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A class of symmetric and non-symmetric band matrices via binomial coefficients

Abstract: Symmetric matrix classes of bandwidth $2r + 1$ was studied in 1972 through binomial coefficients. In this paper, non-symmetric matrix classes with the binomial coefficients are considered where $r + s + 1$ is the bandwidth, r is the lower bandwidth and s is the upper bandwidth. Main results for inverse, determinants and norm-infinity of inverse are presented. The binomial coefficients are used for the derivation of results.

1 Introduction

Toeplitz matrices are square matrices having constant entries along their diagonals. They occur in many research fields. These kinds of matrices can either be finite or infinite. Applications are found in Variational structure, Matrix theory, Theory of differential equations, Time series analysis, Signal and Image processing, Markov Chains, Queueing theory, etc. In Numerical Analysis, Toeplitz matrices are applied in Finite difference schemes, Finite element methods, Spline methods, Boundary value methods, Block unification methods, Partial differential equation and many more. Symmetric matrices have been studied by several researchers (see Refs. [2, 7, 16, 20, 25]).

Arikan and Kilic [1] defined the Toeplitz matrices as special types of square matrices in which the entries on each descending diagonal from left to right are constant. Herein, a Toeplitz matrix $A(r + s + 1, n)$ represents an $n \times n$ band matrix of bandwidth $r + s + 1$ where r is the lower bandwidth and s is the upper bandwidth.

In 1972, the authors [16], derived the formulae

$$\det(A(2r + 1, n)) = (-1)^{n+r-1} \prod_{k=1}^{n} \binom{k + 2r - 1}{r} \binom{k + r - 1}{r}^{-1}, \tag{1}$$

$$\left\{ A(r + s + 1, n)^{-1} \right\}_{i,j} = (-1)^r \binom{i + r - 1}{r} \binom{j + r - 1}{r} \sum_{k=i}^{n} \frac{\binom{k+r-1-i}{r-1}\binom{k+r-1-j}{r-1}}{\binom{k+r-1}{r}\binom{k+2r-1}{r}} \tag{2}$$

and

$$\|A(2r + 1, n)^{-1}\|_\infty = \begin{cases} \frac{\prod_{k=1}^{r}(n+2k-1)^2}{2^{2r}(2r)!} & \text{for odd n,} \\ \frac{\prod_{k=1}^{r}(n+2k-2)^2}{2^{2r}(2r)!} \left(\frac{n+2r}{n} \right) & \text{for even n.} \end{cases} \tag{3}$$

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Also, the authors gave the constructive approach for the inverse of the symmetric matrix $A(2r + 1, n)$ as

$$\left\{ A(2\,r+1, n)^{-1} \right\}_{i,j} = -\frac{(n+1-i)(n+2-i)\cdots(n+r-i)}{(r-1)!(2\,r-1)!}$$
$$\times \sum_{k=0}^{r-1} \left[(-1)^k \binom{r-1}{k} \frac{\prod_{r=0}^{r-1}(i+r)}{(i+k)} \frac{(j+k+r-1)!}{(j+k-r)!} \frac{(n+k)!}{(n+k+r)!} \right], \quad i \ge j, \quad r \ge 1. \tag{4}$$

Recently, Arikan and Kilic [1] studied the matrix

$$a_{ij} = (-1)^{r+i-j} \binom{r+s}{r+i-j},$$

where s is the upper bandwidth and r is the lower bandwidth. The following results were presented in Gaussian q–binomial, generalized Fibonomial and binomial forms.

$$\det C_n = i^{nr(r-1)} \prod_{m=0}^{n-1} \left\{ \begin{array}{c} r+s+m \\ r+m \end{array} \right\}_U \left\{ \begin{array}{c} s+m \\ m \end{array} \right\}_U^{-1},$$

$$h_{kj}^{-1} = (-1)^{r(j+k+r-1)+j} i^{(k-j)(r-s+1)} q^{\frac{1}{2}((r-s)(j-k)+k^2+j)} \left[\begin{array}{c} r+s+j \\ r \end{array} \right]_q^{-1}$$
$$\times \sum_{k \le d \le j} (-1)^d q^{\frac{1}{2}(d^2-d)-kd} \left[\begin{array}{c} s \\ d-k \end{array} \right]_q \left[\begin{array}{c} r+s+k \\ s-d+k \end{array} \right]_q \left[\begin{array}{c} s+k \\ d \end{array} \right]_q^{-1} \left[\begin{array}{c} j-d+r-1 \\ r-1 \end{array} \right]_q \left[\begin{array}{c} r+d \\ r \end{array} \right]_q \tag{5}$$

and

$$\|A^{-1}\|_\infty = \frac{(r+t+1)^{\underline{r}}(s+n-t)^{\underline{n}}}{(r+s)!},$$

where $t = \lfloor \frac{nr}{r+s} \rfloor$ and the falling factorial is defined as $x^{\underline{n}} = x(x-1)\cdots(x-n+1)$.

For more on Gaussian q–binomial, generalized Fibonomial and binomial forms (see Arikan and Kilic [1]). Results were presented for $0 \le k, j \le n - 1$. Unfortunately, the constructive approach has not been studied since 1972. It is quite easy to run into trouble because one may not know if the above summand (5) (in its binomial form) is singular in the interval of summation. Hence, the need to use assumptions or parametric option.

On the other hand, the modeling of inverses of non-symmetric matrices in terms of dimension has been difficult. The author, Murray Dow [8], studied the matrix

$$A(r+s+1, n) = \begin{bmatrix} -3 & 1 & & & & & & \\ 3 & -3 & 1 & & & & & \\ -1 & 3 & -3 & 1 & & & & \\ & -1 & 3 & -3 & 1 & & & \\ & & \cdots & \cdots & \cdots & & & \\ & & & -1 & 3 & -3 & 1 & \\ & & & & -1 & 3 & -3 \end{bmatrix},$$

where $r = 2, \ s = 1$ and gave its determinant as

$$\det A(4, n) = \frac{(n+1)(n+2)}{2}$$

and its inverse is given as

$$A_{ij}^{-1} = \begin{cases} a_0(j)i(i+1) & \text{for odd n,} \\ b_0(j)i^2 + b_1(j)i + b_2(j) & \text{for even n,} \end{cases}$$

$$a_0(j) = \frac{(n + 1 - j)(n + 2 - j)}{2(n + 1)(n + 2)},$$

$$b_0(j) = \frac{j(-3 + j - 2n)}{2(n + 1)(n + 2)},$$

$$b_1(j) = \frac{j(1 + j + 4n + 2n^2)}{2(n + 1)(n + 2)}.$$

However, the infinity-norm of its inverse was not presented in that paper. This is the first time constructive approach is used to model the properties of non-symmetric Teoplitz matrices.

In this paper, a set of band matrices of bandwidth $(r + s + 1)$, where s is the upper bandwidth and r is the lower bandwidth are considered. The coefficients are derived from the binomial expansion $(x - 1)^{r+s}$ and are placed about the diagonal in row and column fashion. The matrix of the type $A(2r + 1, n)$ where $2r + 1$ is the bandwidth has been studied by Hoskins and Ponzo [16], where the matrix is symmetric, for example

$$A(5, n) = \begin{bmatrix} 6 & -4 & 1 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 & 1 \\ & & \cdots & \cdots \\ & & 1 & -4 & 6 & -4 & 1 \\ & & & 1 & -4 & 6 & -4 \\ & & & & 1 & -4 & 6 \end{bmatrix}.$$

Herein, results for the determinants, inverses and infinity-norm of Non-symmetric, Odd and Even banded Toeplitz matrices arising from binomial coefficients are given. Also, constructive approach is used to derive results for different kinds of matrix classes of the type

$$A(2s \pm (t \pm 1), n) = (-1)^{s+j-i} \binom{2s \pm t}{s + j - i} \quad \text{as} \quad t \longrightarrow \infty,$$

where s is the upper bandwidth of these matrices. Theorem 3 and Lemma 5 are similar to the results obtained by Arikan and Kilic [1] but with a novel approach. The matrix classes here are the same as [1] but the used method is different.

2 Main Results

For positive arbitrary integers r, s and t define the matrix a_{ij} with upper bandwidth of s and lower bandwidth of r as

$$a_{ij} = (-1)^{r+i-j} \binom{2r \mp t}{r + i - j} = (-1)^{s+j-i} \binom{2s \pm t}{s + j - i}, \quad r, s \geq 0, \ t \in [0, \infty]. \tag{6}$$

In the equation above, r and s can be arbitrarily chosen as positive integers. An example, which would be treated in this paper, is setting $r = s + 1$. The inverse of the above matrix class is stated below.

Theorem 1.
The inverse of matrix $A(r + s + 1, n)$ where $r + s + 1$ represents the bandwidth is given by

$$\left\{ A(r + s + 1, n)^{-1} \right\}_{i,j} = (-1)^s \binom{i + s - 1}{s} \binom{j + r - 1}{r}$$

$$\times \sum_{k=i}^{n} \frac{\binom{k+r-1-i}{r-1}\binom{k+s-1-j}{s-1}}{\binom{k+s-1}{s}\binom{k+r+s-1}{r}}. \tag{7}$$

The elements derived are those on the main diagonal and below. Other elements are obtained by symmetry, i.e., for values of k where $j \le k \le n$. Related results of this are given by the following Corollaries.

Corollary 1.

Let $A[2\,r \mp (t \mp 1), n]$ be a non-symmetric matrix of this class and r be the lower bandwidth. If $L_r(i) = i(i+1)(i+2)\cdots(i+r-1)$, then, the elements on the diagonal and below are given by

$$
\left\{ A(2\,r \mp (t \mp 1), n)^{-1} \right\}_{i,j} = \pm \frac{(n+1-i)(n+2-i)\cdots(n+r \mp t - i)}{(r-1)!(2\,r \mp (t \pm 1))!}
$$
$$
\times \sum_{v=0}^{r-1} \left[(-1)^v \binom{r-1}{v} \frac{L_r(i)}{(i+v)} \frac{(j+v+r \mp (t \pm 1))!}{(j+v-r)!} \frac{(n+v)!}{(n+v+r \mp t)!} \right], \quad i \ge j, \quad r \ge t \pm 1.
$$
(8)

Corollary 2.

Let $A(2\,s \pm (t \pm 1), n)$ be a non-symmetric matrix of this class and s be the upper bandwidth. If $L_s(j) = j(j+1)\cdots(j+s+1)$, then, the elements on the diagonal and above are given by

$$
\left\{ A(2\,s \pm (t \pm 1), n)^{-1} \right\}_{i,j} = \mp \frac{(n+1-j)(n+2-j)\cdots(n+s \pm t - j)}{(s-1)!(2\,s \pm (t \mp 1))!}
$$
$$
\times \sum_{k=0}^{s-1} \left[(-1)^k \binom{s-1}{k} \left[\frac{L_s(j)}{j+k} \right] \frac{(j+k+s \pm (t \mp 1))!}{(j+k-s)!} \frac{(n+k)!}{(n+k+s \pm t)!} \right], \quad i \le j, \quad s \ge t \mp 1.
$$
(9)

The Hoskins-Ponzo theorem (4) is easily derived when $s = r$, $v = k$ and $t = 0$.

2.1 The Bandwidth Case $2s + 2$, $s \ge 1$ and $2r - 1$, $r \ge 2$

For non-negative values of r and s, define the non-symmetric band matrices $A = \{a_{ij}\}$ in terms of binomial coefficients given by

$$
a_{ij} = (-1)^{r+j-i} \binom{2r-1}{r+j-i} = (-1)^{s+i-j} \binom{2s+1}{s+i-j}.
$$
(10)

For example, when $s = 2$, $r = 3$ and $t = 1$, the resulting matrix is

$$
A(6, n) = \begin{bmatrix}
10 & -5 & 1 & & & & & & \\
-10 & 10 & -5 & 1 & & & & & \\
5 & -10 & 10 & -5 & 1 & & & & \\
-1 & 5 & -10 & 10 & -5 & 1 & & & \\
& -1 & 5 & -10 & 10 & -5 & 1 & & \\
& & \cdots & \cdots & \cdots & & & & \\
& & & -1 & 5 & -10 & 10 & -5 & 1 \\
& & & & -1 & 5 & -10 & 10 & -5 \\
& & & & & -1 & 5 & -10 & 10
\end{bmatrix}.
$$

The inverse of the matrix A^{-1}, is given by the theorems below.

Corollary 3.

Let $A(r + s + 1, n)$ be a non-symmetric matrix of this class and r be the lower bandwidth. If $L_r(i) = i(i + 1)(i + 2) \cdots \cdots \cdots (i + r - 1)$, then, the elements on the diagonal and below are given by

$$\left\{ A(2\,r, n)^{-1} \right\}_{i,j} = \frac{(n + 1 - i)(n + 2 - i) \cdots (n + r - 2 - i)(n + r - 1 - i)}{(r - 1)!(2\,r - 2)!}$$

$$\times \sum_{k=0}^{r-1} \left[(-1)^k \binom{r - 1}{k} \frac{L_r(i)}{(i + k)} \frac{(j + k + r - 2)!}{(j + k - r)!} \frac{(n + k)!}{(n + k + r - 1)!} \right], \quad i \geq j, \quad r \geq 2.$$

(11)

The first three inverses for $i \geq j$ are given by

$$\left\{ A(4, n)^{-1} \right\}_{i,j} = \frac{(n + 1 - i)}{2(n + 1)(n + 2)} \left[(i + 1)\,j\,(j - 1)\,(n + 2) - i\,(j + 1)\,j(n + 1) \right],$$

$$\left\{ A(6, n)^{-1} \right\}_{i,j} = \frac{(n + 1 - i)\,(n + 2 - i)}{48\,(n + 1)\,(n + 2)\,(n + 3)\,(n + 4)} \left[(i + 1)\,(i + 2)\,(j + 1)\,j\,(j - 1)\,(j - 2)\,(n + 4)\,(n + 3) \right.$$

$$-2\,i\,(i + 2)\,(j + 2)\,(j + 1)\,j\,(j - 1)\,(n + 1)\,(n + 4)$$

$$\left. +i\,(i + 1)\,(j + 3)\,(j + 2)\,(j + 1)\,j\,(n + 2)\,(n + 1) \right],$$

and

(12)

$$\left\{ A(8, n)^{-1} \right\}_{i,j} = \frac{(n + 1 - i)\,(-i + n + 2)\,(n + 3 - i)}{4320(n + 1)\,(n + 2)\,(n + 3)\,(n + 4)\,(n + 5)\,(n + 6)}$$

$$\cdot \left[(i + 1)\,(i + 2)\,(i + 3)\,(j + 2)\,(j + 1)\,j\,(j - 1)\,(j - 2)\,(j - 3) \right.$$

$$-3\,i\,(i + 2)\,(i + 3)\,(j + 3)\,(j + 2)\,(j + 1)\,j\,(j - 1)\,(j - 2)$$

$$+3\,i\,(i + 1)\,(i + 3)\,(j + 4)\,(j + 3)\,(j + 2)\,(j + 1)\,j\,(j - 1)$$

$$\left. -i\,(i + 1)\,(i + 2)\,(j + 5)\,(j + 4)\,(j + 3)\,(j + 2)\,(j + 1)\,j \right].$$

Corollary 4.

Let $A(r + s + 1, n)$ be a non-symmetric matrix of this class and s be the upper bandwidth. If $L_s(i) = j(j + 1)(j + 2) \cdots \cdots \cdots (j + s - 1)$, then, the elements on the diagonal and above are given by

$$\left\{ A^{-1}(2\,s + 2, n) \right\}_{i,j} = \frac{(n + 1 - j)(n + 2 - j) \cdots (n + s - j)(n + s + 1 - j)}{(s - 1)!(2\,s)!}$$

$$\times \sum_{k=0}^{s-1} \left[(-1)^k \binom{s - 1}{k} \frac{L_s(j)}{(j + k)} \frac{(i + k + s)!}{(i + k - s)!} \frac{(n + k)!}{(n + k + s + 1)!} \right], \quad i \leq j, \quad s \geq 1.$$

(13)

The first three inverses for $i \leq j$ are given by

$$\left\{ A(4, n)^{-1} \right\}_{i,j} = \frac{(n + 1 - j)(n + 2 - j)(i + 1)i}{2\,(n + 1)(n + 2)},$$

$$\left\{ A(6, n)^{-1} \right\}_{i,j} = \frac{(n + 1 - j)(n + 2 - j)(n + 3 - j)}{24(n + 1)(n + 2)(n + 3)(n + 4)} \left[(j + 1)(i + 2)(i + 1)i(i - 1)(n + 4) \right.$$

$$\left. -j(i + 3)(i + 2)(i + 1)i(n + 1) \right],$$

$$\left\{ A(8, n)^{-1} \right\}_{i,j} = \frac{(n + 1 - j)(n + 2 - j)(n + 3 - j)(n + 4 - j)}{1440(n + 1)(n + 2)(n + 3)(n + 4)(n + 5)(n + 6)}$$

(14)

$$\cdot \left[(j + 1)(j + 2)(i + 3)(i + 2)(i + 1)i(i - 1)(i - 2)(n + 5)(n + 6) \right.$$

$$-2j(j + 2)(i + 4)(i + 3)(i + 2)(i + 1)i(i - 1)(n + 1)(n + 6)$$

$$\left. +j(j + 1)(i + 5)(i + 4)(i + 3)(i + 2)(i + 1)i(n + 1)(n + 2) \right].$$

The determinant of the general Toeplitz matrix classes in Eq. (6) is defined in the theorems below

Theorem 2.

For $r, s, n \geq 0$,

$$\det A(r + s + 1, n) = (-1)^{n+s-1} \prod_{k=1}^{n} \binom{k + s + r - 1}{r} \binom{k + r - 1}{r}^{-1}.$$

From the Lemma above, the determinants of the class of matrices (10) are given more elaborately. The first few terms are

$$\det A(4, n) = \frac{(n + 1)(n + 2)}{1 \cdot 2},$$

$$\det A(6, n) = \frac{(n + 1)(n + 2)^2(n + 3)^2(n + 4)}{1 \cdot 2^2 \cdot 3^2 \cdot 4},$$

$$\det A(8, n) = \frac{(n + 1)(n + 2)^2(n + 3)^3(n + 4)^3(n + 5)^2(n + 6)}{1 \cdot 2^2 \cdot 3^3 \cdot 4^3 \cdot 5^2 \cdot 6}.$$

Attention is then drawn to the Norm-infinity of general Toeplitz matrices' inverse arising from binomial coefficients by the following Corollary.

Corollary 5.

Let r, s and t be non-negative constants then

$$\|A(2s \pm (t \pm 1), n)^{-1}\|_\infty = \frac{\prod_{k=1}^{r}[n + 2k \pm (2t \mp 1)] \prod_{m=1}^{s}[n + 2m \mp (2t \pm 1)]}{2^{r+s}(r + s)!}, \text{ for odd n,}$$

$$= \frac{\prod_{k=1}^{r}[n + 2k \pm (2t \mp 2)] \prod_{m=1}^{s}[n + 2m \mp 2t]}{2^{r+s}(r + s)!}, \text{ for even n.}$$

(15)

When $t = 0$ and $r = s$, Eqn. (3) is easily arrived at. The similar result is given below.

Corollary 6.

Norm-infinity of matrix class (10)'s inverses is stated as

$$\|A(2s + 2, n)^{-1}\|_\infty = \frac{\prod_{k=1}^{r}(n + 2k + 1) \prod_{m=1}^{s}(n + 2m - 3)}{2^{r+s}(r + s)!}, \text{ for odd n,}$$

$$= \frac{\prod_{k=1}^{r}(n + 2k) \prod_{m=1}^{s}(n + 2m - 2)}{2^{r+s}(r + s)!}, \text{ for even n.}$$

(16)

The first few terms for odd n are given as

$$\|A(4, n)^{-1}\|_\infty = \frac{1}{48} (n + 3)(n + 5)(n - 1), \quad n \neq 1,$$

$$\|A(6, n)^{-1}\|_\infty = \frac{1}{3840} (n + 3)(n + 5)(n + 7)(n - 1)(n + 1), \quad n \neq 1,$$

$$\|A(8, n)^{-1}\|_\infty = \frac{1}{645120} (n + 3)^2 (n + 5)(n + 7)(n + 9)(n - 1)(n + 1), \quad n \neq 1.$$

Also, the first few terms for even n are given as

$$\|A(4, n)^{-1}\|_\infty = \frac{1}{48} (n + 2)(n + 4),$$

$$\|A(6, n)^{-1}\|_\infty = \frac{1}{3840} (n + 2)^2 (n + 4)(n + 6) n,$$

$$\|A(8, n)^{-1}\|_\infty = \frac{1}{645120} (n + 2)^2 (n + 4)^2 (n + 6)(n + 8) n.$$

3 Proofs

In this section, Theorems 1, 2 and 3 are proved.

Proof of Theorem 1.

Proof. For positive integers $j \geq 1$, consider

$$
\begin{aligned}
\eta_j(i) &= \frac{i(i+1)(i+2)\cdots(i+s-1)}{s!} \cdot \frac{(j+1-i)(j+2-i)\cdots(j+r-2-i)(j+r-1-i)}{(r-1)!} \\
&= \binom{i+s-1}{s}\binom{j+r-1-i}{r-1}.
\end{aligned}
\tag{17}
$$

Also consider, for any integer $i \geq 1$

$$
\begin{aligned}
\psi_i(j) &= \frac{j(j+1)(j+2)\cdots(j+r-1)}{r!} \cdot \frac{(i+1-j)(i+2-j)\cdots(i+s-2-j)(i+s-1-j)}{(s-1)!} \\
&= \binom{j+r-1}{r}\binom{i+s-1-j}{s-1}.
\end{aligned}
\tag{18}
$$

Then, $\eta_j(i)$ is a polynomial of degree $(r+s-1)$ in the variable i, where $\eta_j(i) = 0$ for $i = 0, -1, -2, \cdots, -s+1$ and $i = j+1, j+2, \cdots, j+r-1$. Also, $\psi_i(j)$ is a polynomial of degree $(r+s-1)$ in the variable j, where $\psi_i(j) = 0$ for $j = 0, -1, -2, \cdots, -r+1$ and $j = i+1, i+2, \cdots, i+s-1$. Give the definition of an n−vector u_j where each component

$$
\begin{aligned}
u_j(i) &= \eta_j(i) \quad for \ \ i < j+r \\
&= 0 \quad for \ \ i \geq j+r.
\end{aligned}
\tag{19}
$$

Also, define the n−vector l_i with components

$$
\begin{aligned}
l_i(j) &= \psi_i(j) \quad for \ \ j < i+s \\
&= 0 \quad for \ \ j \geq i+s.
\end{aligned}
\tag{20}
$$

Then $u_j(i)$ is the polynomial $\eta_j(i)$ [of degree $(r+s-1)$] for $i < j+r$, so that $\gamma^{r+s} u_j(i) = 0$ for $i = 1, 2, \cdots, j-1$. Also, $l_i(j)$ is the polynomial $\psi_i(j)$ [of degree $(r+s-1)$] for $j < i+s$, so that $\delta^{r+s} l_i(j) = 0$ for $j = 1, 2, \cdots, i-1$. If U denotes the upper triangular matrix whose jth column is u_j as given in (19) and L denotes the lower triangular matrix whose ith row is l_i, then $AU = L$. If $A(r+s+1, n) = \{a_{ij}\}$ where

$$
a_{ij} = (-1)^{r+i-j}\binom{r+s}{r+i-j},
$$

and $U = \{u_{ij}\}$ where

$$
\begin{aligned}
u_{ij} &= \binom{i+s-1}{s}\binom{j+r-1-i}{r-1} \quad for \ \ i \leq j, \\
&= 0 \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad for \ \ i > j,
\end{aligned}
$$

then

$$
AU = L.
\tag{21}
$$

To continue the proof, the Theorem below is needed.

Theorem 4:

If the conditions of Theorem 1 are satisfied, then $A^{-1} = UD^{-1}U^T$, if the inverse exist.

Proof. From (21), $L = AU = (U^T)^{-1}D$. Hence, $A = (U^T)^{-1}DU^{-1}$ and $A^{-1} = UD^{-1}U^T$. Here, T denotes the use of transposition. □

Continuing the proof, we determine $A(r + s + 1, n)^{-1}$ by recalling from Theorem 4, that

$$A(r + s + 1, n)^{-1} = UD^{-1}U^T.$$

Thus,

$$\left\{ A(r + s + 1, n)^{-1} \right\}_{i,j} = \sum_{k=1}^{n} \frac{\eta_{ik} \psi_{kj}}{D_{kk}},$$

where

$$\eta_{ik} = \binom{i + s - 1}{s} \binom{k + r - 1 - i}{r - 1},$$

$$\psi_{kj} = \binom{j + r - 1}{r} \binom{k + s - 1 - j}{s - 1}.$$

Hence,

$$\left\{ A(r + s + 1, n)^{-1} \right\}_{i,j} = (-1)^s \binom{i + s - 1}{s} \binom{j + r - 1}{r} \sum_{k=i}^{n} \frac{\binom{k+r-1-i}{r-1} \binom{k+s-1-j}{s-1}}{\binom{k+s-1}{s} \binom{k+r+s-1}{r}}.$$

Remark:
When $r = s$, Eqn. (2) is easily derived. $\square$

Proof of Theorem 2.
Proof. From $AU = L$, the following is obtained

$$|A| = |L| / |U| = \prod_{k=1}^{n} l_{kk} / \prod_{k=1}^{n} u_{kk} = \prod_{k=1}^{n} \left\{ \frac{-\psi_k(k + s)}{\psi_k(k)} \right\}$$

$$= \prod_{k=1}^{n} \left\{ \frac{(-1)^n \binom{k+s+r-1}{r} \binom{-1}{s-1}}{\binom{k+r-1}{r}} \right\} \tag{22}$$

$$= (-1)^{n+s-1} \prod_{k=1}^{n} \frac{\binom{k+s+r-1}{r}}{\binom{k+r-1}{r}}$$

where $l_{kk} = -\psi_k(k + s)$ and $u_{kk} = \gamma^{r+s} l_{kk} = \psi_k(k)$. $\square$

Proof of Corollary 3.
Proof. Before proving Corollary 3, the following Theorem is needed.

Theorem 5.
The row-sums, denoted by S_j of A^{-1} is given by

$$S_j = (-1)^s \frac{\binom{j+r-1}{r} \binom{n-j+s}{s}}{\binom{r+s}{s}}, \quad 1 \le j \le n$$

Proof. By considering $(e_j UD^{-1}U^T e)$, for $j = 1, ..., n$, given in Theorem 4, and, Theorem 3 in [16], the proof similarly follows. $\square$

Corollary 7.

If L and U are the matrices of Theorem 1, then $L^T U = D$ a real diagonal matrix with

$$D_{kk} = (-1)^s \binom{k+s-1}{s}\binom{k+s+r-1}{r} = (-1)^s \binom{k+s+r-1}{r+s}\binom{r+s}{s}. \tag{23}$$

Since $D = L^T U$, recall from Theorem 1 that $D_{kk} = l_{kk} u_{kk}$. From (19), $u_{kk} = \eta_k(k)$ and $l_{kk} = -\psi_k(k+s)$. Thus,

$$u_{kk} = \eta_k(k) = \binom{k+s-1}{r}\binom{k+s-1-k}{s-1} = \binom{k+s-1}{s},$$

also

$$l_{kk} = \psi_k(k+s) = (-1)\binom{k+s+r-1}{r}\binom{k+s-1-k-s}{s-1}$$

$$= (-1)\binom{k+s+r-1}{r}\binom{-1}{s-1}.$$

Then

$$D_{kk} = (-1)\binom{k+s-1}{s}\binom{k+s+r-1}{r}\binom{-1}{s-1}.$$

Also, note that

$$\binom{-1}{s-1} = (-1)^{s-1}.$$

$$D_{kk} = (-1)^s \binom{k+s+r-1}{r}\binom{k+s-1}{s}. \tag{24}$$

Moreover, simple calculations proceeding from (24) imply that

$$D_{kk} = (-1)^s \binom{k+s+r-1}{r+s}\binom{r+s}{s}.$$

<div align="right">□</div>

Data Availability Statement: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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