

## Research Article

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# Incidence matrices and line graphs of mixed graphs

<https://doi.org/10.1515/spma-2022-0176>

received May 11, 2022; accepted October 07, 2022

**Abstract:** In the theory of line graphs of undirected graphs, there exists an important theorem linking the incidence matrix of the root graph to the adjacency matrix of its line graph. For directed or mixed graphs, however, there exists no analogous result. The goal of this article is to present aligned definitions of the adjacency matrix, the incidence matrix, and line graph of a mixed graph such that the mentioned theorem is valid for mixed graphs.

**Keywords:** mixed graph, line graph, Hermitian adjacency matrix

**MSC code:** 05C20, 05C22

## 1 Introduction

Line graphs have been an invaluable concept in graph theory for a long time. The line graph  $L(\mathcal{G})$  of an undirected graph  $\mathcal{G} = (V, E)$  has vertex set  $E$ . Two vertices  $e_1$  and  $e_2$  of  $L(\mathcal{G})$  are adjacent if and only if the edges  $e_1$  and  $e_2$  are adjacent in  $\mathcal{G}$ . Hemminger and Beineke once called this “probably the most interesting of all graph transformations” [1]. As a direct consequence of the line graph definition, we have (compared with Lemma 3.6 in [2])

$$B^*B = A(L(\mathcal{G})) + 2I, \quad (1)$$

where  $B$  denotes the incidence matrix of  $\mathcal{G}$ ,  $B^*$  denotes its conjugated transpose,  $A(L(\mathcal{G}))$  denotes the adjacency matrix of  $L(\mathcal{G})$ , and  $I$  is the identity matrix. Recall that the adjacency matrix of an undirected (resp., directed) graph on  $n$  vertices is the 0-1-matrix such that the entry at position  $(i, j)$  is 1 if there is an edge between vertex no.  $i$  and vertex no.  $j$  (resp. from vertex no.  $i$  to vertex no.  $j$ ), and 0 otherwise. The incidence matrix  $B$  of an undirected graph on  $n$  vertices is the 0-1-matrix such that the entry at position  $(i, j)$  is 1 if vertex no.  $i$  and edge no.  $j$  are incident, and 0 otherwise. Thus, there is a natural algebraic link between the matrix  $B$  that captures the incidence relation of  $\mathcal{G}$  and the matrix  $A(L(\mathcal{G}))$  that captures the adjacency relation of  $L(\mathcal{G})$ .

For directed graphs, the situation appears less satisfying. The line graph  $L(\mathcal{D})$  of a directed digraph  $\mathcal{D} = (V, E)$  (also called the line digraph  $L(\mathcal{D})$ ) has vertex set  $E$ . There is an arc from vertex  $e_1$  to  $e_2$  in  $L(\mathcal{D})$  if and only if the head vertex (i.e., terminal vertex) of the arc  $e_1$  is the same as the tail vertex (i.e., initial vertex) of the arc  $e_2$  in  $\mathcal{D}$ . This definition has the peculiar property that it “ignores adjacencies of arcs at source and sinks” [1]. Clearly, the matrix  $A(L(\mathcal{D}))$  is in general not symmetric. Hence, regardless of how we define the

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incidence matrix of  $\mathcal{D}$ , the left-hand side of (1) is a symmetric matrix, whereas the right-hand side is not. To make things worse, when defining incidence matrices of directed graphs, one traditionally only refers to oriented graphs, i.e., double arcs between vertices are forbidden (compared with [2–4]). Moreover, for mixed graphs, no meaningful definition seems to exist at all. A mixed graph is a graph that has been derived from an undirected graph by orienting some of its edges into arcs, while the unmodified edges remain as digons.

In this article, we will focus on mixed graphs instead of directed graphs since we want the number of edges to remain the same whenever we transition to the underlying undirected simple graph. Our goal is to develop consistently aligned definitions of the adjacency matrix, the incidence matrix, and the line graph of a mixed graph such that equation (1) holds; moreover, the well-known equation (compared with [4])

$$BB^* = A(\mathcal{G}) + D \quad (2)$$

that links the incidence matrix of a graph to its adjacency matrix. Here,  $D$  denotes the diagonal matrix with the respective degrees of all vertices of  $\mathcal{G}$ . It will turn out that, once suitable definitions of adjacency and incidence matrix have been chosen, a matching notion of line graph arises in a perfectly natural way.

Our starting point is a very general concept of adjacency matrix called the  $\alpha$ -Hermitian adjacency matrix (compared with Definition 1 in Section 2) with a complex number  $\alpha$  as parameter. Each value of  $\alpha$  defines its own kind of adjacency matrix. There exists an ongoing discussion between researchers whether some choices of  $\alpha$  may be more favorable than others (compared with Section 2 for more details). To have certain basic results valid for undirected graphs carry over to mixed graphs such that a line graph definition for mixed graphs can be obtained in a natural way, we shall propose to pair the  $\alpha$ -Hermitian adjacency matrix with what we call the  $\beta$ -incidence matrix (compared with Definition 2). A remarkable result of the present article is that, essentially, there exists only one meaningful way of choosing  $\alpha$  and  $\beta$  (in the form of two analogous cases), thus contributing to the mentioned discussion by endorsing a particular choice of  $\alpha$ . In view of this discussion, the presentation of this article will especially point out how each new requirement towards a useful line graph definition for mixed graphs will narrow down possible choices of  $\alpha$  and  $\beta$ . Moreover, for the sake of easy accessibility, we strive to make the content of this article as self-contained as possible.

The rest of the article is organized as follows. In Section 2, the desired notion of line graph will be developed. Section 3 will be devoted to answering the question how many mixed orientations of a graph may result in the same mixed line graph. Section 4 will investigate under which conditions an arbitrary mixed orientation of some (undirected) line graph is actually the mixed line graph of some mixed orientation of the root of the undirected line graph. Finally, Section 5 will wrap up the topic and point out connections to the research topic of gain graphs.

## 2 Incidence matrix vs line graph

In this section, we shall resolve the shortcomings mentioned in Section 1. In the following, whenever we consider adjacency or incidence matrices of some graph, we tacitly refer to a fixed (but otherwise arbitrary) vertex (resp. edge) order. The same reference order is also assumed when constructing derived objects (e.g., other matrices) by iterating over the vertex (resp., edge) set. For the sake of convenience, we may index the entries of vertex order-dependent matrices by the vertices themselves, not by column/row numbers. For example, given a matrix  $M$  and two vertices  $x$  and  $y$  such that  $x$  is indexed as  $i$ th vertex and  $y$  is indexed as  $j$ th vertex, we use  $M_{x,y}$  to refer to the cell at position  $(i, j)$ . Even shorter,  $M_x$  refers to a diagonal entry. In the same spirit, we use mixed vertex/edge indexing for incidence matrices.

Returning to equation (1), we see that for a complex matrix  $B$ , the left-hand side of the equation is Hermitian. Thus, it is natural to use a type of adjacency matrix of mixed graphs that is Hermitian. Several authors have observed that there exists a natural generalization of adjacency matrices as follows (compared with [5,6]):

**Definition 1.** Given a mixed graph  $\mathcal{D}$  and a number  $\alpha \in \mathbb{C} \setminus \mathbb{R}$  with  $|\alpha| = 1$ , we define the  $\alpha$ -Hermitian adjacency matrix  $H^\alpha(\mathcal{D})$  of  $\mathcal{D}$  by

$$(H^\alpha(\mathcal{D}))_{u,v} = \begin{cases} 1 & \text{if there is a digon between } u \text{ and } v, \\ \alpha & \text{if there is an arc from } u \text{ to } v, \\ \bar{\alpha} & \text{if there is an arc from } v \text{ to } u, \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

When there is no ambiguity regarding the reference graph  $\mathcal{D}$  we might just write  $H^\alpha$  instead of  $H^\alpha(\mathcal{D})$ . The  $\alpha$ -Hermitian adjacency matrix will be one cornerstone of what follows.

There has been intensive research on the  $\alpha$ -Hermitian adjacency matrix in general. For example, there is a series of interrelated articles [7–9] proving upper and lower bounds for the difference between the rank of the Hermitian adjacency matrix of a given mixed graph and the rank (resp. twice the matching number, twice the independence number) of the underlying undirected graph. The authors characterize all mixed graphs attaining these bounds, so the bounds are sharp. Other notable recent results include [10], where conditions are stated for strictly decreasing the Hermitian spectral radius of a given mixed graph by removing vertices/edges, and [11], where the Hermitian eigenvalues of certain Cayley digraphs are considered.

Some authors have only considered special values of  $\alpha$  in their research. For instance, in [5], the value  $\alpha = i$  is used. Further, in [12], the value  $\alpha = e^{\frac{\pi}{3}i}$  is endorsed and called the “most natural choice,” whereas [13] presents arguments in favor of  $\alpha = e^{\frac{2\pi}{3}i}$ .

Inspired by the structure of the  $\alpha$ -Hermitian adjacency matrix, we define an incidence matrix as follows:

**Definition 2.** Given a mixed graph  $\mathcal{D}$  and a number  $\beta \in \mathbb{C} \setminus \mathbb{R}$  with  $|\beta| = 1$ , we define the  $\beta$ -incidence matrix  $B^\beta(\mathcal{D})$  of  $\mathcal{D}$  by

$$(B^\beta(\mathcal{D}))_{u,e} = \begin{cases} 1 & \text{if } e \text{ is a digon and } u \text{ is incident with it,} \\ \beta & \text{if } e \text{ is an arc and } u \text{ is its head vertex,} \\ \bar{\beta} & \text{if } e \text{ is an arc and } u \text{ is its tail vertex,} \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

Note that, for both  $H^\alpha$  and  $B^\beta$ , the restriction  $\alpha, \beta \notin \mathbb{R}$  is important to faithfully encode the adjacency and incidence relations of mixed graphs.

Up to this point, the choices of  $\alpha$  and  $\beta$  can be made independently. However, requiring equation (2) to be valid restricts our choices drastically.

**Theorem 3.** Let  $\mathcal{D}$  be a mixed graph,  $B = B^\beta(\mathcal{D})$ ,  $A = H^\alpha(\mathcal{D})$ , and  $D = \text{diag}((\deg(v))_{v \in V(\mathcal{D})})$  the degree diagonal matrix of  $\mathcal{D}$ . Then:

- (i)  $BB^*$  can be derived from  $A + D$  by replacing each entry  $\alpha$  (resp.,  $\bar{\alpha}$ ) with  $\bar{\beta}^2$  (resp.,  $\beta^2$ ), and vice versa.
- (ii) Assuming  $\mathcal{D}$  contains at least one arc,  $BB^* = A + D$  if and only if  $\alpha = \bar{\beta}^2$ .

**Proof.** This is a direct consequence of Definition 2. To determine  $(BB^*)_{u,v}$ , one computes the inner product of row  $u$  and the conjugate of row  $v$  of  $B$ . For  $u = v$ , since  $\beta\bar{\beta} = 1$ , we obtain the degree of  $u$ . For  $u \neq v$ ,  $(BB^*)_{u,v} \neq 0$  if and only if  $u$  and  $v$  are adjacent in  $\mathcal{D}$ . In case of a digon  $uv$ , we have  $(BB^*)_{u,v} = (A + D)_{u,v} = 1$ . In case of an arc from  $u$  to  $v$  (resp.,  $v$  to  $u$ ), we have  $(BB^*)_{u,v} = \alpha$  and  $(A + D)_{u,v} = \bar{\beta}^2$  (resp.,  $(BB^*)_{u,v} = \bar{\alpha}$  and  $(A + D)_{u,v} = \beta^2$ ). Thus,  $BB^*$  and  $A + D$  have identical diagonals and the same zero–nonzero pattern. Assuming  $\mathcal{D}$  contains at least one arc, the two matrices are identical if and only if  $\alpha = \bar{\beta}^2$ .  $\square$

In the traditional setting for undirected graphs, the adjacency and incidence matrix share the same set of possible values for their entries. Hence, it is a natural requirement to have identical parameters  $\alpha = \beta$  for both the adjacency and the incidence matrix (note that  $\alpha = \bar{\beta}$  has analogous consequences). That requirement leaves us with just two possible actual choices for  $\alpha$ :

**Corollary 4.** Let  $\mathcal{D}$  be a mixed graph containing at least one arc,  $B = B^\beta(\mathcal{D})$ ,  $A = H^\alpha(\mathcal{D})$ , and  $D$  the degree diagonal matrix of  $\mathcal{D}$ . Then  $BB^* = A + D$  if and only if  $\alpha = \bar{\beta}^2 \in \{\gamma, \gamma^2\}$ , with  $\gamma = e^{\frac{2\pi i}{3}}$ .

Given any mixed graph  $\mathcal{D}$ , let  $\Gamma(\mathcal{D})$  denote its underlying undirected graph, (i.e., derived by turning of arcs of  $\mathcal{D}$  into digons). Note that, conversely, we shall call any mixed graph  $\mathcal{D}$  with  $\mathcal{X} = \Gamma(\mathcal{D})$  a mixed orientation of the undirected graph  $\mathcal{X}$ . Finally, by  $A(\mathcal{G})$ , we denote the traditional adjacency matrix of an undirected graph  $\mathcal{G}$ . We now consider the goal equation (1).

**Theorem 5.** Let  $\mathcal{D}$  be a mixed graph and  $B = B^\beta(\mathcal{D})$ . Then:

- (i)  $B^*B$  has the same zero-nonzero pattern and main diagonal as the matrix  $A(L(\Gamma(\mathcal{D}))) + 2I$ .
- (ii) Assuming  $\mathcal{D}$  contains at least one vertex with both an incoming and an outgoing arc, there exists a mixed orientation  $\mathcal{Y}$  of  $L(\Gamma(\mathcal{D}))$  such that  $B^*B = H_\alpha(\mathcal{Y}) + 2I$  if and only if either  $\beta = \bar{\alpha}$ ,  $\beta^2 = \alpha$  or  $\beta = \alpha$ ,  $\beta^2 = \bar{\alpha}$ .

**Proof.** Again check the consequences of Definition 2. To determine  $(B^*B)_{e_1, e_2}$ , one computes the inner product of the conjugated column  $e_1$  and the column  $e_2$  of  $B$ . For  $e_1 = e_2$ , one obtains either a term  $1 + 1$  or  $\alpha\bar{\alpha} + \bar{\alpha}\alpha$ ; thus,  $(B^*B)_{e_1, e_1} = 2$ . For  $e_1 \neq e_2$ , we have  $(B^*B)_{e_1, e_2} \neq 0$  if and only if  $e_1$  and  $e_2$  are incident in  $\mathcal{D}$ . Moreover, the value of  $(B^*B)_{e_1, e_2}$  arises from a single nonzero term. This matches the situation arising in equation (1) for the undirected graph  $\Gamma(\mathcal{D})$ . Thus,  $B^*B$ ,  $A(L(\Gamma(\mathcal{D})))$ , and  $H_\alpha(\mathcal{Y})$  have the same zero-nonzero pattern, for every mixed orientation  $\mathcal{Y}$  of  $L(\Gamma(\mathcal{D}))$ . It follows from Definition 2 that  $(B^*B)_{e_1, e_2} \in \{\beta, \bar{\beta}, 1 = \beta\bar{\beta} = \bar{\beta}\beta, \beta^2, \bar{\beta}^2\} =: S$ . The terms  $\beta^2$  and  $\bar{\beta}^2$  arise whenever, at the common vertex of  $e_1$  and  $e_2$ , edge  $e_1$  is an outgoing arc and  $e_2$  is an incoming arc – or vice versa.  $B^*B - 2I$  is an  $\alpha$ -Hermitian adjacency matrix if and only if  $S \subseteq \{1, \alpha, \bar{\alpha}\}$ . Since we require  $\alpha, \beta \notin \mathbb{R}$ , there exist only two viable mappings, namely,  $\beta = \bar{\alpha}$ ,  $\beta^2 = \alpha$  and  $\beta = \alpha$ ,  $\beta^2 = \bar{\alpha}$ .  $\square$

Again, if we require identical parameters  $\alpha = \beta$  for both the adjacency and the incidence matrix, then this leaves us with just two choices:

**Corollary 6.** Let  $\mathcal{D}$  be a mixed graph containing at least one vertex with both an incoming and an outgoing arc,  $B = B^\alpha(\mathcal{D})$  and  $A = H^\alpha(\mathcal{D})$ . Then  $B^*B = H_\alpha(\mathcal{Y}) + 2I$  for some mixed orientation  $\mathcal{Y}$  of  $L(\Gamma(\mathcal{D}))$  if and only if  $\alpha = \bar{\beta}^2 \in \{\gamma, \gamma^2\}$ , with  $\gamma = e^{\frac{2\pi i}{3}}$ .

As we can see from Corollaries 4 and 6, there exist only two natural parameter choices to make equations (1) and (2) work if use the matrices from Definitions 1 and 2. Thus, the following line graph construction results as a natural consequence:

in $\mathcal{D}$	in $L_\gamma(\mathcal{D})$

**Figure 1:** Construction of the  $\gamma$ -line graph  $L_\gamma(\mathcal{D})$ .

**Definition 7.** Given a mixed graph  $\mathcal{D}$ , the  $\gamma$ -line graph  $L_\gamma(\mathcal{D})$  is defined as the mixed orientation of  $L(\Gamma(\mathcal{D}))$  arising according to Figure 1.

Figure 2 contains an example illustrating Definition 7. As an alternative to Definition 7, one can define the  $\gamma^2$ -line graph of a mixed graph by reversing all arcs in the right column of Figure 1. In the following, we shall only be concerned with  $\gamma$ -line graphs, but results for  $\gamma^2$ -line graphs can be obtained in a similar manner.

In view of Definition 7, and Corollaries 4 and 6, the following result is now evident:

**Theorem 8.** Let  $\mathcal{D}$  be a mixed graph,  $B = B^\gamma(\mathcal{D})$ ,  $D = \text{diag}((\deg(v))_{v \in V(\mathcal{D})})$ . Then,

- (i)  $B^*B = H_\gamma(L_\gamma(\mathcal{D})) + 2I$ ,
- (ii)  $BB^* = H_\gamma(\mathcal{D}) + D$ .

The following Theorems 9 and 10 act as converses to Theorem 8.

**Theorem 9.** Let  $\mathcal{X}$  be a mixed orientation of some graph  $\mathcal{G}$ . Let  $R$  be a matrix with nonzero entries from the set  $\{1, \gamma, \gamma^2\}$  having the same zero-nonzero pattern as the incidence matrix of  $\mathcal{G}$ . If

$$RR^* = H_\gamma(\mathcal{X}) + \text{diag}((\deg(v))_{v \in V(\mathcal{G})}), \quad (5)$$

then  $R = B^\gamma(\mathcal{X})$ , i.e., it is the  $\gamma$ -incidence matrix of  $\mathcal{X}$ .

**Proof.** Let entry  $(i, j)$  of  $H_\gamma(\mathcal{X})$  be  $\gamma$ . Consider inner product of row  $i$  of  $R$  and column  $j$  of  $R^*$ . Since  $R$  is a  $\gamma$ -incidence matrix, the inner product contains exactly one non-zero term, of the form  $1 \cdot 1$ ,  $\gamma \cdot \gamma$ , or  $\gamma^2 \cdot \gamma^2$ . Only in the latter case, equation (5) is true; hence, entry  $(i, ij)$  of  $R$  must be  $\gamma^2$ , so the edge  $ij$  of  $\mathcal{G}$  is oriented in  $\mathcal{X}$  just as claimed. The remaining cases follow in the same straightforward manner.  $\square$

**Theorem 10.** Let  $\mathcal{Y}$  be a mixed orientation of the mixed line graph  $L_\gamma(\mathcal{G})$  of some graph  $\mathcal{G}$ . If  $R$  is the  $\gamma$ -incidence matrix of some mixed orientation  $\mathcal{X}$  of  $\mathcal{G}$  such that

$$R^*R = H_\gamma(\mathcal{Y}) + 2I, \quad (6)$$

then  $\mathcal{Y} = L_\gamma(\mathcal{X})$ , i.e.,  $\mathcal{X}$  is a root of  $\mathcal{Y}$ .

**Proof.** Consider entry  $(i, j)$  of  $H_\gamma(\mathcal{Y})$  and assume it equals  $\gamma$ . It must match the inner product of row  $i$  of  $R^*$  and column  $j$  of  $R$ . By the structure of  $R$ , a only nonzero term in this product can only arise if edge  $i$  is adjacent to edge  $j$ . The only possible outcomes are  $\gamma^2 \cdot \gamma^2$ ,  $\gamma \cdot 1$ , and  $1 \cdot \gamma$ . They correspond to the cases  $x \xrightarrow{i} y \xrightarrow{j} z$ ,  $x \xleftarrow{i} y \xleftarrow{j} z$ , and  $x \xrightarrow{i} y \xleftarrow{j} z$ , respectively. Hence, the edge  $ij$  was locally created by a  $\gamma$ -line graph operation. The remaining cases follow in the same straightforward manner.  $\square$

Although providing Theorem 8 has been the initial goal for the development of  $\gamma$ -line graphs, the notion of  $\gamma$ -line graphs occurs to be a natural choice to the degree that other well-known results from

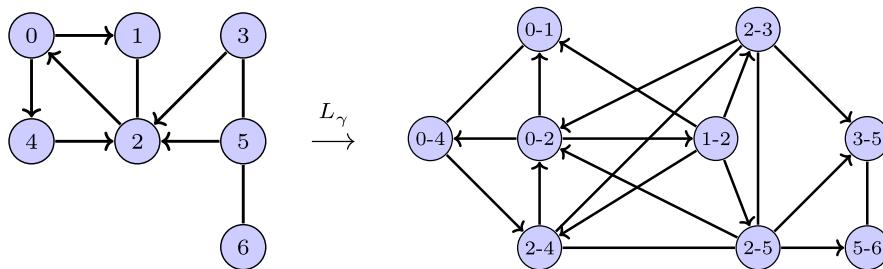


Figure 2:  $\gamma$ -line graph example.

the domain of undirected line graphs carry over easily. The following Theorem 11 on characteristic polynomials  $\chi$  is but one such result:

**Theorem 11.** *Let  $\mathcal{X}$  be a mixed graph. If  $L_y(\mathcal{X})$  is  $k$ -regular with  $n$  vertices and  $m$  edges, then*

$$\chi(H_y(L_y(\mathcal{X})), \lambda) = (\lambda + 2)^{m-n}\chi(H_y(\mathcal{X}), \lambda + 2 - k). \quad (7)$$

**Proof.** Similar to the proof of Theorem 3.8 in [2]. □

### 3 Root orientations

Given a line graph  $H$ , an (undirected) graph  $\mathcal{G}$  satisfying  $L(\mathcal{G}) = H$  is called a root of  $H$ . Likewise, given a  $\gamma$ -line graph  $\mathcal{Y}$ , a mixed graph  $\mathcal{X}$  satisfying  $L_y(\mathcal{X}) = \mathcal{Y}$  is called a root of  $\mathcal{Y}$ . According to Whitney's isomorphism theorem (compared with [14]), any (undirected) line graph usually has exactly one root. The only exception is  $C_3$ , which has two roots, namely, itself and the star  $K_{1,3}$ . Therefore, given a  $\gamma$ -line graph  $\mathcal{Y}$ , we immediately know all graphs  $\mathcal{G}$  such that  $L(\mathcal{G}) = \Gamma(\mathcal{Y})$ , i.e., the underlying graphs of all roots of  $\mathcal{Y}$ . What remains to be decided is which mixed orientations  $\mathcal{X}$  of such an undirected root  $\mathcal{G}$  actually satisfy  $L(\mathcal{X}) = \mathcal{Y}$ .

In view of Theorem 10, equation  $B^*B = H_y(\mathcal{Y}) + 2I$  can be used to derive necessary conditions on the sought mixed orientation  $\mathcal{X}$ , in the sense that the initial choice of orienting any edge of  $\mathcal{G}$  – by way of propagating the conditions along a walk – necessarily determines all other mixed edge orientations. To this end, we present the following construction:

**Construction 12.** Let  $\mathcal{Y}$  be a  $\gamma$ -line graph and let  $\mathcal{G}$  be a root of  $\Gamma(\mathcal{Y})$ . Further, let  $W$  be a walk in  $\mathcal{G}$ , specified by the edges  $e_{i_1}, \dots, e_{i_k}$ , such that  $W$  contains every pair of adjacent edges of  $\mathcal{G}$  as subsequent edges  $e_j$  and  $e_{j+1}$  (for some  $1 \leq j < k$ ). Construct a matrix  $B$  as follows:

- (1) Initialize  $B := 0$ .
- (2) Let  $uv := e_1$ , arbitrarily initialize  $B_{u,uv} \in \{\gamma, \gamma^2, 1\}$  and set  $B_{v,uv} := \overline{B_{u,uv}}$ .
- (3) Set  $i := 1$ .
- (4) While  $i < k$ :
  - (a) Let  $u_1u_2 := e_i$  and let  $u_2u_3 := e_{i+1}$ .
  - (b) Set  $B_{u_2,u_2u_3} := B_{u_2,u_1u_2}(H_y(\mathcal{Y}))_{u_2,u_3}$ .
  - (c) Set  $B_{u_3,u_2u_3} := \overline{B_{u_2,u_2u_3}}$ .
  - (d) Set  $i := i + 1$ .

The core of Construction 12 is step 4(b). Considering the current sub-walk with vertices  $u_1$ ,  $u_2$ , and  $u_3$ , the only way to make equation  $B^*B = H_y(\mathcal{Y}) + 2I$  work at the entry  $(B^*B)_{u_2,u_3}$  is to choose the value of  $B_{u_2,u_2u_3}$  in the proposed way, based on the already computed value  $B_{u_2,u_1u_2}$  and the expected result  $(H_y(\mathcal{Y}))_{u_2,u_3}$ . Altogether, Construction 12 completely determines a candidate  $\gamma$ -incidence matrix  $B$ . Note that any later on encountered sub-walk with the same respective vertices  $u_2$  and  $u_3$  may actually overwrite  $B_{u_2,u_2u_3} \neq 0$  with a different necessary value. Whenever this happens, it is immediately clear that  $B$  cannot be the  $\gamma$ -incidence matrix of a valid mixed root orientation, due to a contradiction of necessary conditions.

The following theorem deals with the question how many valid mixed root orientations may actually exist:

**Theorem 13.** *Any  $\gamma$ -line graph  $\mathcal{Y}$  has at most three mixed root orientations  $\mathcal{X}$  such that  $L_y(\mathcal{X}) = \mathcal{Y}$ .*

**Proof.** We continue our analysis of Construction 12. Assume that step 4b does not alter existing nonzero values for  $B_{u_2,u_2u_3}$  and  $B_{u_2,u_2u_3}$  along the way. Otherwise  $B$  cannot yield a valid mixed root orientation. Next

consider two matrices  $B$  and  $B'$  generated for different walks  $W$  and  $W'$ , respectively. Without loss, we may assume that  $W$  and  $W'$  are closed walks (otherwise extend them such that no necessary conditions are violated). Note that, under the two assumptions mentioned, we may freely rotate or reverse the order of traversal of  $W$  and  $W'$  without changing the resulting matrices  $B$  and  $B'$ . Assume we have  $B_{u,uv} = B'_{u,uv} \neq 0$  for at least one cell  $(u, uv)$ . Without loss, we may assume that  $uv$  is the first edge in both walks. Assume further that  $B_{x,xy} \neq B'_{x,xy}$  for at least one cell  $(x, xy)$ . Concatenating  $W$  and  $W'$  into a single walk  $W''$ , using Construction 12 with  $W''$  will result in alteration of existing nonzero values; hence, neither  $B$  nor  $B'$  can be the  $y$ -incidence matrix of a valid mixed root orientation. As a result, whenever two candidate matrices  $B$  and  $B'$  have the same nonzero value in the same cell, they must be overall identical. Hence, at most three different candidate matrices can be obtained (by varying the orientation of the starting edge  $uv$ ).  $\square$

**Theorem 14.** *Let  $\mathcal{Y}$  be a  $y$ -line graph and let  $\mathcal{G}$  be a root of  $\Gamma(\mathcal{Y})$ . If  $\mathcal{G}$  is bipartite, then there exist exactly three mixed root orientations  $\mathcal{X}$  such that  $L_y(\mathcal{X}) = \mathcal{Y}$ , otherwise there exists exactly one such orientation.*

**Proof.** Let  $\mathcal{X}$  such that  $L_y(\mathcal{X}) = \mathcal{Y}$ , with associated  $y$ -incidence matrix  $B$ . Clearly,  $\mathcal{X}$  and  $B$  can be obtained using any mixed walk  $W$  as required for Construction 12. Choosing a different value  $B_{u,uv}$  in step 2 of Construction 12 means orienting the initial edge  $uv$  of  $\mathcal{X}$  in a different way. This leads to a domino effect in step 4, resulting in a different mixed orientation  $\mathcal{X}'$  and matrix  $B'$ . To start with, we have

$$B'_{u,uv} = yB_{u,uv} \quad (\text{resp.}, y^2). \quad (8)$$

The case in round brackets is analogous, and hence, we focus on the primary case. Consequently,

$$B'_{v,uv} = \overline{B'_{u,uv}} = \overline{yB_{u,uv}} = y^2B_{v,uv}. \quad (9)$$

Let  $w$  be the next vertex along  $W$ . Then

$$B'_{v,vw} = B'_{v,uv}(H_y(\mathcal{Y}))_{vw} = y^2B_{v,uv}(H_y(\mathcal{Y}))_{vw} = y^2B_{v,vw} \quad (10)$$

and

$$B'_{w,vw} = \overline{B'_{v,vw}} = \overline{y^2B_{v,vw}} = yB_{w,vw}. \quad (11)$$

The process continues in this way, alternatingly multiplying by  $y$  and  $y^2$ . Clearly, any odd cycle contained in  $W$  will cause an alteration of some existing nonzero value in  $B$ , hence violate necessary conditions. Assume that the graph  $\mathcal{G}$  is not bipartite. Then it contains an odd cycle, and  $W$  can be chosen to contain an odd cycle, too. Since the starting edge  $uv$  is arbitrary, we see that no changes to  $\mathcal{X}$  are possible without violating necessary conditions. Hence,  $\mathcal{X}$  is the only mixed orientation of  $\mathcal{G}$  such that  $L_y(\mathcal{X}) = \mathcal{Y}$ .

Now assume that  $\mathcal{G}$  is bipartite. Consider equations (9) and (11). As  $W$  contains every pair of adjacent edges of  $\mathcal{G}$  as subsequent edges  $e_j$  and  $e_{j+1}$  it follows that these equations will equally treat all neighbors  $v$  of  $u$  and in turn all neighbors of  $w$  of  $v$ . Thus, comparing the matrix  $B'$  to  $B$ , the entire row  $v$  of  $B$  gets multiplied by the factor  $y^2$  and row  $w$  of  $B$  by the factor  $y$ . In general, it follows that all rows corresponding to vertices having odd distance from  $u$  in  $\mathcal{G}$  need to be multiplied by  $y$ , whereas all rows corresponding to even distance vertices need to be multiplied by  $y^2$ . All such row multiplications can be collected in a diagonal matrix  $\mathcal{D}$ , with diagonal entries  $y, y^2$  only, so that  $DB = B'$ . Thus,  $(B')^*B' = (DB)^*(DB) = B^*D^*DB$ . Hence, if  $B$  satisfies equation (1), then  $B'$  does as well, effectively giving us another two valid mixed root orientations (one for each starting factor in (8)).  $\square$

**Corollary 15.** *Let  $\mathcal{Y}$  be a  $y$ -line graph, let  $\mathcal{G}$  be a root of  $\Gamma(\mathcal{Y})$ , and assume that  $\mathcal{G}$  is bipartite. Given two distinct mixed orientations  $\mathcal{X}$  and  $\mathcal{X}'$  of  $\mathcal{G}$  such that  $L_y(\mathcal{X}) = L_y(\mathcal{X}') = \mathcal{Y}$ , there exists a diagonal matrix  $\mathcal{D}$  with diagonal entries  $y$  and  $y^2$  only such that  $DB = B'$  (where  $B$  and  $B'$  are the  $y$ -incidence matrices of  $\mathcal{X}$  and  $\mathcal{X}'$ , respectively).*

**Proof.** This follows from the proof of Theorem 14. Since  $\mathcal{X} \neq \mathcal{X}'$  there exist exactly three different mixed root orientations. These can necessarily be transformed into one another by way of the mentioned process.  $\square$

## 4 Line graph orientations

Next, we shall shift our perspective a little. Given an arbitrary mixed orientation  $\mathcal{Y}$  of some line graph  $H = L(\mathcal{G})$ , we will now investigate under which conditions we can guarantee that there exists a mixed orientation  $\mathcal{X}$  of  $\mathcal{G}$  such that  $L(\mathcal{X}) = \mathcal{Y}$ . First let us recall the following definition:

**Definition 16.** (compared with [15]) Given an undirected graph  $\mathcal{G}$ , a system  $Q_1, \dots, Q_k$  of cliques of  $\mathcal{G}$  is called a complete system of cliques (or Krausz partition) if the following conditions are satisfied:

- (i) For  $i \neq j$ , we have  $|Q_i \cap Q_j| \leq 1$ .
- (ii) Every vertex of  $\mathcal{G}$  is contained in exactly two of the cliques.
- (iii) If  $|Q_i \cap Q_j| = 1$ , then  $|Q_i| + |Q_j| = \deg(u) + 2$ , where  $\{u\} = Q_i \cap Q_j$ .

Note that Definition 16 permits trivial cliques, i.e., some cliques may be isomorphic to  $K_1$ . Next we state a classic characterization of line graphs:

**Theorem 17.** (compared with [15]) *An undirected graph is a line graph if and only if it admits a complete system of cliques.*

We remark that, in view of the line graph operation, any clique  $Q_i$  mentioned that Definition 16 either arises locally from a maximal star subgraph contained in the root or from a triangle subgraph.

**Definition 18.** Let  $\mathcal{D}$  be a mixed graph and  $H = H^\alpha(\mathcal{D})$ . With respect to  $\mathcal{D}$  and  $H$ , the value  $\mathfrak{h}_\alpha(W)$  of a mixed walk  $W$  with vertices  $v_1, v_2, \dots, v_k$  is defined as follows:

$$\mathfrak{h}_\alpha(W) = (H_{v_1 v_2} H_{v_2 v_3} H_{v_3 v_4} \cdots H_{v_{k-1} v_k}) \in \{\alpha^r\}_{r \in \mathbb{Z}}. \quad (12)$$

**Theorem 19.** *Let  $\mathcal{X}$  be a mixed graph and  $\mathcal{Y} = L_\gamma(\mathcal{X})$ . Further, let  $Q_1, \dots, Q_k$  be a complete system of cliques of  $\Gamma(\mathcal{Y})$  and let  $C$  be a cycle in some nontrivial  $Q_i$ , where  $Q_i$  arises from a star subgraph in the root  $\Gamma(\mathcal{X})$  of  $\Gamma(\mathcal{Y})$ . Then the mixed cycle  $\vec{C}$  in  $\mathcal{Y}$  that corresponds to  $C$  has weight  $\mathfrak{h}_\alpha(\vec{C}) = 1$  in  $\mathcal{Y}$ .*

**Proof.** Under the assumptions of the theorem, the clique  $Q_i$  of  $\Gamma(\mathcal{Y})$  corresponds to the star subgraph induced by the edges incident some vertex  $r$  in  $\Gamma(\mathcal{X})$ . Suppose that  $C$  is traversed by  $rs_1, rs_2, \dots, rs_m, rs_1$  (where  $s_1, \dots, s_m \in V(\mathcal{X})$ ). Let  $\vec{C}$  be the mixed cycle in  $\mathcal{Y}$  that corresponds to  $C$ . By using any traversal direction, we compute the weight of  $\vec{C}$  as follows:

$$\begin{aligned} \mathfrak{h}_\gamma(\vec{C}) &= H_\gamma(\mathcal{Y})_{rs_1, rs_2} H_\gamma(\mathcal{Y})_{rs_2, rs_3} \cdots H_\gamma(\mathcal{Y})_{rs_m, rs_1} \\ &= ([B]_{rs_1}^* [B]_{rs_2}) ([B]_{rs_2}^* [B]_{rs_3}) \cdots ([B]_{rs_m}^* [B]_{rs_1}) \\ &= \overline{B_{r, rs_1}} B_{r, rs_2} \overline{B_{r, rs_2}} B_{r, rs_3} \cdots \overline{B_{r, rs_m}} B_{r, rs_1} = 1, \end{aligned} \quad (13)$$

where  $[B]_e$  denotes the column of the incidence matrix  $B$  of  $\mathcal{X}$  that corresponds to the edge  $e$ .  $\square$

**Example 20.** When checking whether some mixed orientations of a given line graph  $L(\mathcal{G})$  is actually the mixed line graph of some mixed orientation of the root  $\mathcal{G}$ , the necessary condition stated in Theorem 19 can be used as a first check. For example, consider the mixed graph shown in Figure 3(b). In its undirected counterpart  $L(\mathcal{G})$ , the triangle with vertices 0-1, 0-2, and 0-4 has arisen from the star around the vertex 0 when applying the line graph operation to the root shown in Figure 3(a). Hence, traversing that triangle as a mixed cycle  $\vec{C}$  of the considered mixed orientation should yield  $\mathfrak{h}_\gamma(\vec{C}) = 1$ , by Theorem 19. One readily verifies that this is not the case, and hence, this mixed graph cannot be a mixed line graph of any mixed orientation of the given root.

As can be seen from Theorem 19, the existence of feasible root orientations can be linked to the algebraic properties of the cycles in the candidate  $y$ -line graph. The goal of the remainder of this section is to prove that, for a line graph  $L(\mathcal{T})$  of a tree  $\mathcal{T}$ , it suffices to have a mixed  $y$ -monograph orientation  $\mathcal{Y}$  in order to guarantee the existence of a mixed root orientation  $\mathcal{X}$  of  $\mathcal{T}$  such that  $L_y(\mathcal{X}) = \mathcal{Y}$ . This will be shown in Theorem 31. In preparation of this theorem, we first require some auxiliary results on monographs.

**Definition 21.** (Compared with [13]) Let  $\mathcal{D}$  be a mixed graph.

- (i)  $\mathcal{D}$  is called an  $\alpha$ -monograph if  $h_\alpha(\vec{C}) = 1$  for all its cycles  $C$ , where  $\vec{C}$  denotes an arbitrary closed traversal walk on  $C$ .
- (ii) The  $\alpha$ -store  $S^\alpha(u)$  of  $u \in V(\mathcal{D})$  is defined as follows:

$$S^\alpha(u) = \{h_\alpha(W) : W \text{ is a closed walk in } \mathcal{D} \text{ from } u \text{ to } u\}. \quad (14)$$

Monographs capture the idea of transporting values along the edges of a mixed graph. One starts by assigning a seed value to some initial vertex. Spreading along a forward arc, the value at its terminal vertex will be  $\alpha$  times the value at its initial vertex. For a backward arc, the factor is  $\bar{\alpha}$ . For a digon, the factor is 1. The required factors are easily looked up in the  $\alpha$ -Hermitian adjacency matrix. Trivially, trees are  $\alpha$ -monographs.

Clearly,  $1 \in S^\alpha(u)$  so that  $|S^\alpha(u)| \geq 1$ . It is not hard to see that the store content is independent of the reference vertex  $u$ , i.e.,  $S^\alpha(u) = S^\alpha(v)$  for any  $u, v \in V(\mathcal{D})$ , as long as  $\mathcal{D}$  is connected. Thus,  $\alpha$ -monographs can be characterized as follows:

**Theorem 22.** (Compared with [13]) Let  $\mathcal{D}$  be a connected mixed graph. Then the following statements are equivalent:

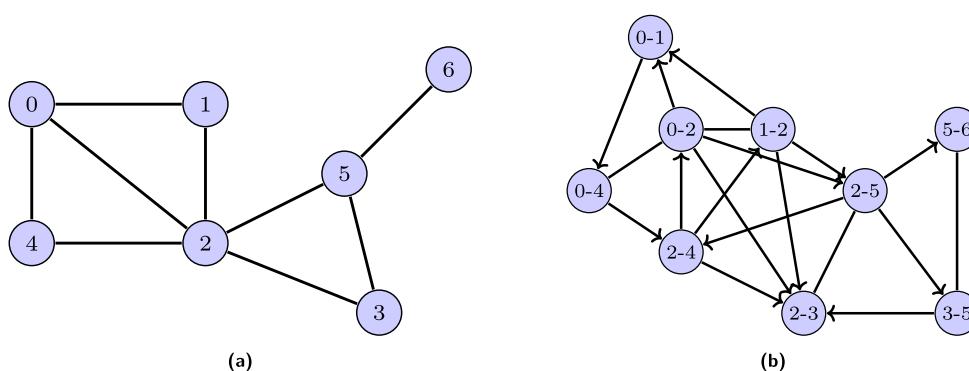
- (i)  $\mathcal{D}$  is an  $\alpha$ -monograph.
- (ii)  $h_\alpha(W') = h_\alpha(W'')$  for every pair  $W', W''$  of mixed walks sharing the same start and end vertices.
- (iii)  $|S^\alpha(u)| = 1$  for every  $u \in V(\mathcal{D})$ .

In view of Theorem 22, we may define a store function  $S^\alpha : V \rightarrow \{\alpha^j : j \in \mathbb{Z}\}$  that assigns to each vertex  $u$  of  $\mathcal{D}$  the unique element in  $S^\alpha(u)$ . This function is unique up to a normative factor  $\alpha^k$ .

**Theorem 23.** Let  $\mathcal{X}$  be an  $\alpha$ -monograph and  $u \in V(\mathcal{X})$ . Given a store function  $S^\alpha$ , define the matrix  $\Delta = \text{diag}((S^\alpha(v))_{v \in V(\mathcal{X})})$ . Then

$$\Delta H_\alpha(\mathcal{X})\Delta^* = A(\Gamma(\mathcal{X})). \quad (15)$$

**Proof.** Since  $\Delta$  is an invertible diagonal matrix,  $\Delta H_\alpha(\mathcal{X})\Delta^*$  and  $H_\alpha(\mathcal{X})$  have the same zero-nonzero pattern. Therefore, it suffices to prove that  $\Delta H_\alpha(\mathcal{X})\Delta^*$  is a 0-1-matrix. Let  $xy$  be any edge of  $\mathcal{X}$ . Since  $\mathcal{X}$  is an



**Figure 3:** Mixed orientation of a line graph that is not a mixed line graph. (a) Root graph  $\mathcal{G}$  and (b) mixed orientation of  $L(\mathcal{G})$ .

$\alpha$ -monograph, we have  $\Delta_x \Delta_y = (H_\alpha(\mathcal{X}))_{y,x} = \overline{(H_\alpha(\mathcal{X}))_{x,y}}$  by the definition of the store. So we obtain  $(\Delta H_\alpha(\mathcal{X}) \Delta^*)_{x,y} = \Delta_x (H_\alpha(\mathcal{X}))_{x,y} \overline{\Delta_y} = 1$ .  $\square$

**Remark 24.** Application of Theorem 23 to  $\gamma$ -monographs (analogously, to  $\gamma^2$ -monographs) yields an edge switching procedure that will turn any mixed graph  $\mathcal{X}$  into its unoriented counterpart. Recall that  $\gamma^2 = \bar{\gamma}$ . Performing the multiplication  $\Delta H_\gamma(\mathcal{X}) \Delta^*$ , for every vertex  $x$ , we effectively multiply its associated row in  $H_\gamma(\mathcal{X})$  by  $S^\alpha(x)$  and its associated column by  $\overline{S^\alpha(x)}$ . Thus, edges adjacent to  $x$  are subjected to the switching pattern depicted in Figure 4. Subsequently applying this pattern to all vertices of  $\mathcal{X}$  (in any order) yields  $\Gamma(\mathcal{X})$ .

**Theorem 25.** Let  $\mathcal{G}$  be a graph. Further, let  $\alpha \in \mathbb{C} \setminus \mathbb{R}$  with  $|\alpha| = 1$ . Given any matrix  $\Delta = \text{diag}((\Delta_v)_{v \in V(\mathcal{X})})$  such that  $\Delta_v = \alpha^k \Delta_u$  for some  $k \in \{-1, 0, 1\}$  is satisfied for all  $uv \in E(\mathcal{G})$ , we have:

- (i)  $\Delta^* A(\mathcal{G}) \Delta$  is the  $\alpha$ -adjacency matrix of a mixed graph  $\mathcal{X}$ ,
- (ii)  $\Gamma(\mathcal{X}) = \mathcal{G}$ , and
- (iii)  $\mathcal{X}$  is an  $\alpha$ -monograph.

**Proof.** Clearly,  $H := \Delta^* A(\mathcal{G}) \Delta$  is a Hermitian matrix with the same zero-nonzero-pattern as  $A(\mathcal{G})$ . By the condition imposed on  $\Delta$ , it follows that all nonzero entries in  $H$  must be from the set  $\{1, \alpha, \bar{\alpha}\}$ . Hence, (i) and (ii) have been proven. With respect to (iii), let  $C = v_1 v_2 v_3 \dots v_k v_1$  be any cycle in  $\mathcal{G}$ . Then, the weight of  $C$  is given as follows:

$$h_\alpha(C) = H_{v_1, v_2} H_{v_2, v_3} \dots H_{v_k, v_1} = (\overline{\Delta_{v_1} \Delta_{v_2}}) (\overline{\Delta_{v_2} \Delta_{v_3}}) \dots (\overline{\Delta_{v_k} \Delta_{v_1}}) = 1. \quad \square \quad (16)$$

**Corollary 26.** Every graph has a nontrivial mixed orientation that yields an  $\alpha$ -monograph.

In the case of  $\gamma$ -monographs, the matrices  $\Delta$  encountered in Theorems 23 and 25 play an important role in describing the relation between a mixed root graph  $\mathcal{G}$  and its mixed line graph  $L_\gamma(\mathcal{G})$ .

**Definition 27.** Let  $\mathcal{X}$  be a  $\gamma$ -monograph. Any diagonal matrix  $\Delta$  with entries from the set  $\{1, \gamma, \gamma^2\}$  satisfying  $\Delta H_\gamma(\mathcal{X}) \Delta^* = A(\Gamma(\mathcal{X}))$  shall be called an orientation matrix of  $\mathcal{X}$ .

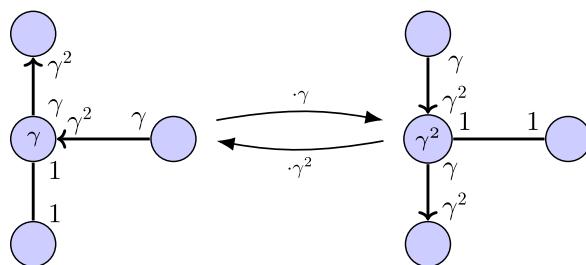
**Theorem 28.** Let  $\mathcal{X}$  be a  $\gamma$ -monograph and  $B$  its  $\gamma$ -incidence matrix. Further, let  $\Delta$  be an orientation matrix of  $\mathcal{X}$ . Define the matrix  $\delta = \text{diag}((\delta_{uv})_{uv \in E(\Gamma(\mathcal{X}))})$  by  $\delta_{uv} = \Delta_u B_{u,v} \Delta_v$ . Then,

- (i)  $\Delta B \delta^*$  is the incidence matrix of  $\Gamma(\mathcal{X})$ .
- (ii)  $\delta$  is an orientation matrix of  $L_\gamma(\mathcal{X})$ .

**Proof.** Since  $\Delta$  and  $\delta$  are invertible diagonal matrices,  $B' := \Delta B \delta^*$  and  $B$  have the same zero-nonzero pattern. Therefore, regarding claim (i), it suffices to prove that  $B'$  is a 0-1-matrix. Since for any  $uv \in E(\mathcal{X})$  we have

$$(\Delta B \delta^*)_{u,v} = \Delta_u B_{u,v} \delta^*_{v,u} = \Delta_u B_{u,v} \overline{\Delta_v} \overline{B_{u,v}} = 1, \quad (17)$$

it follows that this is indeed the case.



**Figure 4:** Pattern for switching an  $\alpha$ -monograph into its undirected counterpart.

To prove that  $\delta$  is an orientation matrix of  $L_y(\mathcal{X})$ , we need to assert that  $\delta H_y(L_y(\mathcal{X}))\delta^* = A(\Gamma(L_y(\mathcal{X})))$ . To this end, we verify

$$A(\Gamma(L_y(\mathcal{X}))) = A(L(\Gamma(\mathcal{X}))) = (\Delta B\delta^*)^*(\Delta B\delta^*) - 2I = \delta B^*\Delta^*\Delta B\delta^* - 2I = \delta(B^*B - 2I)\delta^* = \delta H_y(L_y(\mathcal{X}))\delta^*. \quad (18)$$

□

**Theorem 29.** Let  $\mathcal{X}$  be a mixed graph and  $\mathcal{Y} = L_y(\mathcal{X})$ . Further, let  $\vec{C}$  be a mixed cycle in  $\mathcal{X}$  and  $\vec{C}'$  its corresponding cycle in  $\mathcal{Y}$ . Then  $h_y(\vec{C}, \mathcal{X}) = h_y(\vec{C}', \mathcal{Y})$ , i.e., the weight of  $\vec{C}$  in  $\mathcal{X}$  and the weight of  $\vec{C}'$  in  $\mathcal{Y}$  are equal (using analogous traversal direction).

**Proof.** Assume that  $\vec{C}$  is traversed as  $u_1, u_2, \dots, u_k, u_1$  (with  $u_i \in V(\mathcal{X})$ ). Let  $[B]_{u_i}$  denote the row of the incidence matrix  $B$  of  $\mathcal{X}$  that corresponds to the vertex  $u_i$ . Then,

$$\begin{aligned} h_y(\vec{C}, \mathcal{X}) &= H_y(\mathcal{X})_{u_1, u_2} H_y(\mathcal{X})_{u_2, u_3} \cdots H_y(\mathcal{X})_{u_k, u_1} \\ &= ([B]_{u_1}([B]_{u_2})^*) ([B]_{u_2}([B]_{u_3})^*) \cdots ([B]_{u_k}([B]_{u_1})^*) \\ &= ((B)_{u_1, u_1 u_2} \overline{(B)_{u_2, u_1 u_2}}) ((B)_{u_2, u_2 u_3} \overline{(B)_{u_3, u_2 u_3}}) \cdots ((B)_{u_k, u_k u_1} \overline{(B)_{u_1, u_k u_1}}). \end{aligned} \quad (19)$$

Let  $[B^*]_{u_i u_j}$  denote the row of the matrix  $B^*$  that corresponds to the edge  $u_i u_j$ . Observe that

$$[B^*]_{u_i u_j} ([B^*]_{u_i u_k})^* = (B^* B)_{u_i u_j, u_j u_k} = (H_y(L_y(\mathcal{X})))_{u_i u_j, u_j u_k} \quad (20)$$

for  $u_i \neq u_j \neq u_k$ . Moving the first term in the final product of (19) to the back and making use of (20), we obtain

$$\begin{aligned} h_y(\vec{C}, \mathcal{X}) &= (\overline{(B)_{u_2, u_1 u_2}}(B)_{u_2, u_2 u_3}) (\overline{(B)_{u_3, u_2 u_3}}(B)_{u_3, u_3 u_4}) \cdots (\overline{(B)_{u_1, u_k u_1}}(B)_{u_1, u_1 u_2}) \\ &= ([B^*]_{u_1 u_2} ([B^*]_{u_2 u_3})^*) ([B^*]_{u_2 u_3} ([B^*]_{u_3 u_4})^*) \cdots ([B^*]_{u_k u_1} ([B^*]_{u_1 u_2})^*) \\ &= (H_y(L_y(\mathcal{X})))_{u_1 u_2, u_2 u_3} (H_y(L_y(\mathcal{X})))_{u_2 u_3, u_3 u_4} \cdots (H_y(L_y(\mathcal{X})))_{u_k u_1, u_1 u_2} \\ &= h_y(\vec{C}', \mathcal{Y}). \end{aligned} \quad (21)$$

□

**Corollary 30.** A mixed graph  $\mathcal{X}$  is an  $\alpha$ -monograph if and only if  $L_y(\mathcal{X})$  is an  $\alpha$ -monograph.

**Proof.** Every cycle in  $\mathcal{X}$  has a corresponding cycle in  $L_y(\mathcal{X})$ ; hence, if  $L_y(\mathcal{X})$  is an  $\alpha$ -monograph, then the same is also true for  $\mathcal{X}$ , by Theorem 29. Conversely, let  $\mathcal{X}$  be an  $\alpha$ -monograph and consider a mixed cycle  $\vec{C}$  in  $L_y(\mathcal{X})$ . If  $\vec{C}$  lies in a single clique  $Q_i$  of the Krausz partition of  $\Gamma(L_y(\mathcal{X}))$ , then either it arises from a star in the root graph  $\Gamma(\mathcal{X})$  (so that Theorem 19 can be applied) or it arises from a triangle (so that Theorem 29 can be applied). In any case, one obtains  $h_y(\vec{C}) = 1$ . Now assume that  $\vec{C}$  spans more than one clique of the Krausz partition. Note that, in view of Theorem 19, within any single clique  $Q_i$  arising from a star subgraph, the values  $h_y$  of any two mixed walks with the same start and endpoint are identical. Hence, we may alter  $\vec{C}$  into a cycle  $\vec{C}'$  of the same value such that it does not contain more than two vertices from the same clique in a row. As a consequence,  $\vec{C}'$  corresponds to a cycle in the graph  $\mathcal{X}$ , hence the result follows by Theorem 29. □

Given some mixed orientation  $\mathcal{Y}$  of the line graph of a tree, a necessary condition for the existence of a mixed root is that all cycles in  $\mathcal{Y}$  must satisfy the condition stated in Theorem 19. It turns out that this condition is actually sufficient:

**Theorem 31.** Let  $\mathcal{Y}$  be a mixed  $y$ -monograph such that  $\Gamma(\mathcal{Y}) = L(\mathcal{T})$  for some tree  $\mathcal{T}$ . Then  $\mathcal{Y} = L_y(\mathcal{X})$  for some mixed graph  $\mathcal{X}$  with  $\Gamma(\mathcal{X}) = \mathcal{T}$ .

**Proof.** Our goal is to construct a mixed orientation  $\mathcal{X}$  of  $\mathcal{T}$  such that  $\mathcal{Y} = L_y(\mathcal{X})$ . By Theorem 23, since  $\mathcal{Y}$  is a  $y$ -monograph, it has an orientation matrix  $\delta$ , i.e.,

$$A(\Gamma(\mathcal{Y})) = \delta H_{\gamma}(\mathcal{Y})\delta^*. \quad (22)$$

Below we outline a recursive procedure that defines a diagonal matrix  $\Delta$  and the  $\gamma$ -incidence matrix  $R$  of a mixed orientation of  $\mathcal{T}$  satisfying the claim of the theorem. To this end, let  $B$  be the incidence matrix of  $\mathcal{T}$ . Fix some vertex  $u \in V(\mathcal{T})$  and a seed value  $\Delta_u \in \{1, \gamma, \gamma^2\}$ . Now, consider any path  $P$  in  $\mathcal{T}$  with vertices  $u = v_1, v_2, \dots, v_k = v$  from  $u$  to  $v \in V(\mathcal{T})$  and use the defining equation:

$$\delta_{v_i v_{i+1}} = \Delta_{v_i} R_{v_i, v_i v_{i+1}} \quad (23)$$

to determine  $R_{v_i, v_i v_2}$ :

$$R_{v_1, v_1 v_2} = \frac{\delta_{v_1 v_2}}{\Delta_{v_1}}. \quad (24)$$

Requiring  $R$  to be a  $\gamma$ -incidence matrix, we set  $R_{v_2, v_1 v_2} = \overline{R_{v_1, v_1 v_2}}$ . By using equation (23) once again, we deduce

$$\delta_{v_1 v_2} = \delta_{v_2 v_1} = \Delta_{v_2} R_{v_2, v_1 v_2}, \quad (25)$$

and therefore, we can compute  $\Delta_{v_2}$  from previously known values as follows:

$$\Delta_{v_2} = (\delta_{v_1 v_2})^2 \overline{\Delta_{v_1}}. \quad (26)$$

Continuing along the vertices of  $P$ , we can repeatedly apply the same pattern of computations as in equations (24)–(26) to obtain a recursive formula:

$$\Delta_{v_i} = (\delta_{v_{i-1} v_i})^2 \overline{\Delta_{v_{i-1}}}, \quad (27)$$

and finally,

$$\Delta_{v_i} = (\delta_{v_{i-1} v_i})^2 (\overline{\delta_{v_{i-2} v_{i-1}}})^2 \cdots \begin{cases} \Delta_{v_1} & \text{if } i \text{ is odd} \\ \overline{\Delta_{v_1}} & \text{if } i \text{ is even.} \end{cases} \quad (28)$$

Since every vertex  $v$  of  $\mathcal{T}$  can be reached by a unique path  $P$  from  $u$  to  $v$  in  $\mathcal{T}$ , the matrix  $\Delta$  is thus complete and well-defined. Minding the seed value, it is clear from (28) that  $\Delta_{v_i} \in \{1, \gamma, \gamma^2\}$ . Hence, by virtue of Theorem 25, we have

$$\Delta^* A(\mathcal{T}) \Delta = H_{\gamma}(\mathcal{X}) \quad (29)$$

for some mixed orientation  $\mathcal{X}$  of  $\mathcal{T}$ .

With respect to the partially defined matrix  $R$ , note that for every edge  $v_1 v_2$  of  $\mathcal{T}$ , we have defined two entries  $R_{v_1, v_1 v_2} = \overline{R_{v_2, v_1 v_2}} \in \{1, \gamma, \gamma^2\}$  in the column indexed by  $v_1 v_2$ . Augment  $R_{x,y} = 0$  for all entries of  $R$  not defined so far. Then, by construction,  $R = \Delta^* B \delta$  is the incidence matrix of some mixed orientation of  $\mathcal{T}$ . Keeping in mind equation (29), we compute

$$RR^* = \Delta^* B \delta \delta^* B^* \Delta = \Delta^* A(\mathcal{T}) + \text{diag}((\deg(v))_{v \in V(\mathcal{T})}) \Delta = H_{\gamma}(\mathcal{X}) + \text{diag}((\deg(v))_{v \in V(\mathcal{T})}), \quad (30)$$

and thus  $R$  is actually the  $\gamma$ -incidence matrix of  $\mathcal{X}$ , as per Theorem 9. Moreover, by using (22), we conclude from

$$R^* R = (\delta^* B^* \Delta)(\Delta^* B \delta) = \delta^* (B^* B) \delta = \delta^* A(\Gamma(\mathcal{Y})) \delta + 2I = \delta^* \delta H_{\gamma}(\mathcal{Y}) \delta^* \delta + 2I = H_{\gamma}(\mathcal{Y}) + 2I \quad (31)$$

that indeed  $\mathcal{Y} = L_{\gamma}(\mathcal{X})$ , as per Theorem 10.  $\square$

**Corollary 32.** *Let  $\mathcal{Y}$  be a mixed graph such that  $\Gamma(\mathcal{Y}) = L(\mathcal{T})$  for some tree  $\mathcal{T}$ . Then  $\mathcal{T}$  has exactly three different mixed orientations  $\mathcal{X}$  such that  $\mathcal{Y} = L_{\gamma}(\mathcal{X})$ .*

It is possible to generalize Theorem 31 as follows:

**Corollary 33.** *Given any  $\gamma$ -monograph  $\mathcal{Y}'$  such that  $\Gamma(\mathcal{Y}') = L(\mathcal{G})$  for some undirected root graph  $\mathcal{G}$ , consider a spanning tree  $\mathcal{T}$  of  $\mathcal{G}$  and construct the matrices  $\Delta$ ,  $\delta$ , and  $R$  as outlined in the proof of Theorem 31. However, instead of augmenting  $R$  with zeroes, redefine it as  $R := \Delta^* B' \delta$ , where  $B'$  is the incidence matrix of  $\mathcal{G}$  (not  $\mathcal{T}$ ). If for every edge  $xy \in E(\mathcal{G}) \setminus E(\mathcal{T})$  we have  $\Delta_x \Delta_y = (\delta_{xy})^2$ , then the mixed graph  $\mathcal{Y}'$  with  $H_y(\mathcal{Y}') = \Delta^* A(\mathcal{G}) \Delta$  is a root of  $\mathcal{Y}'$ .*

**Proof.** First one needs to verify that  $R$  is a proper  $\gamma$ -incidence matrix. As  $R$  has the same zero-nonzero pattern as  $B$ , it suffices to check that  $R_{x,xy} = \overline{R_{y,xy}}$  holds for each of the extra edges  $xy \in E(\mathcal{G}) \setminus E(\mathcal{T})$ . Note that the redefinition of  $R$  merely augments any entries not yet specified. It follows from  $\Delta_x \Delta_y = (\delta_{xy})^2$  that  $R_{x,xy} = \Delta_x B_{x,xy} \overline{\delta_{xy}} = (\delta_{xy}^2 \overline{\Delta_y}) \overline{\delta_{xy}} = \delta_{xy} \overline{\Delta_y} = \overline{R_{y,xy}}$ . We see that equations (27) and (28) are now satisfied for arbitrary paths in  $\mathcal{G}$ . Consequently, the remainder of the proof of Theorem 31 can now be lifted to the entire graph  $\mathcal{G}$ .  $\square$

Note that, whenever the conditions stated in Theorem 33 are met, it permits the construction of three valid mixed roots, and hence, Theorem 14 applies.

## 5 Conclusion and outlook

Returning to the aforementioned discussion whether some choices of  $\alpha$  for the  $\alpha$ -Hermitian adjacency matrix may prove better than others, it has turned out that the combination of  $\gamma$ -Hermitian adjacency matrix and  $\gamma$ -incidence matrix (resp.,  $\gamma^2$ ) gives rise to a natural notion of line graphs of mixed graphs. The theory that follows from studying the properties of mixed line graphs shows both interesting parallels to – and digressions from – the theory of undirected line graphs. Most notably, if  $\mathcal{Y} = L_y(\mathcal{X})$  for mixed graphs  $\mathcal{X}$  and  $\mathcal{Y}$ , then  $\Gamma(\mathcal{Y}) = L(\mathcal{G})$  (with  $\mathcal{G} = \Gamma(\mathcal{X})$ ), but besides  $\mathcal{X}$ , there may be further mixed orientations  $\mathcal{X}'$  of  $\mathcal{G}$  with the property that  $\mathcal{Y} = L_y(\mathcal{X}')$ . Keeping in mind a broad audience that might be interested in using  $\alpha$ -Hermitian adjacency matrices in connection with mixed line graphs, the results in this article have been purposely presented in a way to make the article self-contained. However, for the sake of deriving further, and even deeper results on mixed line graphs one can turn to the theory of gain graphs. These graphs constitute a generalization of mixed graphs, along with a generalization of the  $\alpha$ -Hermitian adjacency matrix (compared with [16]). A  $\mathbb{T}_n$ -gain graph can be defined as follows. Let  $\mathbb{T}_n$  denote the multiplicative group of all  $n$ th roots of unity. Then, a  $\mathbb{T}_n$ -gain graph is a triple  $\Phi = (\Gamma, \mathbb{T}_n, \phi)$  consisting of an underlying graph,  $\Gamma = (V, E)$ , the group  $\mathbb{T}_n$  and the gain function  $\phi : E \rightarrow \mathbb{T}_n$  such that for every  $uv \in E$ ,  $\phi(uv) = \phi(vu)^{-1}$ . The  $\phi$ -Hermitian adjacency matrix of  $\Phi$  is defined by  $H_\phi = [h_{uv}] \in C^{n \times n}$ , where

$$h_{uv} = \begin{cases} \phi(uv) & \text{if } u \text{ is adjacent to } v, \\ 0 & \text{otherwise.} \end{cases} \quad (32)$$

Obviously, the  $\phi$ -Hermitian adjacency matrix of a mixed graph is a generalization of the  $\alpha$ -Hermitian adjacency matrix (compared with [17] and [18] for recent results). Furthermore, the  $\gamma$ -Hermitian adjacency matrix is exactly the  $\phi$ -Hermitian adjacency matrix of  $\mathbb{T}_3$ -gain graph. Like in this article, the authors of [19–21] introduced different matrices (called incidence  $\mathbb{T}_n$ -phase) that can serve as incidence matrices to  $\mathbb{T}_n$ -gain graphs (see also [5]). This paved the way to define the gain line graph, gain Laplacian, and gain signless Laplacian adjacency matrix of  $\mathbb{T}_n$ -gain graphs. In fact, different  $\mathbb{T}_n$ -phases can produce the same gain line graph.

**Acknowledgements:** The authors are indebted to the two anonymous referees for their valuable feedback.

**Conflict of interest:** The authors state no conflict of interest.

**Data availability statement:** Data sharing is not applicable to this article as no data sets were generated or analyzed during this study.

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