Short Note

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Two $n \times n$ G-classes of matrices having finite intersection

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Abstract: Let $\mathbf{M}_n$ be the set of all $n \times n$ real matrices. A nonsingular matrix $A \in \mathbf{M}_n$ is called a G-matrix if there exist nonsingular diagonal matrices D_1 and D_2 such that $A^{-T} = D_1 A D_2$. For fixed nonsingular diagonal matrices D_1 and D_2, let $\mathbb{G}(D_1, D_2) = \{A \in \mathbf{M}_n : A^{-T} = D_1 A D_2\}$, which is called a G-class. The purpose of this short article is to answer the following open question in the affirmative: do there exist two $n \times n$ G-classes having finite intersection when $n \geq 3$?

Keywords: G-matrix, G-class, inertia of matrices

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1 Introduction

All matrices in this note have real number entries. Let $\mathbf{M}_n$ be the set of all $n \times n$ real matrices. A nonsingular matrix $A \in \mathbf{M}_n$ is called a G-matrix if there exist nonsingular diagonal matrices D_1 and D_2 such that $A^{-T} = D_1 A D_2$, where A^{-T} denotes the transpose of the inverse of A. These matrices form a rich class and were originally studied in [1] by Fiedler and Hall; they include the orthogonal and J-orthogonal matrices. For a survey of the basic properties of G-matrices and connections to other classes of matrices, the reader can refer to [1] and [2] and references therein. Here, we just mention two other connections.

Cauchy matrices have the form $C = [c_{ij}]$, where $c_{ij} = \frac{1}{x_i + y_j}$ for some numbers x_i and y_j. We shall restrict to square, say $n \times n$, Cauchy matrices – such matrices are defined only if $x_i + y_j \neq 0$ for all pairs of indices i and j, and it is well known that C is nonsingular if and only if all the numbers x_i are mutually distinct and all the numbers y_j are mutually distinct. It turns out that by observation of Fiedler [3], every nonsingular Cauchy matrix is a G-matrix. So, in particular, G-matrices arise naturally as very well-defined structured nonsingular Cauchy matrices. Furthermore, G-matrices arise also in the context of "combined matrices" $C(A) = A \circ A^{-T}$, where $\circ$ denotes the Hadamard product, see [3]. For example, if A is a G-matrix, then $C(A) = A \circ (D_1 A D_2) = D_1(A \circ A)D_2$; so if say D_1 and D_2 are nonnegative, then $C(A)$ is nonnegative. The combined matrices appear in the chemical literature where they represent the relative gain array [4].

For fixed nonsingular diagonal matrices D_1 and D_2, let the class of $n \times n$ G-matrices be

$$\mathbb{G}(D_1, D_2) = \{A \in \mathbf{M}_n : A^{-T} = D_1 A D_2\}.$$

We call such a class of matrices a G-class of matrices.

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In this note, it is shown that for every n, there exist two $n \times n$ G-classes having finite, nonempty intersection. This answers an open question in [2] in the affirmative.

We note that the nonsingular diagonal matrices D_1 and D_2 satisfying $A^{-T} = D_1AD_2$ are, in general, not uniquely determined as we can multiply one of them by a nonzero real number and divide the other by the same number. On the other hand, for nonsingular $n \times n$ diagonal matrices D_1 and D_2, the following known result from [1] shows that if $A^{-T} = D_1AD_2$, then D_1 and D_2 have the same inertia matrix. For the definitions of the inertia and the corresponding inertia matrix of a general Hermitian matrix, the reader can refer to [5, pp. 281–282]. Simply put, the inertia matrix of a Hermitian matrix A is the diagonal matrix

$$\mathrm{diag}(1, \dots, 1, -1, \dots, -1, 0, \dots, 0),$$

where the number of $1's$, $-1's$, and $0's$ is the number of positive, negative, and zero eigenvalues, respectively, of A.

Proposition 1.1. *Suppose A is a G-matrix and $A^{-T} = D_1AD_2$, where D_1 and D_2 are nonsingular diagonal matrices. Then, the inertia of D_1 is equal to the inertia of D_2.*

2 Solution of the open question

By a signature matrix, we mean a diagonal matrix where each diagonal entry is ± 1. Let D be a nonsingular diagonal matrix with the inertia matrix J (a signature matrix having all its positive entries in the upper left corner). Then, there exists a permutation matrix P such that $D = |D|P^TJP$, where $|D|$ is obtained by taking the absolute value on entries of D.

For a fixed signature matrix J, $\Gamma_n(J) = \{A \in \mathbf{M}_n : A^TJA = J\}$. In fact,

$$\Gamma_n(J) = \mathbb{G}(J, J).$$

We mention that the matrices in $\Gamma_n(J)$ are precisely the J-orthogonal matrices discussed in [6–9]. Also, note that when J is I or $-I$, $\Gamma_n(J) = O_n$, the set of all $n \times n$ orthogonal matrices.

In [2], the authors proved the following theorem (Theorem 2.2 of [2]).

Theorem 2.1. *Let D_1 and D_2 be nonsingular diagonal matrices with the same inertia matrix J. Then, there exist permutation matrices P and Q such that*

$$\mathbb{G}(D_1, D_2) = \{|D_1|^{-1/2}P^TAQ|D_2|^{-1/2} : A \in \Gamma_n(J)\}.$$

This characterization shows that $\mathbb{G}(D_1, D_2)$ is in fact nonempty.

Finally, we mention one other preliminary result.

Theorem 2.2. [2, Theorem 3.1] *Let D_1, D_2, D_3, and D_4 be nonsingular diagonal matrices, all of which have the same inertia matrix I or $-I$. Then,*

$$\mathbb{G}(D_1, D_2) = \mathbb{G}(D_3, D_4)$$

if and only if there exists a positive number d such that $D_3 = dD_1$ and $D_4 = \frac{1}{d}D_2$.

With that background, we can now answer the open question. We remark that an example is already given in [2] in the case when $n = 2$. However, from that example, no inductive procedure is apparent. But now we are able to give patterns in D_1, D_2, D_3, and D_4 that are amenable to induction.

Theorem 2.3. *There exist two $n \times n$ G-classes having finite, nonempty intersection when $n \geq 3$.*

Proof. Let

$$D_1 = \operatorname{diag}\left(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}\right), \quad D_2 = \operatorname{diag}(1, 2, 3, \dots, n),$$
$$D_3 = \operatorname{diag}\left(\frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n+2}\right), \quad D_4 = \operatorname{diag}(3, 4, \dots, n+2).$$

The inertia matrix of each of D_1, D_2, D_3, and D_4 is I. By using Theorem 2.2,

$$\mathbb{G}(D_1, D_2) \neq \mathbb{G}(D_3, D_4).$$

Let $A \in \mathbb{G}(D_1, D_2) \cap \mathbb{G}(D_3, D_4)$. Since here the inertia matrix of each D_i is $J = I$, $\Gamma_n(J) = O_n$, and the permutation matrices P and Q are not needed; therefore, by using Theorem 2.1, there are $V, W \in O_n$ such that

$$A = D_1^{\frac{-1}{2}} V D_2^{\frac{-1}{2}} = D_3^{\frac{-1}{2}} W D_4^{\frac{-1}{2}},$$

which implies that

$$W = D_3^{\frac{1}{2}} D_1^{\frac{-1}{2}} V D_2^{\frac{-1}{2}} D_4^{\frac{1}{2}}$$

(Since $W \in O_n$, this means that also $D_3^{1/2} D_1^{-1/2} V D_2^{-1/2} D_4^{1/2} \in O_n$.).

From $W = D_3^{\frac{1}{2}} D_1^{\frac{-1}{2}} V D_2^{\frac{-1}{2}} D_4^{\frac{1}{2}}$, with $W = [w_{ij}]$ and $V = [v_{ij}]$, it follows that

$$w_{ij} = \frac{\sqrt{i}}{\sqrt{i+2}} \frac{\sqrt{j+2}}{\sqrt{j}} v_{ij}.$$

For all $i < j$, we have $0 < \frac{\sqrt{i}}{\sqrt{i+2}} \frac{\sqrt{j+2}}{\sqrt{j}} < 1$, and consequently, when $v_{ij} = 0$, $w_{ij} = 0$, and $v_{ij} \neq 0$, $w_{ij}^2 < v_{ij}^2$.

From the diagonal entries of $WW^T = I$, we obtain for $1 \le i \le n$,

$$1 = (WW^T)_{ii} = \sum_{j=1}^{n} w_{ij}^2 = \sum_{j=1}^{n} \frac{i(j+2)}{(i+2)j} v_{ij}^2. \qquad (\,*_i\,)$$

From the entries of $VV^T = I$, we obtain for $1 \le i \le n$,

$$1 = (VV^T)_{ii} = \sum_{j=1}^{n} v_{ij}^2, \qquad (**_i)$$

and for each i and t with $1 \le i \ne t \le n$,

$$0 = (VV^T)_{i,t} = \sum_{j=1}^{n} v_{ij} v_{tj}. \qquad (***_{i,t}).$$

Now, we show that the off-diagonal entries of row 1 and column 1 of V are zero. In $(\,*_1\,)$, if at least one of $v_{1j} \neq 0 \quad (j = 2, \dots, n)$, then the right-hand sides of $(\,*_1\,)$ and $(**_1)$ are not equal, which is a contradiction. Therefore, $v_{1j} = 0$, $(j = 2, \dots, n)$ and $v_{11} = \pm 1$. Now, relations $(***_{1,t})$ $(1 < t \le n)$ imply $v_{21} = v_{31} = \cdots = v_{n1} = 0$.

So far, we have:

$$V = \begin{pmatrix} \pm 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & \star & \\ 0 & & & \end{pmatrix}.$$

The case where $i = 2$ uses the above structure of V and proceeds similar to the case where $i = 1$. We arrive at

$$V = \begin{pmatrix} \pm 1 & 0 & 0 & \dots & 0 \\ 0 & \pm 1 & 0 & \dots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & \star & \\ 0 & 0 & & & \end{pmatrix}.$$

The induction hypothesis is that all the off-diagonal entries in V in the first $k-1$ rows and columns are zero, and each diagonal entry is ± 1. Since $v_{k1}, v_{k2}, \dots, v_{k,k-1}$ are zero, in $(*_k)$, if at least one of $v_{kj} \neq 0$, $(j = k+1, \dots, n)$, then the right-hand sides of $(*_k)$ and $(**_k)$ are not equal, which is a contradiction. Therefore, $v_{k,k+1} = v_{k,k+2} = \dots = v_{kn} = 0$, and so $v_{kk} = \pm 1$. Now, relations $(***_{k,t})$ $(k < t \leq n)$ imply $v_{k+1,k} = v_{k+2,k} = \dots = v_{nk} = 0$. So, the off-diagonal entries of row k and column k of V are zero. Thus, by induction, $V = \operatorname{diag}(\pm 1)$, and hence

$$A = D_1^{\frac{-1}{2}} V D_2^{\frac{-1}{2}} = \operatorname{diag}(\pm 1).$$

So A can only be of the form $\operatorname{diag}(\pm 1)$. Therefore, the intersection of $\mathbb{G}(D_1, D_2)$ and $\mathbb{G}(D_3, D_4)$ is finite, and it has 2^n matrices. □

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References

[1] M. Fiedler and F. J. Hall, *G-matrices*, Linear Algebra Appl. **436** (2012), 731–741.
[2] S. M. Motlaghian, A. Armandnejad, and F. J. Hall, *A note on some classes of G-matrices*, Operators Matrices **16** (2022) 251–263.
[3] M. Fiedler, *Notes on Hilbert and Cauchy matrices*, Linear Algebra Appl. **432** (2010), 351–356.
[4] T. J. McAvoy, *Interaction Analysis: Principles and Applications*, vol. 6 of Monograph Series, Instrument Society of America, Research Triangle Park, NC, 1983.
[5] R. A. Horn and C. R. Johnson, *Matrix Analysis*, second edition, Cambridge University Press, New York, NY, 2013.
[6] F. J. Hall, Z. Li, C. T. Parnass, and M. Rozložník, *Sign patterns of J-orthogonal matrices*, Spec. Matrices **5** (2017), 225–241.
[7] F. J. Hall and M. Rozložník, *G-matrices, J-orthogonal matrices, and their sign patterns*, Czechoslovak Math J. **66** (2016), no. 3, 653–670.
[8] S. M. Motlaghian, A. Armandnejad, and F. J. Hall, *Topological properties of J-orthogonal matrices*, Linear and Multilinear Algebra **66** (2018), no. 12, 2524–2533.
[9] S. M. Motlaghian, A. Armandnejad, and F. J. Hall, *Topological properties of J-orthogonal matrices, part II*, Linear and Multilinear Algebra **69** (2021), no. 3, 438–447.