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# Some integral inequalities for operator monotonic functions on Hilbert spaces

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**Abstract:** Let $f$ be an operator monotonic function on $I$ and $A, B \in \mathcal{SA}_I(H)$, the class of all selfadjoint operators with spectra in $I$. Assume that $p : [0,1] \to \mathbb{R}$ is non-decreasing on $[0,1]$. In this paper we obtained, among others, that for $A \le B$ and $f$ an operator monotonic function on $I$,

$$0 \le \int_0^1 p(t) f((1-t)A + tB)\,dt - \int_0^1 p(t)\,dt \int_0^1 f((1-t)A + tB)\,dt$$
$$\le \frac{1}{4}[p(1) - p(0)][f(B) - f(A)]$$

in the operator order.
Several other similar inequalities for either $p$ or $f$ is differentiable, are also provided. Applications for power function and logarithm are given as well.

**Keywords:** Operator monotonic functions, Integral inequalities, Čebyšev inequality, Grüss inequality, Ostrowski inequality

**MSC:** 47A63, 26D15, 26D10.

## 1 Introduction

Consider a complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$. An operator $T$ is said to be positive (denoted by $T \ge 0$) if $\langle Tx, x\rangle \ge 0$ for all $x \in H$ and also an operator $T$ is said to be *strictly positive* (denoted by $T > 0$) if $T$ is positive and invertible. A real valued continuous function $f(t)$ on $(0,\infty)$ is said to be operator monotone if $f(A) \ge f(B)$ holds for any $A \ge B > 0$.

In 1934, K. Löwner [7] had given a definitive characterization of operator monotone functions as follows:

**Theorem 1.** *A function $f : (0,\infty) \to \mathbb{R}$ is operator monotone in $(0,\infty)$ if and only if it has the representation*

$$f(t) = a + bt + \int_0^\infty \frac{t}{t+s}\,dm(s)$$

*where $a \in \mathbb{R}$ and $b \ge 0$ and a positive measure $m$ on $(0,\infty)$ such that*

$$\int_0^\infty \frac{dm(s)}{t+s} < \infty.$$

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We recall the important fact proved by Löwner and Heinz that states that the power function $f:(0,\infty)\to\mathbb{R}$, $f(t)=t^{\alpha}$ is an operator monotone function for any $\alpha\in[0,1]$.

In [3], T. Furuta observed that for $\alpha_j\in[0,1]$, $j=1,\dots,n$ the functions

$$g(t):=\left(\sum_{j=1}^{n}t^{-\alpha_j}\right)^{-1}\text{ and } h(t)=\sum_{j=1}^{n}\left(1+t^{-1}\right)^{-\alpha_j}$$

are operator monotone in $(0,\infty)$.

Let $f(t)$ be a continuous function $(0,\infty)\to(0,\infty)$. It is known that $f(t)$ is operator monotone if and only if $g(t)=t/f(t)=:f^{*}(t)$ is also operator monotone, see for instance [3] or [8].

Consider the family of functions defined on $(0,\infty)$ and $p\in[-1,2]\setminus\{0,1\}$ by

$$f_p(t):=\frac{p-1}{p}\left(\frac{t^p-1}{t^{p-1}-1}\right)$$

and

$$f_0(t):=\frac{t}{1-t}\ln t,$$

$$f_1(t):=\frac{t-1}{\ln t}\text{ (logarithmic mean).}$$

We also have the functions of interest

$$f_{-1}(t)=\frac{2t}{1+t}\text{ (harmonic mean)},\ f_{1/2}(t)=\sqrt{t}\text{ (geometric mean).}$$

In [2] the authors showed that $f_p$ is operator monotone for $1\le p\le 2$.

In the same category, we observe that the function

$$g_p(t):=\frac{t-1}{t^p-1}$$

is an operator monotone function for $p\in(0,1]$, [3].

It is well known that the logarithmic function ln is operator monotone and in [3] the author obtained that the functions

$$f(t)=t(1+t)\ln\left(1+\frac{1}{t}\right),\ g(t)=\frac{1}{(1+t)\ln\left(1+\frac{1}{t}\right)}$$

are also operator monotone functions on $(0,\infty)$.

Let $f$ be an operator monotonic function on an interval of real numbers $I$ and $A,B\in\mathcal{SA}_I(H)$, the class of all selfadjoint operators with spectra in $I$. Assume that $p:[0,1]\to\mathbb{R}$ is non-decreasing on $[0,1]$. In this paper we obtain, among others, that for $A\le B$ and $f$ an operator monotonic function on $I$,

$$\begin{aligned}0&\le\int_0^1 p(t)f((1-t)A+tB)\,dt-\int_0^1 p(t)\,dt\int_0^1 f((1-t)A+tB)\,dt\\&\le\frac{1}{4}[p(1)-p(0)][f(B)-f(A)]\end{aligned}$$

in the operator order.

Several other similar inequalities for either $p$ or $f$ is differentiable, are also provided. Applications for power function and logarithm are given as well.

## 2 Main Results

For two *Lebesgue integrable* functions $h,g:[a,b]\to\mathbb{R}$, consider the *Čebyšev functional*:

$$C(h,g):=\frac{1}{b-a}\int_a^b h(t)g(t)dt-\frac{1}{b-a}\int_a^b h(t)dt\frac{1}{b-a}\int_a^b g(t)dt. \tag{2.1}$$

It is well known that, if $h$ and $g$ have the same monotonicity on $[a, b]$, then

$$\frac{1}{b-a}\int_a^b h(t)g(t)dt \geq \frac{1}{b-a}\int_a^b h(t)dt\frac{1}{b-a}\int_a^b g(t)dt, \tag{2.2}$$

which is known in the literature as *Čebyšev's inequality.*

In 1935, Grüss [4] showed that

$$|C(h, g)| \leq \frac{1}{4}(M-m)(N-n), \tag{2.3}$$

provided that there exists the real numbers $m, M, n, N$ such that

$$m \leq h(t) \leq M \quad \text{and} \quad n \leq g(t) \leq N \quad \text{for a.e. } t \in [a, b]. \tag{2.4}$$

The constant $\frac{1}{4}$ is best possible in (2.1) in the sense that it cannot be replaced by a smaller quantity.

Let $f$ be a continuous function on $I$. If $(A, B) \in \mathcal{SA}_I(H)$, the class of all selfadjoint operators with spectra in $I$ and $t \in [0, 1]$, then the convex combination $(1-t)A + tB$ is a selfadjoint operator with the spectrum in $I$ showing that $\mathcal{SA}_I(H)$ is a convex set in the Banach algebra $\mathcal{B}(H)$ of all bounded linear operators on $H$. By the continuous functional calculus of selfadjoint operator we also conclude that $f((1-t)A + tB)$ is a selfadjoint operator in $\mathcal{B}(H)$.

For $A, B \in \mathcal{SA}_I(H)$, we consider the auxiliary function $\varphi_{(A,B)} : [0, 1] \to \mathcal{B}(H)$ defined by

$$\varphi_{(A,B)}(t) := f((1-t)A + tB). \tag{2.5}$$

For $x \in H$ we can also consider the auxiliary function $\varphi_{(A,B);x} : [0, 1] \to \mathbb{R}$ defined by

$$\varphi_{(A,B);x}(t) := \left\langle \varphi_{(A,B)}(t)x, x\right\rangle = \langle f((1-t)A + tB)x, x\rangle. \tag{2.6}$$

**Theorem 2.** *Let $A, B \in \mathcal{SA}_I(H)$ with $A \leq B$ and $f$ an operator monotonic function on $I$. If $p : [0, 1] \to \mathbb{R}$ is monotonic nondecreasing on* $[0, 1]$, *then*

$$\begin{aligned} 0 &\leq \int_0^1 p(t)f((1-t)A + tB)\,dt - \int_0^1 p(t)\,dt\int_0^1 f((1-t)A + tB)\,dt \\ &\leq \frac{1}{4}[p(1) - p(0)][f(B) - f(A)]. \end{aligned} \tag{2.7}$$

*If $p : [0, 1] \to \mathbb{R}$ is monotonic nonincreasing on* $[0, 1]$, *then*

$$\begin{aligned} 0 &\leq \int_0^1 p(t)\,dt\int_0^1 f((1-t)A + tB)\,dt - \int_0^1 p(t)f((1-t)A + tB)\,dt \\ &\leq \frac{1}{4}[p(0) - p(1)][f(B) - f(A)]. \end{aligned} \tag{2.8}$$

*Proof.* Let $0 \leq t_1 < t_2 \leq 1$ and $A \leq B$. Then

$$(1-t_2)A + t_2B - (1-t_1)A - t_1B = (t_2 - t_1)(B - A) \geq 0$$

and by operator monotonicity of $f$ we get

$$f((1-t_2)A + t_2B) \geq f((1-t_1)A + t_1B),$$

which is equivalent to

$$\begin{aligned} \varphi_{(A,B);x}(t_2) &= \langle f((1-t_2)A + t_2B)x, x\rangle \\ &\geq \langle f((1-t_1)A + t_1B)x, x\rangle = \varphi_{(A,B);x}(t_1) \end{aligned}$$

that shows that the scalar function $\varphi_{(A,B);x} : [0,1] \to \mathbb{R}$ is monotonic nondecreasing for $A \le B$ and for any $x \in H$.

If we write the inequality (2.2) for the functions $p$ and $\varphi_{(A,B);x}$ we get

$$\int_0^1 p(t)\langle f((1-t)A+tB)x,x\rangle dt \ge \int_0^1 p(t)\,dt \int_0^1 \langle f((1-t)A+tB)x,x\rangle dt,$$

which can be written as

$$\left\langle \left(\int_0^1 p(t) f((1-t)A+tB)\,dt\right)x,x\right\rangle \ge \left\langle \left(\int_0^1 p(t)\,dt \int_0^1 f((1-t)A+tB)\right) dtx,x\right\rangle$$

for $x \in H$, and the first inequality in (2.7) is obtained.

We also have that

$$\begin{aligned}\langle f(A)x,x\rangle &= \varphi_{(A,B);x}(0) \le \varphi_{(A,B);x}(t) = \langle f((1-t)A+tB)x,x\rangle \\ &\le \varphi_{(A,B);x}(1) = \langle f(B)x,x\rangle\end{aligned}$$

and

$$p(0) \le p(t) \le p(1)$$

for all $t \in [0,1]$.

By writing Grüss' inequality for the functions $\varphi_{(A,B);x}$ and $p$, we get

$$\begin{aligned}0 &\le \int_0^1 p(t)\langle f((1-t)A+tB)x,x\rangle dt - \int_0^1 p(t)\,dt \int_0^1 \langle f((1-t)A+tB)x,x\rangle dt \\ &\le \frac{1}{4}[p(1)-p(0)][\langle f(B)x,x\rangle - \langle f(A)x,x\rangle]\end{aligned}$$

for $x \in H$ and the second inequality in (2.7) is obtained. □

A continuous function $g : \mathcal{SA}_I(H) \to \mathcal{B}(H)$ is said to be *Gâteaux differentiable* in $A \in \mathcal{SA}_I(H)$ along the direction $B \in \mathcal{B}(H)$ if the following limit exists in the strong topology of $\mathcal{B}(H)$

$$\nabla g_A(B) := \lim_{s\to 0} \frac{g(A+sB)-g(A)}{s} \in \mathcal{B}(H). \tag{2.9}$$

If the limit (2.9) exists for all $B \in \mathcal{B}(H)$, then we say that $g$ is *Gâteaux differentiable* in $A$ and we can write $g \in \mathcal{G}(A)$. If this is true for any $A$ in an open set $\mathcal{S}$ from $\mathcal{SA}_I(H)$ we write that $g \in \mathcal{G}(\mathcal{S})$.

If $g$ is a continuous function on $I$, by utilising the continuous functional calculus the corresponding function of operators will be denoted in the same way.

For two distinct operators $A, B \in \mathcal{SA}_I(H)$ we consider the segment of selfadjoint operators

$$[A,B] := \{(1-t)A+tB \mid t \in [0,1]\}.$$

We observe that $A, B \in [A,B]$ and $[A,B] \subset \mathcal{SA}_I(H)$.

**Lemma 1.** *Let $f$ be a continuous function on $I$ and $A, B \in \mathcal{SA}_I(H)$, with $A \ne B$. If $f \in \mathcal{G}([A,B])$, then the auxiliary function $\varphi_{(A,B)}$ is differentiable on $(0,1)$ and*

$$\varphi'_{(A,B)}(t) = \nabla f_{(1-t)A+tB}(B-A). \tag{2.10}$$

*In particular,*

$$\varphi'_{(A,B)}(0+) = \nabla f_A(B-A) \tag{2.11}$$

*and*

$$\varphi'_{(A,B)}(1-) = \nabla f_B(B-A). \tag{2.12}$$

*Proof.* Let $t \in (0,1)$ and $h \neq 0$ small enough such that $t+h \in (0,1)$. Then

$$\begin{aligned}\frac{\varphi_{(A,B)}(t+h)-\varphi_{(A,B)}(t)}{h} &= \frac{f((1-t-h)A+(t+h)B)-f((1-t)A+tB)}{h} \\ &= \frac{f((1-t)A+tB+h(B-A))-f((1-t)A+tB)}{h}.\end{aligned} \tag{2.13}$$

Since $f \in \mathcal{G}([A,B])$, hence by taking the limit over $h \to 0$ in (2.13) we get

$$\begin{aligned}\varphi'_{(A,B)}(t) &= \lim_{h\to 0}\frac{\varphi_{(A,B)}(t+h)-\varphi_{(A,B)}(t)}{h} \\ &= \lim_{h\to 0}\frac{f((1-t)A+tB+h(B-A))-f((1-t)A+tB)}{h} \\ &= \nabla f_{(1-t)A+tB}(B-A),\end{aligned}$$

which proves (2.10).

Also, we have

$$\begin{aligned}\varphi'_{(A,B)}(0+) &= \lim_{h\to 0+}\frac{\varphi_{(A,B)}(h)-\varphi_{(A,B)}(0)}{h} \\ &= \lim_{h\to 0+}\frac{f((1-h)A+hB)-f(A)}{h} \\ &= \lim_{h\to 0+}\frac{f(A+h(B-A))-f(A)}{h} \\ &= \nabla f_A(B-A)\end{aligned}$$

since $f$ is assumed to be Gâteaux differentiable in $A$. This proves (2.11).

The equality (2.12) follows in a similar way. □

**Lemma 2.** *Let $f$ be an operator monotonic function on $I$ and $A, B \in \mathcal{SA}_I(H)$, with $A \leq B$, $A \neq B$. If $f \in \mathcal{G}([A,B])$, then*

$$\nabla f_{(1-t)A+tB}(B-A) \geq 0 \text{ for all } t \in (0,1). \tag{2.14}$$

*Also*

$$\nabla f_A(B-A),\ \nabla f_B(B-A) \geq 0. \tag{2.15}$$

*Proof.* Let $x \in H$. The auxiliary function $\varphi_{(A,B);x}$ is monotonic nondecreasing in the usual sense on $[0,1]$ and differentiable on $(0,1)$, and for $t \in (0,1)$

$$\begin{aligned}0 \leq \varphi'_{(A,B);x}(t) &= \lim_{h\to 0}\frac{\varphi_{(A,B),x}(t+h)-\varphi_{(A,B),x}(t)}{h} \\ &= \lim_{h\to 0}\left\langle\frac{\varphi_{(A,B)}(t+h)-\varphi_{(A,B)}(t)}{h}x, x\right\rangle \\ &= \left\langle\lim_{h\to 0}\frac{\varphi_{(A,B)}(t+h)-\varphi_{(A,B)}(t)}{h}x, x\right\rangle \\ &= \left\langle\nabla f_{(1-t)A+tB}(B-A)x, x\right\rangle.\end{aligned}$$

This shows that

$$\nabla f_{(1-t)A+tB}(B-A) \geq 0$$

for all $t \in (0,1)$.

The inequalities (2.15) follow by (2.11) and (2.12). □

The following inequality obtained by Ostrowski in 1970, [9] also holds

$$|C(h,g)| \leq \frac{1}{8}(b-a)(M-m)\|g'\|_\infty, \tag{2.16}$$

provided that $h$ is *Lebesgue integrable* and satisfies (2.4) while $g$ is absolutely continuous and $g' \in L_\infty[a,b]$. The constant $\frac{1}{8}$ is best possible in (2.16).

**Theorem 3.** *Let $A$, $B \in \mathcal{SA}_I(H)$ with $A \le B$, $f$ be an operator monotonic function on $I$ and $p : [0,1] \to \mathbb{R}$ monotonic nondecreasing on* $[0,1]$.

*(i) If $p$ is differentiable on* $(0,1)$, *then*

$$\begin{aligned} 0 &\le \int_0^1 p(t) f((1-t)A + tB)\,dt - \int_0^1 p(t)\,dt \int_0^1 f((1-t)A + tB)\,dt \\ &\le \frac{1}{8} \sup_{t \in (0,1)} p'(t) [f(B) - f(A)]. \end{aligned} \tag{2.17}$$

*(ii) If $f \in \mathcal{G}([A,B])$, then*

$$\begin{aligned} 0 &\le \int_0^1 p(t) f((1-t)A + tB)\,dt - \int_0^1 p(t)\,dt \int_0^1 f((1-t)A + tB)\,dt \\ &\le \frac{1}{8} [p(1) - p(0)] \sup_{t \in (0,1)} \left\| \nabla f_{(1-t)A+tB}(B - A) \right\| 1_H. \end{aligned} \tag{2.18}$$

*Proof.* Let $x \in H$. If we use the inequality (2.16) for $g = p$ and $h = \varphi_{(A,B);x}$, then

$$\begin{aligned} 0 &\le \int_0^1 p(t) \langle f((1-t)A + tB)x, x\rangle\,dt - \int_0^1 p(t)\,dt \int_0^1 \langle f((1-t)A + tB)x, x\rangle\,dt \\ &\le \frac{1}{8} \sup_{t \in (0,1)} p'(t) [\langle f(B)x, x\rangle - \langle f(A)x, x\rangle], \end{aligned}$$

for any $x \in H$, which is equivalent to (2.17).

If we use the inequality (2.16) for $h = p$ and $g = \varphi_{(A,B);x}$ then by Lemmas 1 and 2

$$\begin{aligned} 0 &\le \int_0^1 p(t) \langle f((1-t)A + tB)x, x\rangle\,dt - \int_0^1 p(t)\,dt \int_0^1 \langle f((1-t)A + tB)x, x\rangle\,dt \\ &\le \frac{1}{8} [p(1) - p(0)] \sup_{t \in (0,1)} \left\langle \nabla f_{(1-t)A+tB}(B - A)x, x \right\rangle, \end{aligned} \tag{2.19}$$

for any $x \in H$, which is an inequality of interest in itself.

Observe that for all $t \in (0,1)$,

$$\left\langle \nabla f_{(1-t)A+tB}(B - A)x, x \right\rangle \le \left\| \nabla f_{(1-t)A+tB}(B - A) \right\| \|x\|^2$$

for any $x \in H$, which implies that

$$\sup_{t \in (0,1)} \left\langle \nabla f_{(1-t)A+tB}(B - A)x, x \right\rangle \le \sup_{t \in (0,1)} \left\| \nabla f_{(1-t)A+tB}(B - A) \right\| \langle 1_H x, x\rangle \tag{2.20}$$

for any $x \in H$.

By making use of (2.19) and (2.20) we derive

$$\begin{aligned} 0 &\le \int_0^1 p(t) \langle f((1-t)A + tB)x, x\rangle\,dt - \int_0^1 p(t)\,dt \int_0^1 \langle f((1-t)A + tB)x, x\rangle\,dt \\ &\le \frac{1}{8} [p(1) - p(0)] \sup_{t \in (0,1)} \left\| \nabla f_{(1-t)A+tB}(B - A) \right\| \langle 1_H x, x\rangle \end{aligned}$$

for any $x \in H$, which is equivalent to (2.18). □

Another, however less known result, even though it was obtained by Čebyšev in 1882, [1], states that

$$|C(h,g)| \le \frac{1}{12} \|h'\|_\infty \|g'\|_\infty (b-a)^2, \tag{2.21}$$

provided that $h', g'$ exist and are continuous on $[a,b]$ and $\|h'\|_\infty = \sup_{t\in[a,b]} |h'(t)|$. The constant $\frac{1}{12}$ cannot be improved in the general case.

The case of *euclidean norms* of the derivative was considered by A. Lupaş in [5] in which he proved that

$$|C(h,g)| \le \frac{1}{\pi^2} \|h'\|_2 \|g'\|_2 (b-a), \tag{2.22}$$

provided that $h, g$ are absolutely continuous and $h', g' \in L_2[a,b]$. The constant $\frac{1}{\pi^2}$ is the best possible.

Using the above inequalities (2.21) and (2.22) and a similar procedure to the one employed in the proof of Theorem 3, we can also state the following result:

**Theorem 4.** *Let $A, B \in \mathcal{SA}_I(H)$ with $A \le B$, $f$ be an operator monotonic function on $I$ and $p : [0,1] \to \mathbb{R}$ monotonic nondecreasing on* $[0,1]$. *If $p$ is differentiable and $f \in \mathcal{G}([A,B])$, then*

$$\begin{aligned} 0 &\le \int_0^1 p(t) f((1-t)A+tB)\,dt - \int_0^1 p(t)\,dt \int_0^1 f((1-t)A+tB)\,dt \\ &\le \frac{1}{12} \sup_{t\in(0,1)} p'(t) \sup_{t\in(0,1)} \left\|\nabla f_{(1-t)A+tB}(B-A)\right\| 1_H \end{aligned} \tag{2.23}$$

*and*

$$\begin{aligned} 0 &\le \int_0^1 p(t) f((1-t)A+tB)\,dt - \int_0^1 p(t)\,dt \int_0^1 f((1-t)A+tB)\,dt \\ &\le \frac{1}{\pi^2} \left(\int_0^1 \left[p'(t)\right]^2 dt\right)^{1/2} \left(\int_0^1 \left\|\nabla f_{(1-t)A+tB}(B-A)\right\|^2 dt\right)^{1/2} 1_H, \end{aligned} \tag{2.24}$$

*provided the integrals in the second term are finite.*

## 3 Some Examples

We consider the function $f : (0,\infty) \to \mathbb{R}$, $f(t) = -t^{-1}$ which is *operator monotone* on $(0,\infty)$.

If $0 < A \le B$ and $p : [0,1] \to \mathbb{R}$ is monotonic nondecreasing on $[0,1]$, then by (2.7)

$$\begin{aligned} 0 &\le \int_0^1 p(t)\,dt \int_0^1 ((1-t)A+tB)^{-1}\,dt - \int_0^1 p(t)((1-t)A+tB)^{-1}\,dt \\ &\le \frac{1}{4}[p(1)-p(0)]\left(A^{-1}-B^{-1}\right). \end{aligned} \tag{3.1}$$

Moreover, if $p$ is differentiable on $(0,1)$, then by (2.17) we obtain

$$\begin{aligned} 0 &\le \int_0^1 p(t)\,dt \int_0^1 ((1-t)A+tB)^{-1}\,dt - \int_0^1 p(t)((1-t)A+tB)^{-1}\,dt \\ &\le \frac{1}{8} \sup_{t\in(0,1)} p'(t)\left(A^{-1}-B^{-1}\right). \end{aligned} \tag{3.2}$$

The function $f(t) = -t^{-1}$ is operator monotonic on $(0,\infty)$, operator Gâteaux differentiable and

$$\nabla f_T(S) = T^{-1}ST^{-1}$$

for $T, S > 0$.

If $p : [0, 1] \to \mathbb{R}$ is monotonic nondecreasing on $[0, 1]$, then by (2.18) we get

$$0 \le \int_0^1 p(t)\,dt \int_0^1 ((1-t)A + tB)^{-1}\,dt - \int_0^1 p(t)((1-t)A + tB)^{-1}\,dt \le \frac{1}{8}[p(1) - p(0)] \sup_{t\in(0,1)} \left\| ((1-t)A + tB)^{-1}(B - A)((1-t)A + tB)^{-1} \right\| 1_H \tag{3.3}$$

for $0 < A \le B$.

If $p$ is monotonic nondecreasing and differentiable on $(0, 1)$, then by (2.23) and (2.24) we get

$$0 \le \int_0^1 p(t)\,dt \int_0^1 ((1-t)A + tB)^{-1}\,dt - \int_0^1 p(t)((1-t)A + tB)^{-1}\,dt \le \frac{1}{12} \sup_{t\in(0,1)} p'(t) \sup_{t\in(0,1)} \left\| ((1-t)A + tB)^{-1}(B - A)((1-t)A + tB)^{-1} \right\| 1_H \tag{3.4}$$

and

$$0 \le \int_0^1 p(t)\,dt \int_0^1 ((1-t)A + tB)^{-1}\,dt - \int_0^1 p(t)((1-t)A + tB)^{-1}\,dt \le \frac{1}{\pi^2} \left( \int_0^1 [p'(t)]^2\,dt \right)^{1/2} \left( \int_0^1 \left\| ((1-t)A + tB)^{-1}(B - A)((1-t)A + tB)^{-1} \right\|^2 dt \right)^{1/2} 1_H, \tag{3.5}$$

for $0 < A \le B$.

We note that the function $f(t) = \ln t$ is operator monotonic on $(0, \infty)$.

If $0 < A \le B$ and $p : [0, 1] \to \mathbb{R}$ is monotonic nondecreasing on $[0, 1]$, then by (2.7) we have

$$0 \le \int_0^1 p(t) \ln((1-t)A + tB)\,dt - \int_0^1 p(t)\,dt \int_0^1 \ln((1-t)A + tB)\,dt \le \frac{1}{4}[p(1) - p(0)](\ln B - \ln A). \tag{3.6}$$

Moreover, if $p$ is differentiable on $(0, 1)$, then by (2.17) we obtain

$$0 \le \int_0^1 p(t) \ln((1-t)A + tB)\,dt - \int_0^1 p(t)\,dt \int_0^1 \ln((1-t)A + tB)\,dt \le \frac{1}{8} \sup_{t\in(0,1)} p'(t)(\ln B - \ln A). \tag{3.7}$$

The ln function is operator Gâteaux differentiable with the following explicit formula for the derivative (cf. Pedersen [10, p. 155]):

$$\nabla \ln_T(S) = \int_0^\infty (s1_H + T)^{-1} S (s1_H + T)^{-1}\,ds \tag{3.8}$$

for $T, S > 0$.

If $p : [0, 1] \to \mathbb{R}$ is monotonic nondecreasing on $[0, 1]$, then by (2.18) we get

$$0 \le \int_0^1 p(t) \ln((1-t)A + tB)\,dt - \int_0^1 p(t)\,dt \int_0^1 \ln((1-t)A + tB)\,dt \le \frac{1}{8}[p(1) - p(0)] \sup_{t\in(0,1)} \left\| \int_0^\infty (s1_H + (1-t)A + tB)^{-1}(B - A)(s1_H + (1-t)A + tB)^{-1}\,ds \right\| 1_H \tag{3.9}$$

and if $p$ is differentiable on $(0,1)$ , then

$$0 \le \int_0^1 p(t) \ln((1-t)A + tB)\,dt - \int_0^1 p(t)\,dt \int_0^1 \ln((1-t)A + tB)\,dt \quad (3.10)$$
$$\le \frac{1}{12} \sup_{t\in(0,1)} p'(t) \sup_{t\in(0,1)} \left\| \int_0^\infty (s1_H + (1-t)A + tB)^{-1} (B-A) (s1_H + (1-t)A + tB)^{-1} ds \right\| 1_H$$

for $0 < A \le B$.

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