at j. Thus, 2-downer edges are connected sequentially and compose a cycle in G in the end. □

2.1 Change in status by removing a 2-downer edge

First, we see that when a vertex on a 2-downer edge cycle Γ is removed, other 2-downer edges on Γ in G are not 2-downer in the remaining graph $G(v)$.

Lemma 8. *Let G be a graph, $A \in \mathcal{S}(G)$, and $\lambda \in \sigma(A)$ with $m_A(\lambda) \geq 2$. Let Γ be a primitive 2-downer edge cycle in G for λ relative to A. If a vertex v on Γ is removed from G, then the rest of the edges on Γ in $G(v)$ are not 2-downer for λ relative to $A(v)$.*

Proof. If there is a 2-downer edge on $\Gamma(v)$ in $G(v)$, then there must be a 2-downer cycle in $G(v)$. Then there is a 2-downer edge e_{ik} in which i is on $\Gamma(v)$ and k is outside $\Gamma(v)$.

Since e_{ik} is 2-downer in $G(v)$, i is downer in $G(v)$ and k is downer in $G(v, i)$ by Theorem 4. Then,

$$m_{A(v,i,k)}(\lambda) = m_A(\lambda) - 3. \tag{3}$$

However, k is not downer in $G(i)$ from the assumption that Γ is a primitive 2-downer edge cycle in G and k is not on Γ. Then k is neutral or Parter in $G(i)$. Thus, $m_{A(i,k,v)}(\lambda) > m_A(\lambda) - 3$ has to hold. That is a contradiction to (3). Therefore, the rest of the edges on Γ in $G(v)$ are not 2-downer edges in $G(v)$. □

Let Γ be a primitive 2-downer edge cycle for λ in a general simple graph G, and Γ' be a subgraph of Γ obtained by removing an edge e_{ij} on Γ. Then let $G' = G(e_{ij})$.

Lemma 9. *Let G be a graph, $A \in \mathcal{S}(G)$, and $\lambda \in \sigma(A)$ with $m_A(\lambda) \geq 2$. Let Γ be a primitive 2-downer edge cycle in G for λ relative to A. If a 2-downer edge e_{ij} on Γ is removed from G, then the statuses of vertices and edges on Γ' are at least neutral in G' for λ relative to $A(e_{ij})$.*

Proof. Let v be a vertex on Γ, then v is downer for λ. If v is removed from G, all the edges on Γ' are not 2-downer for λ by Lemma 8. Then

$$m_{A(v,e_{ij})}(\lambda) \neq m_A(\lambda) - 3.$$

Since $m_{A(e_{ij})}(\lambda) = m_A(\lambda) - 2$, v cannot be downer in $A(e_{ij})$. Then the status of vertices on Γ' is at least neutral in G'.

If an edge on Γ' is downer or 2-downer for λ, the status of incident vertices has to be (N, D) or (D, D) (cf. Table 1). From the preceding argument, the statuses of vertices on Γ' are not downer. Therefore, the edges on Γ' are at least neutral for λ in G'. ☐

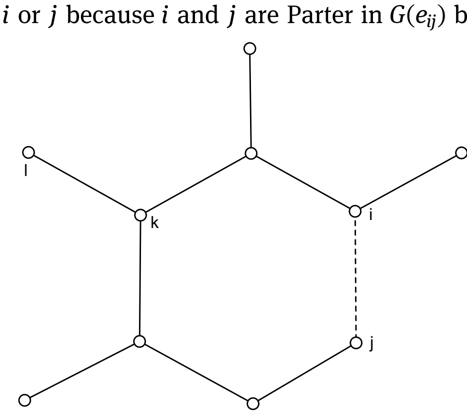
It is known that when a neutral vertex is removed from G, a downer vertex in G stays in the remaining graph, and *vice versa* [4].

Lemma 10. [4] *Let A be an n-by-n Hermitian matrix. If i is neutral, then $j \neq i$ is downer for A if and only if j is downer for $A(i)$.*

Theorem 11. *Let G be a graph, $A \in \mathcal{S}(G)$, and $\lambda \in \sigma(A)$ with $m_A(\lambda) \geq 2$. Let Γ be a primitive 2-downer edge cycle in G for λ relative to A, and Γ' be a subgraph of Γ obtained by removing an edge e_{ij}* on Γ. *The statuses of all edges in G' incident to the vertices on Γ' are at least neutral for λ relative to $A(e_{ij})$.*

Proof. From Lemma 9, the edges on Γ' are at least neutral for λ. We examine the status of an edge outside Γ' that is incident to a vertex on Γ'.

We know that the statuses of vertices on Γ' are Parter or neutral from Lemma 9. If a vertex on Γ' is Parter, it is obvious that edges incident to it are at least neutral from Table 1. Therefore, we suppose that there is a neutral vertex k on Γ' that is not i or j because i and j are Parter in $G(e_{ij})$ by (1).



To attain a contradiction, we suppose that there is a downer edge incident to k in $G(e_{ij})$. Let the edge be e_{kl}. Then the status of l has to be downer in $G(e_{ij})$ (cf. Table 1). Then

$$m_{A(e_{ij},k,l)}(\lambda) = m_A(\lambda) - 3, \tag{4}$$

because a downer vertex is still downer after removing a neutral vertex by Lemma 10.

On the other hand, k is downer in G since k is on Γ, and the status of l is not downer in $G(k)$ since Γ is primitive. So l is Parter or neutral for λ in $G(k)$. If l is Parter in $G(k)$, $m_{A(k,l,e_{ij})}(\lambda) > m_A(\lambda) - 3$, then (4) does not hold. Thus, in the case for (4) to hold, l must be neutral in $G(k)$ and e_{ij} has to be 2-downer in $G(k, l)$, then we note that

$$m_{A(k,l,i,j)}(\lambda) = m_A(\lambda) - 3 \tag{5}$$

holds by Theorem 4.

Next, we focus on the status of the vertex i in $G(k)$.

If i is Parter in $G(k)$, $m_{A(k,i)}(\lambda) = m_A(\lambda)$, then $m_{A(k,i,j,l)}(\lambda) > m_A(\lambda) - 3$, so i has to be neutral or downer in $G(k)$ for (5) to hold. We note that l is neutral in $G(k)$ from the prior argument. If i is neutral in $G(k)$, then in $G(k, i)$, l is neutral or Parter, because l cannot be downer in $G(k, i)$ from Lemma 10. Then we have $m_{A(k,i,l,j)}(\lambda) > m_A(\lambda) - 3$. So (5) does not hold. Therefore, for (5) to hold, i has to be downer in $G(k)$. By Lemma 8, e_{ij} is not a 2-downer edge in $G(k)$, so j is not downer in $G(k, i)$, then j is neutral or Parter in $G(k, i)$.

If j is neutral in $G(k, i)$, then l can be Parter or neutral in $G(k, i, j)$, because l cannot be downer from Lemma 10. Then $m_{A(k,i,j,l)}(\lambda) \neq m_A(\lambda) - 3$.

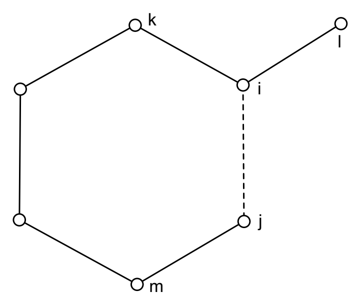
If j is Parter in $G(k, i)$, then $m_{A(k,i,j)}(\lambda) = m_A(\lambda) - 1$, so $m_{A(k,i,j,l)}(\lambda) \neq m_A(\lambda) - 3$. Therefore, (5) cannot hold. It is a contradiction.

So, we have a conclusion that e_{kl} cannot be a downer edge in $G(e_{ij})$. □

If edges in G' are incident to the removed 2-downer edge in G, we can see their statuses more accurately.

Corollary 12. *Let G be a graph, $A \in \mathcal{S}(G)$, and $\lambda \in \sigma(A)$ with $m_A(\lambda) \geq 2$. Let Γ be a primitive 2-downer edge cycle in G for λ relative to A and Γ' be a subgraph of Γ obtained by removing an edge e_{ij} on Γ. The edge incident to i or j on Γ' is Parter or 2-Parter, and other edges incident to i or j are neutral in G' for λ relative to $A(e_{ij})$.*

Proof. The status of i and j is Parter in $G(e_{ij})$ by (1). Let k and m be vertices on Γ' adjacent to i and j, respectively. We refer to the following figure.



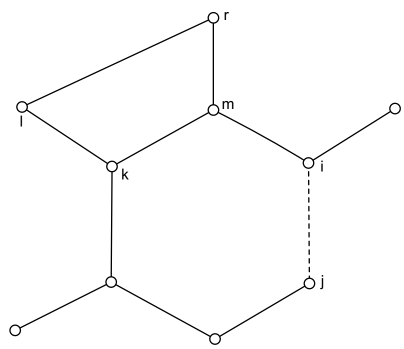
If e_{ik} is neutral in $G(e_{ij})$, then i is Parter in $G(e_{ij}, e_{ik})$ by (1). However, since Γ is primitive, there is no downer neighbor at i in $G(e_{ij}, e_{ik})$. So i cannot be Parter in $G(e_{ij}, e_{ik})$ by Theorem 6. It is a contradiction. So, e_{ik} is Parter or 2-Parter for λ in $G(e_{ij})$. By a similar argument, e_{jm} is Parter or 2-Parter for λ in $G(e_{ij})$.

Let e_{il} be an edge incident to i that is not on Γ'. Then l is not downer neighbor at i in $G(e_{ij})$, since Γ is primitive. So e_{il} cannot be a 2-Parter edge from Theorem 5. Next, we show that e_{il} cannot be a Parter edge for λ in $G(e_{ij})$. If e_{il} is a Parter edge with (P, N) in $G(e_{ij})$, i is neutral and l is downer in $G(e_{ij}, e_{il})$. Further, when i is removed from $G(e_{ij}, e_{il})$, l is downer in $G(e_{ij}, e_{il}, i)$ by Lemma 10. It means that l is a downer neighbor at i in $G(e_{ij})$, a contradiction. So, e_{il} cannot be a Parter edge in $G(e_{ij})$. Then e_{il} is neutral by Theorem 11. □

Next, we investigate the status of vertices adjacent to Γ'.

Theorem 13. *Let G be a graph, $A \in \mathcal{S}(G)$, and $\lambda \in \sigma(A)$ with $m_A(\lambda) \geq 2$. Let Γ be a primitive 2-downer edge cycle for λ relative to A, and Γ' be a graph obtained by removing an edge e_{ij} on Γ. The statuses of all vertices adjacent to Γ' in G' are at least neutral for λ relative to $A(e_{ij})$. Furthermore, the edges between the vertices adjacent to Γ' are also at least neutral for λ relative to $A(e_{ij})$.*

Proof. From Lemma 9, vertices on Γ' are at least neutral in G'. Let k be a vertex on Γ' and l be a vertex that is not on Γ' and adjacent to k.



To reach a contradiction, if we suppose that l is downer in G' for λ, then $m_{A(e_{ij},l)}(\lambda) = m_A(\lambda) - 3$. Then, $m_{A(l,e_{ij})}(\lambda) = m_A(\lambda) - 3$, so l has to be downer for λ in G and e_{ij} be 2-downer in $G(l)$.

We note that k is not downer in $G(l)$, because l is not downer in $G(k)$. Since k is not downer in $G(l)$, Γ is not a 2-downer edge cycle in $G(l)$.

If e_{ij} is a 2-downer edge for λ in $G(l)$, then there is a 2-downer edge e_{mr} in which the vertex m (possibly i or j) is on Γ and the vertex r is adjacent to m that is not on Γ. Then we have $m_{A(l,m,r)}(\lambda) = m_A(\lambda) - 3$ since l is downer in G and e_{mr} is 2-downer in $G(l)$ from the assumption. Then $m_{A(m,r)}(\lambda) = m_A(\lambda) - 2$ must hold to be $m_{A(l,m,r)}(\lambda) = m_A(\lambda) - 3$, then r is a downer neighbor at m in G. But r cannot be a downer neighbor at m in G, since Γ is primitive. It is a contradiction, so e_{ij} is not a 2-downer edge in $G(l)$. Therefore, l is not downer in $G(e_{ij})$, then l is at least neutral in $G(e_{ij})$.

When the vertices adjacent to Γ' are at least neutral in G', the edges between them are at least neutral from Table 1. So, the edge e_{lr} in the figure is also at least neutral in G'. □

2.2 The same status by removing an edge

Next, we observed that when an edge of a certain status for λ relative to A is removed from G, there are edges or vertices whose statuses stay same in the resulting graph.

Theorem 14. *Let G be a graph, $A \in \mathcal{S}(G)$, and $\lambda \in \sigma(A)$ with $m_A(\lambda) \geq 2$. Let e_{ij} be a 2-downer edge for λ relative to A. Let $\widetilde{G}$ be a graph obtained by removing the edge e_{ij} from G. The statuses of neutral or Parter vertices in G for λ relative to A stay in $\widetilde{G}$ for λ relative to $A(e_{ij})$.*

Proof. Since e_{ij} is a 2-downer edge in G, i and j are downer for λ in G, and $m_{A(i,j)}(\lambda) = m_A(\lambda) - 2$. Let k be a Parter (resp. neutral) vertex in G. We note that k is Parter (resp. neutral) in $A(i,j)$, because Parter (resp. neutral) vertices stay after removing a downer vertex. Then,

$$m_{A(i,j,k)}(\lambda) = m_A(\lambda) - 1, \text{(resp. } m_A(\lambda) - 2). \tag{6}$$

On the other hand, i is downer for λ in $A(k)$, because downer vertices stay after removing a Parter (resp. neutral) vertex (cf. Table 1 [8]). Then j has to be downer in $A(k,i)$ for (6) to hold. Thus, j is a downer neighbor at i in $A(k)$. Then e_{ij} is a 2-downer edge in $A(k)$. Therefore, 2-downer edges stay after removing a Parter (resp. neutral) vertex from G.

Through converse consideration, a Parter (resp. neutral) vertex stays after removing a 2-downer edge from G, because the change in multiplicity is consistent after removing a 2-downer edge first and a Parter (resp. neutral) vertex. □

In Theorem 14, it was observed that the status of a Parter (resp. neutral) vertex does not change after removing a 2-downer edge for λ from G.

Furthermore, we noticed that there is another case in which the status of the vertex does not change after removing an identified status of edge.

Theorem 15. *Let G be a graph, $A \in \mathcal{S}(G)$, and $\lambda \in \sigma(A)$. If a 2-Parter edge or a Parter edge for λ relative to A is removed from G, then the statuses of downer vertices for λ in G stay in the resulting graph for λ relative to the corresponding matrix.*

Proof. Let e_{ij} be a 2-Parter edge in G. Let k be a downer vertex in G. If we assume that the status of k changes to neutral or Parter after removing e_{ij} from G, then

$$m_{A(e_{ij},k)}(\lambda) \geq m_A(\lambda) + 2. \tag{7}$$

However, since k is downer for λ in G, $m_{A(k)}(\lambda) = m_A(\lambda) - 1$. Then, $m_{A(k,e_{ij})}(\lambda) \leq m_A(\lambda) + 1$. That is a contradiction to (7). Therefore, k stays downer in $G(e_{ij})$.

Next, let e_{ij} be a Parter edge in G and k be a downer vertex in G. A pair of statuses of i and j can be (P, N) or (N, N) from Table 1.

If we suppose the status of k changes to Parter in $G(e_{ij})$, then

$$m_{A(e_{ij},k)}(\lambda) = m_A(\lambda) + 2. \tag{8}$$

However, since k is downer for λ in G, $m_{A(k)}(\lambda) = m_A(\lambda) - 1$. Then, $m_{A(k,e_{ij})}(\lambda) \leq m_A(\lambda) + 1$. That is a contradiction to (8). So, k cannot be Parter for λ in $G(e_{ij})$.

Next, we suppose that k changes to neutral in $G(e_{ij})$. Wlog, let j be neutral for λ in G since i or j is neutral when e_{ij} is a Parter edge. In $G(e_{ij})$, j is downer by (1), then

$$m_{A(e_{ij},j,k)}(\lambda) = m_A(\lambda), \tag{9}$$

because the status of k stays neutral in $G(e_{ij}, j)$, it can be said by using Lemma 10. However, we note that

$$m_{A(e_{ij},j,k)}(\lambda) = m_{A(j,k)}(\lambda) = m_A(\lambda) - 1, \tag{10}$$

because k is downer and j is neutral in G, and k stays downer in $G(j)$. This is a contradiction to (9). So, k cannot become neutral in $G(e_{ij})$. Therefore, k stays downer in $G(e_{ij})$. □

If we conversely see Theorem 15, then we have the next result. The change in the multiplicity of an eigenvalue is independent of the order of removing a vertex and an edge.

Corollary 16. *Let G be a graph, $A \in \mathcal{S}(G)$, and $\lambda \in \sigma(A)$. If a downer vertex for λ relative to A is removed from G, then 2-Parter edges or Parter edges for λ in G stay in the resulting graph for λ relative to the corresponding matrix.*

Next, we observe that when a 2-downer edge is removed from G, some edges do not change in their statuses.

Theorem 17. *Let G be a graph, $A \in \mathcal{S}(G)$, and $\lambda \in \sigma(A)$ with $m_A(\lambda) \geq 2$. Let e_{ij} be a 2-downer edge in G for λ relative to A. Let $\widetilde{G}$ be a graph obtained by removing the edge e_{ij} from G. The statuses of 2-Parter edges or Parter edges in G for λ relative to A stay in $\widetilde{G}$ for λ relative to $A(e_{ij})$.*

Proof. Let e_{kl} be a 2-Parter edge in G. Then the statuses of k and l are Parter. When a 2-downer edge e_{ij} is removed from G, the status of a Parter vertex stays in $\widetilde{G}$ by Theorem 14. So, e_{kl} is 2-Parter or neutral in $G(e_{ij})$ from Table 1. However, we show that e_{kl} cannot be neutral in $G(e_{ij})$. If we suppose that e_{kl} is neutral in $G(e_{ij})$, then

$$m_{A(e_{ij},e_{kl})}(\lambda) = m_A(\lambda) - 2. \tag{11}$$

However, since e_{kl} is 2-Parter for λ,

$$m_{A(e_{kl},e_{ij})}(\lambda) \geq m_A(\lambda). \tag{12}$$

considering Lemma 1. So, (11) is contradictory to (12). Therefore, e_{kl} is a 2-Parter edge for λ in $G(e_{ij})$.

Next, we consider Parter edges in G. Let e_{kl} be a Parter edge for λ in G. There are two types of Parter edges with (P, N) and (N, N) as a pair of the statuses of the adjacent vertices from Table 1. When the 2-downer edge e_{ij} is removed from G, Parter vertices and neutral vertices in G stay in $G(e_{ij})$ by Theorem 14. So the status of e_{kl} is a Parter edge or a neutral edge in $G(e_{ij})$ with (P, N) or (N, N) by Table 1. However, we show that it cannot be a neutral edge in $G(e_{ij})$.

If we suppose that e_{kl} is neutral in $G(e_{ij})$, then

$$m_{A(e_{ij},e_{kl})}(\lambda) = m_A(\lambda) - 2. \tag{13}$$

But, since e_{kl} is a Parter edge in G,

$$m_{A(e_{kl},e_{ij})}(\lambda) \geq m_A(\lambda) - 1, \tag{14}$$

then (13) is contradictory to (14). Therefore, e_{kl} is a Parter edge in $G(e_{ij})$. □

Considering Theorem 17 conversely, when a 2-Parter edge or a Parter edge is removed from G, 2-downer edges stay in the resulting graph. Thus, we have the next result.

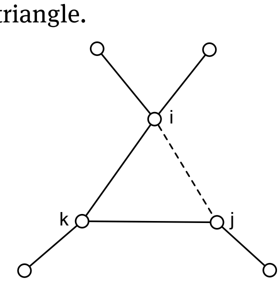
Corollary 18. *Let G be a graph, $A \in \mathcal{S}(G)$, and $\lambda \in \sigma(A)$ with $m_A(\lambda) \geq 2$. If a 2-Parter edge or a Parter edge for λ relative to A is removed from G, original 2-downer edge cycles for λ in G stay in the resulting graph for λ relative to the corresponding matrix.*

3 Removing a cut 2-downer edge triangle

Next, we focus on a 2-downer edge cycle with three vertices, called a *2-downer edge triangle* here. When three edges on a triangle are removed from G, if the number of components of G increases, then we call it a *cut triangle* in G. Now, we give a simple observation for a cut triangle that is used later.

Lemma 19. *Let Γ be a cut triangle in G, then there is an edge on Γ, which is a cut edge after removing the rest of the edges on Γ from G.*

Proof. Let the vertices of Γ be i, j, and k. Wlog, we suppose G is a connected graph. Let $G' = G(e_{ij})$, then G' is connected. We refer to the following figure as a part of G that includes Γ. If e_{ik} is not a cut edge in G', then e_{jk} is a cut edge in $G(e_{ij}, e_{ik})$ since Γ is a cut triangle.



If e_{ik} is a cut edge in G', then e_{kj} is an edge that is included in one component of $G(e_{ij}, e_{ik})$. Then we note that e_{ik} is also a cut edge in $G(e_{ij}, e_{jk})$. □

When a cut edge in G is removed, the change in multiplicity of an identified eigenvalue is observed in [7, Lemma 19].

Lemma 20. [7, Lemma 19] *Let G be a graph and $A \in \mathcal{H}(G)$. If e_{ij} is a cut-edge in G and $\lambda \in \sigma(A)$, then*

$$m_A(\lambda) - 1 \le m_{A(e_{ij})}(\lambda) \le m_A(\lambda) + 2.$$

This lemma indicates that if there is a cut edge in G, it cannot be a 2-downer edge for an eigenvalue of $A \in \mathcal{H}(G)$.

The condition for a cut edge to be a downer edge is shown in [7, Theorem 20].

Lemma 21. [7, Theorem 20] *Let G be a graph, $A \in \mathcal{H}(G)$, and $\lambda \in \sigma(A)$. A cut-edge e_{ij} in G is downer for λ relative to A if and only if the statuses of i and j in G are (D, D) for λ relative to A.*

Lemma 22. *Let G be a graph, $A \in \mathcal{S}(G)$, and $\lambda \in \sigma(A)$ with $m_A(\lambda) \ge 2$. Let Γ be a cut primitive 2-downer edge triangle in G for λ relative to A. If one edge on Γ is removed from G, then the rest of the edges on Γ are Parter edges in the remaining graph for λ relative to the corresponding matrix.*

Proof. Let the vertices of Γ be i, j, and k. From Corollary 12, if an edge e_{ij} is removed, the rest of the edges e_{ik} and e_{jk} are Parter or 2-Parter.

We note that i and j are Parter vertices in $G(e_{ij})$. If e_{ik} is a Parter edge, then k has to be neutral for λ, and if e_{ik} is 2-Parter, then k has to be Parter for λ (cf. Table 1). Thus, the statuses of two edges e_{ik} and e_{jk} have to be the same when e_{ij} is removed. We note that if e_{ik} and e_{jk} are Parter in $G(e_{ij})$, then e_{ij} and e_{jk} are Parter in $G(e_{ik})$.

We can observe that if one edge on Γ is removed, the other two edges on Γ cannot be 2-Parter edges in $G(e_{ij})$.

Wlog, we suppose that e_{jk} is a cut edge in $G(e_{ij}, e_{ik})$ by Lemma 19. To reach a contradiction, we suppose that e_{ik} and e_{jk} are 2-Parter for λ in $G(e_{ij})$. Then j is a downer neighbor at k in $G(e_{ij})$ by Theorem 5 since e_{jk} is 2-Parter in $G(e_{ij})$. Then, k is downer for λ in $G(e_{ij}, e_{ik})$. We note that j is a downer neighbor at k also in $G(e_{ij}, e_{ik})$. Then e_{jk} has to be a 2-downer edge by Theorem 4, and it is a cut edge in $G(e_{ij}, e_{ik})$. So it is a contradiction to Lemma 20, because a cut edge cannot be a 2-downer edge. Therefore, e_{ik} and e_{jk} cannot be 2-Parter in $G(e_{ij})$, then they are Parter edges for λ in $G(e_{ij})$. □

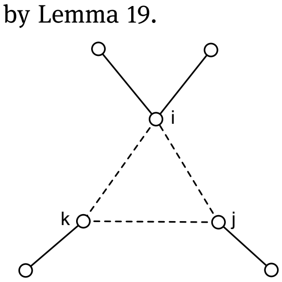
We investigate the change in multiplicity of an identified eigenvalue λ, when all edges on a cut primitive 2-downer edge triangle are removed.

Theorem 23. *Let G be a graph, $A \in \mathcal{S}(G)$, and $\lambda \in \sigma(A)$ with $m_A(\lambda) \ge 2$. Let Γ be a cut primitive 2-downer edge triangle in G for λ relative to A. Let $\widetilde{G}$ be a subgraph of G obtained by removing all edges on Γ from G and $\widetilde{A}$ a corresponding matrix. Then*

$$m_{\tilde{A}}(\lambda) = m_A(\lambda) - 2,$$

and the statuses of three vertices on Γ are neutral in $\widetilde{G}$ for λ relative to $\widetilde{A}$.

Proof. Let the vertices of Γ be i, j, and k. Since Γ is a cut 2-downer edge triangle, we may suppose that wlog e_{jk} is a cut edge after removing e_{ij} and e_{ik} by Lemma 19.



From Lemma 22, when the edge e_{ij} is removed from G, e_{ik} and e_{jk} are Parter edges for λ, then i and j are Parter by (1) and k is neutral in $G(e_{ij})$ since e_{ik} is a Parter edge with vertices (P, N) in $G(e_{ij})$. Then if e_{ik} is removed from $G(e_{ij})$, k is downer by (1), then we next show that j becomes downer in $G(e_{ij}, e_{ik})$. When e_{ik} is removed from G, e_{ij} and e_{jk} are Parter by Lemma 22, and j is neutral because when an edge is a Parter edge, if one vertex is Parter, then the other vertex has to be neutral. So, $m_{A(e_{ik},j)}(\lambda) = m_A(\lambda) - 2$. On the other hand, $m_{A(e_{ik},j)}(\lambda) = m_{A(e_{ij},e_{ik},j)}(\lambda) = m_A(\lambda) - 2$. Since $m_{A(e_{ij},e_{ik})}(\lambda) = m_A(\lambda) - 1$, j has to be downer in $G(e_{ij}, e_{ik})$.

Since the statuses of j and k are downer for λ in $G(e_{ij}, e_{ik})$, and e_{jk} is a cut edge in $G(e_{ij}, e_{ik})$, then e_{kj} has to be a downer edge by Lemma 21. We conclude

$$m_{\tilde{A}}(\lambda) = m_{A(e_{ij},e_{ik},e_{jk})}(\lambda) = m_A(\lambda) - 2.$$

We note that e_{jk} is a downer edge with (D, D) in $G(e_{ij}, e_{ik})$, and i is neutral in $G(e_{ij}, e_{ik})$ for λ. If the edge e_{jk} is removed from $G(e_{ij}, e_{ik})$, then the statuses of j and k change to neutral.

Next, we observe i is still neutral in $G(e_{ij}, e_{ik}, e_{jk})$. When i is removed from $G(e_{ij}, e_{ik})$, k and j are downer by Lemma 10 and e_{jk} is a cut edge, so e_{jk} is downer in $G(e_{ij}, e_{ik}, i) = G(i)$. Then $m_{A(e_{ij},e_{ik},i,e_{jk})}(\lambda) = m_{A(i,e_{jk})}(\lambda) = m_A(\lambda) - 2$. Since e_{jk} was downer in $G(e_{ij}, e_{ik})$, $m_{A(e_{ij},e_{ik},e_{jk})}(\lambda) = m_A(\lambda) - 2$.

Therefore, i is also neutral in $G(e_{ij}, e_{ik}, e_{jk})$. □

Next, we investigate the change in the multiplicity of an identified eigenvalue λ when all vertices on a cut 2-downer edge triangle are removed.

Theorem 24. *Let G be a graph, $A \in \mathcal{S}(G)$, and $\lambda \in \sigma(A)$ with $m_A(\lambda) \geq 2$. Let Γ be a cut 2-downer edge triangle in G for λ relative to A. Let $\widetilde{G}$ be a subgraph of G obtained by removing all vertices on Γ from G and $\widetilde{A}$ a corresponding matrix. Then*

$$m_{\tilde{A}}(\lambda) = m_A(\lambda) - 2.$$

Proof. Wlog, we may suppose that e_{jk} is a cut edge in $G(e_{ij}, e_{ik})$ by Lemma 19. Then we note that e_{jk} is also a cut edge in $G(i)$. Since j and k are downer in $G(i)$ and e_{jk} is a cut edge in $G(i)$, e_{jk} is a downer edge in $G(i)$.

If e_{jk} is removed from $G(i)$, then $m_{A(i,e_{jk})}(\lambda) = m_A(\lambda) - 2$. Then k and j is neutral in $G(i, e_{jk})$, and k and j belong to different components. So, $m_{A(i,e_{jk},j,k)}(\lambda) = m_{A(i,j,k)}(\lambda) = m_A(\lambda) - 2$. □

We note that in Theorem 24, the cut 2-downer edge triangle does not have to be primitive in G.

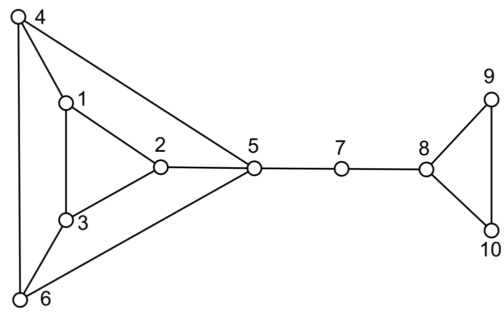
4 Example

Example 1. We give an example to sketch Theorems 11 and 13.

Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \end{bmatrix},$$

whose graph G is as follows.

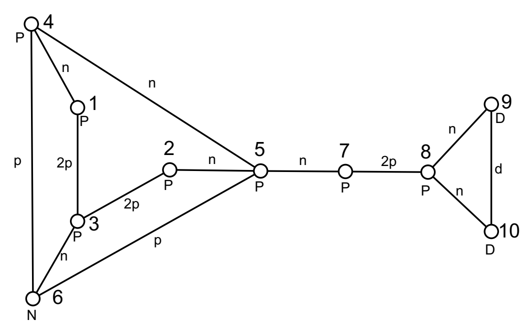


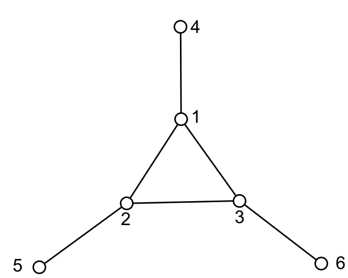
A has an eigenvalue $\lambda = 1$ with multiplicity 3. The triangle Γ whose vertices are 1, 2, 3 is a 2-downer edge triangle for $\lambda = 1$. When one edge e_{12} on Γ is removed from G, the status of edges and vertices on G' are shown in the following figure. The statuses of edges and vertices are indicated in small letters and capital letters, respectively.

Example 2. We can find a simple example to sketch Theorems 23 and 24. Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix},$$

whose graph G is as follows.





A has an eigenvalue $\lambda = \frac{1+\sqrt{5}}{2}$ with multiplicity 2. The center triangle is a cut primitive 2-downer edge triangle for λ. When edges e_{12}, e_{23}, and e_{13} are removed from G, let

$$\widetilde{A} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

$\widetilde{A}$ does not have $\lambda = \frac{1+\sqrt{5}}{2}$ as an eigenvalue of $\widetilde{A}$. So, the multiplicity of λ decreases by 2 in $\widetilde{A}$.

If vertices 1, 2, and 3 are removed from G, isolated vertices 4, 5, and 6 do not have λ as an eigenvalue. So, the multiplicity of λ decreases by 2.

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