Appendix to: Optimal Macroeconomic Policy in Nonlinear Models: A VSTAR Perspective

Vito Polito
University of Sheffield, CESifo, NETSPAR

September 16, 2025

A Transformation from the VSTAR to the VSTARX

To transform a VSTAR into a VSTARX it is necessary to determine a transformation matrix that can be applied to the original VSTAR to separate the vector of policy instruments from the other variables. To determine this transformation matrix, the initial step is to partition the VSTAR. In particular, partitioning equation (1) to separate the economy from the policy instruments gives:

$$\begin{bmatrix} \mathbf{x}_{t+1} \\ \mathbf{u}_{t+1} \end{bmatrix} = \left[1 - g_M\left(\mathbf{s}_u'\mathbf{y}_t\right)\right] \left\{ \begin{array}{c} \left[1 - l_M\left(\mathbf{s}_x'\mathbf{y}_t\right)\right] \begin{bmatrix} \lambda_{x1} \\ \lambda_{u1} \end{bmatrix} + \begin{bmatrix} \mathbf{\Lambda}_{x1}(L) \\ \mathbf{\Lambda}_{u1}(L) \end{bmatrix} \mathbf{y}_t \\ +l_M\left(\mathbf{s}_x'\mathbf{y}_t\right) \left\{ \begin{bmatrix} \lambda_{x2} \\ \lambda_{u2} \end{bmatrix} + \begin{bmatrix} \mathbf{\Lambda}_{x2}(L) \\ \mathbf{\Lambda}_{u2}(L) \end{bmatrix} \mathbf{y}_t \right\} \end{array} \right\} + \tag{A.1}$$

$$g_M\left(\mathbf{s}_u'\mathbf{y}_t\right) \left\{ \begin{array}{c} \left[1 - l_M\left(\mathbf{s}_x'\mathbf{y}_t\right)\right] \begin{bmatrix} \lambda_{x3} \\ \lambda_{u3} \end{bmatrix} + \begin{bmatrix} \mathbf{\Lambda}_{x3}(L) \\ \mathbf{\Lambda}_{u3}(L) \end{bmatrix} \mathbf{y}_t \\ +l_M\left(\mathbf{s}_x'\mathbf{y}_t\right) \left\{ \begin{bmatrix} \lambda_{x4} \\ \lambda_{u4} \end{bmatrix} + \begin{bmatrix} \mathbf{\Lambda}_{x4}(L) \\ \mathbf{\Lambda}_{u4}(L) \end{bmatrix} \mathbf{y}_t \right\} \end{array} \right\} + \begin{bmatrix} \mathbf{v}_{xt+1} \\ \mathbf{v}_{ut+1} \end{bmatrix},$$

where $\mathbf{\Lambda}_{xj}(L) = \begin{bmatrix} \mathbf{\Lambda}_{xxj}(L) & \mathbf{\Lambda}_{xuj}(L) \end{bmatrix}$, $\mathbf{\Lambda}_{uj}(L) = \begin{bmatrix} \mathbf{\Lambda}_{uxj}(L) & \mathbf{\Lambda}_{uuj}(L) \end{bmatrix}$, $j = 1, ..., 4$. The covariance matrix (2) is partitioned conformably as:

$$\mathbf{\Omega}_t = \begin{bmatrix} \mathbf{\Omega}_{xxt} & \mathbf{\Omega}_{xut} \\ \mathbf{\Omega}_{uxt} & \mathbf{\Omega}_{uut} \end{bmatrix} = \left[1 - g_V\left(\mathbf{s}_u'\mathbf{y}_t\right)\right] \left\{ \begin{array}{c} \left[1 - l_V\left(\mathbf{s}_x'\mathbf{y}_t\right)\right] \begin{bmatrix} \mathbf{\Omega}_{xx1} & \mathbf{\Omega}_{xu1} \\ \mathbf{\Omega}_{ux1} & \mathbf{\Omega}_{uu1} \end{bmatrix} \\ +l_V\left(\mathbf{s}_x'\mathbf{y}_t\right) \begin{bmatrix} \mathbf{\Omega}_{xx2} & \mathbf{\Omega}_{xu2} \\ \mathbf{\Omega}_{ux2} & \mathbf{\Omega}_{uu2} \end{bmatrix} \end{array} \right\} + \tag{A.2}$$

$$g_V\left(\mathbf{s}_u'\mathbf{y}_t\right) \left\{ \begin{array}{c} \left[1 - l_V\left(\mathbf{s}_x'\mathbf{y}_t\right)\right] \begin{bmatrix} \mathbf{\Omega}_{xx3} & \mathbf{\Omega}_{xu3} \\ \mathbf{\Omega}_{ux3} & \mathbf{\Omega}_{uu3} \end{bmatrix} \\ +l_V\left(\mathbf{s}_x'\mathbf{y}_t\right) \begin{bmatrix} \mathbf{\Omega}_{xx4} & \mathbf{\Omega}_{xu4} \\ \mathbf{\Omega}_{ux4} & \mathbf{\Omega}_{uu4} \end{bmatrix} \end{array} \right\}.$$

The VSTAR residuals in (A.1) can be written as

$$\begin{bmatrix} \mathbf{v}_{xt} \\ \mathbf{v}_{ut} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{G}_{12t} \\ \mathbf{G}_{21t} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \epsilon_{xt} \\ \epsilon_{ut} \end{bmatrix}$$

where $\begin{bmatrix} \epsilon'_{xt} & \epsilon'_{ut} \end{bmatrix}'$ is a vector of uncorrelated error terms, with $\epsilon_{xt} \sim i.i.d.\,(\mathbf{0}, \sigma_x^2)$ and $\epsilon_{ut} \sim i.i.d.\,(\mathbf{0}, \sigma_u^2)$. Thus $\mathbf{v}_{xt} = \epsilon_{xt} + \mathbf{G}_{12t}\epsilon_{ut}$ and $\mathbf{v}_{ut} = \mathbf{G}_{21t}\epsilon_{xt} + \epsilon_{ut}$.

A.1 Assumption A1

Set $\mathbf{G}_{12t} = \mathbf{0}$, so that $\mathbf{v}_{xt} = \epsilon_{xt}$. Under this restriction (A.2) becomes

$$\begin{aligned} \mathbf{\Omega}_t &= \begin{bmatrix} \mathbf{\Omega}_{xxt} & \mathbf{0} \\ \mathbf{\Omega}_{uxt} & \mathbf{\Omega}_{uut} \end{bmatrix} = \left[1 - g_V\left(\mathbf{s}'_u\mathbf{y}_t\right)\right] \left\{ \begin{array}{c} \left[1 - l_V\left(\mathbf{s}'_x\mathbf{y}_t\right)\right] \begin{bmatrix} \mathbf{\Omega}_{xx1} & \mathbf{0} \\ \mathbf{\Omega}_{ux1} & \mathbf{\Omega}_{uu1} \end{bmatrix} \\ +l_V\left(\mathbf{s}'_x\mathbf{y}_t\right) \begin{bmatrix} \mathbf{\Omega}_{xx2} & \mathbf{0} \\ \mathbf{\Omega}_{ux2} & \mathbf{\Omega}_{uu2} \end{bmatrix} \end{array} \right\} + \\ & \quad g_V\left(\mathbf{s}'_u\mathbf{y}_t\right) \left\{ \begin{array}{c} \left[1 - l_V\left(\mathbf{s}'_x\mathbf{y}_t\right)\right] \begin{bmatrix} \mathbf{\Omega}_{xx3} & \mathbf{0} \\ \mathbf{\Omega}_{ux3} & \mathbf{\Omega}_{uu3} \end{bmatrix} \\ +l_V\left(\mathbf{s}'_x\mathbf{y}_t\right) \begin{bmatrix} \mathbf{\Omega}_{xx4} & \mathbf{0} \\ \mathbf{\Omega}_{ux4} & \mathbf{\Omega}_{uu4} \end{bmatrix} \end{array} \right\}. \end{aligned}$$

Thus

$$\begin{aligned} \mathbf{\Omega}_{xxt} = \epsilon_{xt}^2 &= \left[1 - g_V\left(\mathbf{s}'_u\mathbf{y}_t\right)\right] \left\{\left[1 - l_V\left(\mathbf{s}'_x\mathbf{y}_t\right)\right] \mathbf{\Omega}_{xx1} + l_V\left(\mathbf{s}'_x\mathbf{y}_t\right) \mathbf{\Omega}_{xx2}\right\} + \\ & \quad g_V\left(\mathbf{s}'_u\mathbf{y}_t\right) \left\{\left[1 - l_V\left(\mathbf{s}'_x\mathbf{y}_t\right)\right] \mathbf{\Omega}_{xx3} + l_V\left(\mathbf{s}'_x\mathbf{y}_t\right) \mathbf{\Omega}_{xx4}\right\} \\ \mathbf{\Omega}_{uxt} = \mathbf{G}_{21t}\epsilon_{xt}^2 &= \left[1 - g_V\left(\mathbf{s}'_u\mathbf{y}_t\right)\right] \left\{\left[1 - l_V\left(\mathbf{s}'_x\mathbf{y}_t\right)\right] \mathbf{\Omega}_{ux1} + l_V\left(\mathbf{s}'_x\mathbf{y}_t\right) \mathbf{\Omega}_{ux2}\right\} + \\ & \quad g_V\left(\mathbf{s}'_u\mathbf{y}_t\right) \left\{\left[1 - l_V\left(\mathbf{s}'_x\mathbf{y}_t\right)\right] \mathbf{\Omega}_{ux3} + l_V\left(\mathbf{s}'_x\mathbf{y}_t\right) \mathbf{\Omega}_{ux4}\right\} \end{aligned}$$

which implies $\mathbf{\Omega}_{uxt} = \mathbf{G}_{21t}\mathbf{\Omega}_{xxt}$ and

$$\begin{aligned} \mathbf{G}_{21t} &= \mathbf{\Omega}_{uxt}\mathbf{\Omega}_{xxt}^{-1} \\ &= \left\{ \begin{array}{c} \left[1 - g_V\left(\mathbf{s}'_u\mathbf{y}_t\right)\right] \left\{\left[1 - l_V\left(\mathbf{s}'_x\mathbf{y}_t\right)\right] \mathbf{\Omega}_{ux1} + l_V\left(\mathbf{s}'_x\mathbf{y}_t\right) \mathbf{\Omega}_{ux2}\right\} + \\ g_V\left(\mathbf{s}'_u\mathbf{y}_t\right) \left\{\left[1 - l_V\left(\mathbf{s}'_x\mathbf{y}_t\right)\right] \mathbf{\Omega}_{ux3} + l_V\left(\mathbf{s}'_x\mathbf{y}_t\right) \mathbf{\Omega}_{ux4}\right\} \end{array} \right\} \times \\ & \quad \left\{ \begin{array}{c} \left[1 - g_V\left(\mathbf{s}'_u\mathbf{y}_t\right)\right] \left\{\left[1 - l_V\left(\mathbf{s}'_x\mathbf{y}_t\right)\right] \mathbf{\Omega}_{xx1} + l_V\left(\mathbf{s}'_x\mathbf{y}_t\right) \mathbf{\Omega}_{xx2}\right\} + \\ g_V\left(\mathbf{s}'_u\mathbf{y}_t\right) \left\{\left[1 - l_V\left(\mathbf{s}'_x\mathbf{y}_t\right)\right] \mathbf{\Omega}_{xx3} + l_V\left(\mathbf{s}'_x\mathbf{y}_t\right) \mathbf{\Omega}_{xx4}\right\} \end{array} \right\}^{-1}. \end{aligned}$$

Thus the transformation matrix

$$\mathbf{H}_t^{-1} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{G}_{21t} & \mathbf{I} \end{bmatrix}$$

can then be used to map the VSTAR into a VSTARX. To this end, pre-multiply both sides of (A.1) by $\mathbf{H}_t^{-1}$ to obtain:

$$
\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{G}_{21t} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x}_{t+1} \\ \mathbf{u}_{t+1} \end{bmatrix} =
$$

$$
[1 - g_M(\mathbf{s}_u'\mathbf{y}_t)] \left\{ \begin{array}{c} [1 - l_M(\mathbf{s}_x'\mathbf{y}_t)] \left\{ \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{G}_{21t} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \lambda_{x1} \\ \lambda_{u1} \end{bmatrix} + \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{G}_{21t} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{\Lambda}_{x1}(L) \\ \mathbf{\Lambda}_{u1}(L) \end{bmatrix} \mathbf{y}_t \right\} \\ + l_M(\mathbf{s}_x'\mathbf{y}_t) \left\{ \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{G}_{21t} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \lambda_{x2} \\ \lambda_{u2} \end{bmatrix} + \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{G}_{21t} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{\Lambda}_{x2}(L) \\ \mathbf{\Lambda}_{u2}(L) \end{bmatrix} \mathbf{y}_t \right\} \end{array} \right\} +
$$

$$
g_M(\mathbf{s}_u'\mathbf{y}_t) \left\{ \begin{array}{c} [1 - l_M(\mathbf{s}_x'\mathbf{y}_t)] \left\{ \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{G}_{21t} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \lambda_{x3} \\ \lambda_{u3} \end{bmatrix} + \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{G}_{21t} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{\Lambda}_{x3}(L) \\ \mathbf{\Lambda}_{u3}(L) \end{bmatrix} \mathbf{y}_t \right\} \\ + l_M(\mathbf{s}_x'\mathbf{y}_t) \left\{ \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{G}_{21t} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \lambda_{x4} \\ \lambda_{u4} \end{bmatrix} + \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{G}_{21t} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{\Lambda}_{x4}(L) \\ \mathbf{\Lambda}_{u4}(L) \end{bmatrix} \mathbf{y}_t \right\} \end{array} \right\} +
$$

$$
\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{G}_{21t} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{v}_{xt+1} \\ \mathbf{v}_{ut+1} \end{bmatrix}.
$$

This is solved as

$$
\begin{bmatrix} \mathbf{x}_{t+1} \\ \mathbf{u}_{t+1} - \mathbf{G}_{12t}\mathbf{x}_{t+1} \end{bmatrix} =
$$

$$
[1 - g_M(\mathbf{s}_u'\mathbf{y}_t)] \left\{ \begin{array}{c} [1 - l_M(\mathbf{s}_x'\mathbf{y}_t)] \left\{ \begin{bmatrix} \lambda_{x1} \\ \lambda_{u1} - \mathbf{G}_{12t}\lambda_{x1} \end{bmatrix} + \begin{bmatrix} \mathbf{\Lambda}_{x1}(L) \\ \mathbf{\Lambda}_{u1}(L) - \mathbf{G}_{12t}\mathbf{\Lambda}_{x1}(L) \end{bmatrix} \mathbf{y}_t \right\} + \\ + l_M(\mathbf{s}_x'\mathbf{y}_t) \left\{ \begin{bmatrix} \lambda_{x2} \\ \lambda_{u2} - \mathbf{G}_{12t}\lambda_{x2} \end{bmatrix} + \begin{bmatrix} \mathbf{\Lambda}_{x2}(L) \\ \mathbf{\Lambda}_{u2}(L) - \mathbf{G}_{12t}\mathbf{\Lambda}_{x2}(L) \end{bmatrix} \mathbf{y}_t \right\} \end{array} \right\}
$$

$$
+ g_M(\mathbf{s}_u'\mathbf{y}_t) \left\{ \begin{array}{c} [1 - l_M(\mathbf{s}_x'\mathbf{y}_t)] \left\{ \begin{bmatrix} \lambda_{x3} \\ \lambda_{u3} - \mathbf{G}_{12t}\lambda_{x3} \end{bmatrix} + \begin{bmatrix} \mathbf{\Lambda}_{x3}(L) \\ \mathbf{\Lambda}_{u3}(L) - \mathbf{G}_{12t}\mathbf{\Lambda}_{x3}(L) \end{bmatrix} \mathbf{y}_t \right\} + \\ + l_M(\mathbf{s}_x'\mathbf{y}_t) \left\{ \begin{bmatrix} \lambda_{x4} \\ \lambda_{u4} - \mathbf{G}_{12t}\lambda_{x4} \end{bmatrix} + \begin{bmatrix} \mathbf{\Lambda}_{x4}(L) \\ \mathbf{\Lambda}_{u4}(L) - \mathbf{G}_{12t}\mathbf{\Lambda}_{x4}(L) \end{bmatrix} \mathbf{y}_t \right\} \end{array} \right\} +
$$

$$
\begin{bmatrix} \mathbf{v}_{xt+1} \\ \mathbf{v}_{ut+1} - \mathbf{G}_{12t}\mathbf{v}_{xt+1} \end{bmatrix}.
$$

After moving the term $\mathbf{G}_{12t}\mathbf{x}_{t+1}$ on the right side and multiplying through, the equations for the economy vector can be written in state-space form as a VSTARX:

$$
\begin{aligned}
&\begin{bmatrix} \mathbf{x}_{t+1} \\ \mathbf{u}_{t+1} \end{bmatrix} = \\
&[1-g_M(\mathbf{s}_u'\mathbf{y}_t)] \left\{ \begin{array}{l}
[1-l_M(\mathbf{s}_x'\mathbf{y}_t)] \left\{ \begin{array}{c}
\begin{bmatrix} \lambda_{x1} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\Lambda}_{xx1}(L) & \boldsymbol{\Lambda}_{xu1}(L) - \boldsymbol{\Lambda}_{xu11} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix} \\
+ \begin{bmatrix} \boldsymbol{\Lambda}_{xu11} \\ \mathbf{0} \end{bmatrix} \mathbf{u}_t
\end{array} \right\} + \\
l_M(\mathbf{s}_x'\mathbf{y}_t) \left\{ \begin{array}{c}
\begin{bmatrix} \lambda_{x2} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\Lambda}_{xx2}(L) & \boldsymbol{\Lambda}_{xu2}(L) - \boldsymbol{\Lambda}_{xu21} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix} \\
+ \begin{bmatrix} \boldsymbol{\Lambda}_{xu21} \\ \mathbf{0} \end{bmatrix} \mathbf{u}_t
\end{array} \right\}
\end{array} \right\} \\
&+ \; g_M(\mathbf{s}_u'\mathbf{y}_t) \left\{ \begin{array}{l}
[1-l_M(\mathbf{s}_x'\mathbf{y}_t)] \left\{ \begin{array}{c}
\begin{bmatrix} \lambda_{x3} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\Lambda}_{xx3}(L) & \boldsymbol{\Lambda}_{xu3}(L) - \boldsymbol{\Lambda}_{xu31} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix} \\
+ \begin{bmatrix} \boldsymbol{\Lambda}_{xu31} \\ \mathbf{0} \end{bmatrix} \mathbf{u}_t
\end{array} \right\} + \\
l_M(\mathbf{s}_x'\mathbf{y}_t) \left\{ \begin{array}{c}
\begin{bmatrix} \lambda_{x4} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\Lambda}_{xx4}(L) & \boldsymbol{\Lambda}_{xu4}(L) - \boldsymbol{\Lambda}_{xu41} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix} \\
+ \begin{bmatrix} \boldsymbol{\Lambda}_{xu41} \\ \mathbf{0} \end{bmatrix} \mathbf{u}_t
\end{array} \right\}
\end{array} \right\} + \\
&\begin{bmatrix} \mathbf{v}_{xt+1} \\ \mathbf{0} \end{bmatrix}.
\end{aligned}
$$

In the above, $\boldsymbol{\Lambda}_{xuj}(L) = \sum_{k=1}^{q} \boldsymbol{\Lambda}_{xujk} L^k$, $j = 1, ..., 4$. Thus the term $\boldsymbol{\Lambda}_{xuj1}$ is the matrix of coefficients pertinent to $\mathbf{u}_t$ in each state. This state-space representation for the economy is equivalently written as in (4)-(5). Note that setting in the above all transition functions equal to zero yields a dynamic system equivalent to the structural representation obtained by applying A1 to the VAR model, as described in Section 2.2.

A.2 Assumption A2

Set $\mathbf{G}_{21t} = \mathbf{0}$, which implies $\mathbf{v}_{ut} = \epsilon_{ut}$. Under this restriction (A.1) becomes

$$
\begin{aligned}
\boldsymbol{\Omega}_t &= \begin{bmatrix} \boldsymbol{\Omega}_{xxt} & \boldsymbol{\Omega}_{xut} \\ \mathbf{0} & \boldsymbol{\Omega}_{uut} \end{bmatrix} = [1-g_V(\mathbf{s}_u'\mathbf{y}_t)] \left\{ \begin{array}{c}
[1-l_V(\mathbf{s}_x'\mathbf{y}_t)] \begin{bmatrix} \boldsymbol{\Omega}_{xx1} & \boldsymbol{\Omega}_{xu1} \\ \mathbf{0} & \boldsymbol{\Omega}_{uu1} \end{bmatrix} \\
+ l_V(\mathbf{s}_x'\mathbf{y}_t) \begin{bmatrix} \boldsymbol{\Omega}_{xx2} & \boldsymbol{\Omega}_{xu2} \\ \mathbf{0} & \boldsymbol{\Omega}_{uu2} \end{bmatrix}
\end{array} \right\} + \\
&\quad g_V(\mathbf{s}_u'\mathbf{y}_t) \left\{ \begin{array}{c}
[1-l_V(\mathbf{s}_x'\mathbf{y}_t)] \begin{bmatrix} \boldsymbol{\Omega}_{xx3} & \boldsymbol{\Omega}_{xu3} \\ \mathbf{0} & \boldsymbol{\Omega}_{uu3} \end{bmatrix} \\
+ l_V(\mathbf{s}_x'\mathbf{y}_t) \begin{bmatrix} \boldsymbol{\Omega}_{xx4} & \boldsymbol{\Omega}_{xu4} \\ \mathbf{0} & \boldsymbol{\Omega}_{uu4} \end{bmatrix}
\end{array} \right\}.
\end{aligned}
$$

It follows that:

$$\begin{aligned}\boldsymbol{\Omega}_{xut} = \mathbf{G}_{12t}\epsilon_{ut}^2 = [1 - g_V(\mathbf{s}_u'\mathbf{y}_t)]\left\{[1 - l_V(\mathbf{s}_x'\mathbf{y}_t)]\boldsymbol{\Omega}_{xu1} + l_V(\mathbf{s}_x'\mathbf{y}_t)\boldsymbol{\Omega}_{xu2}\right\} + \\ g_V(\mathbf{s}_u'\mathbf{y}_t)\left\{[1 - l_V(\mathbf{s}_x'\mathbf{y}_t)]\boldsymbol{\Omega}_{xu3} + l_V(\mathbf{s}_x'\mathbf{y}_t)\boldsymbol{\Omega}_{xu4}\right\}, \\ \boldsymbol{\Omega}_{uut} = \epsilon_{ut}^2 = [1 - g_V(\mathbf{s}_u'\mathbf{y}_t)]\left\{[1 - l_V(\mathbf{s}_x'\mathbf{y}_t)]\boldsymbol{\Omega}_{uu1} + l_V(\mathbf{s}_x'\mathbf{y}_t)\boldsymbol{\Omega}_{uu2}\right\} + \\ g_V(\mathbf{s}_u'\mathbf{y}_t)\left\{[1 - l_V(\mathbf{s}_x'\mathbf{y}_t)]\boldsymbol{\Omega}_{uu3} + l_V(\mathbf{s}_x'\mathbf{y}_t)\boldsymbol{\Omega}_{uu4}\right\},\end{aligned}$$

which yields: $\boldsymbol{\Omega}_{xut} = \mathbf{G}_{12t}\boldsymbol{\Omega}_{uut}$. Thus $\mathbf{G}_{12t} = \boldsymbol{\Omega}_{xut}\boldsymbol{\Omega}_{uut}^{-1}$ with

$$\begin{aligned}\boldsymbol{\Omega}_{xut} &= [1 - g_V(\mathbf{s}_u'\mathbf{y}_t)]\left\{[1 - l_V(\mathbf{s}_x'\mathbf{y}_t)]\boldsymbol{\Omega}_{xu1} + l_V(\mathbf{s}_x'\mathbf{y}_t)\boldsymbol{\Omega}_{xu2}\right\} + \\ &\quad g_V(\mathbf{s}_u'\mathbf{y}_t)\left\{[1 - l_V(\mathbf{s}_x'\mathbf{y}_t)]\boldsymbol{\Omega}_{xu3} + l_V(\mathbf{s}_x'\mathbf{y}_t)\boldsymbol{\Omega}_{xu4}\right\} \\ \boldsymbol{\Omega}_{uut} &= [1 - g_V(\mathbf{s}_u'\mathbf{y}_t)]\left\{[1 - l_V(\mathbf{s}_x'\mathbf{y}_t)]\boldsymbol{\Omega}_{uu1} + l_V(\mathbf{s}_x'\mathbf{y}_t)\boldsymbol{\Omega}_{uu2}\right\} + \\ &\quad g_V(\mathbf{s}_u'\mathbf{y}_t)\left\{[1 - l_V(\mathbf{s}_x'\mathbf{y}_t)]\boldsymbol{\Omega}_{uu3} + l_V(\mathbf{s}_x'\mathbf{y}_t)\boldsymbol{\Omega}_{uu4}\right\}.\end{aligned}$$

The solution for $\mathbf{G}_{12t}$ can be used to construct the transformation matrix

$$\mathbf{H}_t^{-1} = \begin{bmatrix} \mathbf{I} & -\mathbf{G}_{12t} \\ \mathbf{0} & \mathbf{I} \end{bmatrix},$$

which can then be used to map the VSTAR into a VSTARX. To this end, pre-multiply both sides of (A.1) by $\mathbf{H}_t^{-1}$ to obtain:

$$\begin{aligned}&\begin{bmatrix} \mathbf{I} & -\mathbf{G}_{12t} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}\begin{bmatrix} \mathbf{x}_{t+1} \\ \mathbf{u}_{t+1} \end{bmatrix} = \\ &[1 - g_M(\mathbf{s}_u'\mathbf{y}_t)]\left\{\begin{array}{l}[1 - l_M(\mathbf{s}_x'\mathbf{y}_t)]\left\{\begin{bmatrix} \mathbf{I} & -\mathbf{G}_{12t} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}\begin{bmatrix} \lambda_{x1} \\ \lambda_{u1} \end{bmatrix} + \begin{bmatrix} \mathbf{I} & -\mathbf{G}_{12t} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}\begin{bmatrix} \boldsymbol{\Lambda}_{x1}(L) \\ \boldsymbol{\Lambda}_{u1}(L) \end{bmatrix}\mathbf{y}_t\right\} \\ +l_M(\mathbf{s}_x'\mathbf{y}_t)\left\{\begin{bmatrix} \mathbf{I} & -\mathbf{G}_{12t} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}\begin{bmatrix} \lambda_{x2} \\ \lambda_{u2} \end{bmatrix} + \begin{bmatrix} \mathbf{I} & -\mathbf{G}_{12t} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}\begin{bmatrix} \boldsymbol{\Lambda}_{x2}(L) \\ \boldsymbol{\Lambda}_{u2}(L) \end{bmatrix}\mathbf{y}_t\right\}\end{array}\right\} \\ &+g_M(\mathbf{s}_u'\mathbf{y}_t)\left\{\begin{array}{l}[1 - l_M(\mathbf{s}_x'\mathbf{y}_t)]\left\{\begin{bmatrix} \mathbf{I} & -\mathbf{G}_{12t} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}\begin{bmatrix} \lambda_{x3} \\ \lambda_{u3} \end{bmatrix} + \begin{bmatrix} \mathbf{I} & -\mathbf{G}_{12t} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}\begin{bmatrix} \boldsymbol{\Lambda}_{x3}(L) \\ \boldsymbol{\Lambda}_{u3}(L) \end{bmatrix}\mathbf{y}_t\right\} \\ +l_M(\mathbf{s}_x'\mathbf{y}_t)\left\{\begin{bmatrix} \mathbf{I} & -\mathbf{G}_{12t} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}\begin{bmatrix} \lambda_{x4} \\ \lambda_{u4} \end{bmatrix} + \begin{bmatrix} \mathbf{I} & -\mathbf{G}_{12t} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}\begin{bmatrix} \boldsymbol{\Lambda}_{x4}(L) \\ \boldsymbol{\Lambda}_{u4}(L) \end{bmatrix}\mathbf{y}_t\right\}\end{array}\right\} + \\ &\begin{bmatrix} \mathbf{I} & -\mathbf{G}_{12t} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}\begin{bmatrix} \mathbf{v}_{xt+1} \\ \mathbf{v}_{ut+1} \end{bmatrix}.\end{aligned}$$

This is solved as

$$
\begin{bmatrix} \mathbf{x}_{t+1} - \mathbf{G}_{12t}\mathbf{u}_{t+1} \\ \mathbf{u}_{t+1} \end{bmatrix} =
$$

$$
\left[1 - g_M\left(\mathbf{s}_u'\mathbf{y}_t\right)\right] \left\{ \begin{array}{r} \left[1 - l_M\left(\mathbf{s}_x'\mathbf{y}_t\right)\right] \left\{ \begin{bmatrix} \lambda_{x1} - \mathbf{G}_{12t}\lambda_{u1} \\ \lambda_{u1} \end{bmatrix} + \begin{bmatrix} \mathbf{\Lambda}_{x1}(L) - \mathbf{G}_{12t}\mathbf{\Lambda}_{u1}(L) \\ \mathbf{\Lambda}_{u1}(L) \end{bmatrix} \mathbf{y}_t \right\} + \\ l_M\left(\mathbf{s}_x'\mathbf{y}_t\right) \left\{ \begin{bmatrix} \lambda_{x2} - \mathbf{G}_{12t}\lambda_{u2} \\ \lambda_{u2} \end{bmatrix} + \begin{bmatrix} \mathbf{\Lambda}_{x2}(L) - \mathbf{G}_{12t}\mathbf{\Lambda}_{u2}(L) \\ \mathbf{\Lambda}_{u2}(L) \end{bmatrix} \mathbf{y}_t \right\} \end{array} \right\}
$$

$$
+ g_M\left(\mathbf{s}_u'\mathbf{y}_t\right) \left\{ \begin{array}{r} \left[1 - l_M\left(\mathbf{s}_x'\mathbf{y}_t\right)\right] \left\{ \begin{bmatrix} \lambda_{x3} - \mathbf{G}_{12t}\lambda_{u3} \\ \lambda_{u3} \end{bmatrix} + \begin{bmatrix} \mathbf{\Lambda}_{x3}(L) - \mathbf{G}_{12t}\mathbf{\Lambda}_{u3}(L) \\ \mathbf{\Lambda}_{u3}(L) \end{bmatrix} \mathbf{y}_t \right\} + \\ l_M\left(\mathbf{s}_x'\mathbf{y}_t\right) \begin{bmatrix} \lambda_{x4} - \mathbf{G}_{12t}\lambda_{u4} \\ \lambda_{u4} \end{bmatrix} + \begin{bmatrix} \mathbf{\Lambda}_{x4}(L) - \mathbf{G}_{12t}\mathbf{\Lambda}_{u4}(L) \\ \mathbf{\Lambda}_{u4}(L) \end{bmatrix} \mathbf{y}_t \end{array} \right\} +
$$

$$
\begin{bmatrix} \mathbf{v}_{xt+1} - \mathbf{G}_{12t}\mathbf{v}_{ut+1} \\ \mathbf{v}_{ut+1} \end{bmatrix}.
$$

The equations for the economy vector can be written in state-space form as a VSTARX:

$$
\begin{bmatrix} \mathbf{x}_{t+1} \\ \mathbf{u}_{t+1} \end{bmatrix} =
$$

$$
\left[1 - g_M\left(\mathbf{s}_u'\mathbf{y}_t\right)\right] \left\{ \begin{array}{r} \left[1 - l_M\left(\mathbf{s}_x'\mathbf{y}_t\right)\right] \left\{ \begin{bmatrix} \lambda_{x1} - \mathbf{G}_{12t}\lambda_{u1} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{\Lambda}_{x1}(L) - \mathbf{G}_{12t}\mathbf{\Lambda}_{u1}(L) \\ \mathbf{0} \end{bmatrix} \mathbf{y}_t \right\} + \\ l_M\left(\mathbf{s}_x'\mathbf{y}_t\right) \left\{ \begin{bmatrix} \lambda_{x2} - \mathbf{G}_{12t}\lambda_{u2} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{\Lambda}_{x2}(L) - \mathbf{G}_{12t}\mathbf{\Lambda}_{u2}(L) \\ \mathbf{0} \end{bmatrix} \mathbf{y}_t \right\} \end{array} \right\}
$$

$$
+ g_M\left(\mathbf{s}_u'\mathbf{y}_t\right) \left\{ \begin{array}{r} \left[1 - l_M\left(\mathbf{s}_x'\mathbf{y}_t\right)\right] \left\{ \begin{bmatrix} \lambda_{x3} - \mathbf{G}_{12t}\lambda_{u3} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{\Lambda}_{x3}(L) - \mathbf{G}_{12t}\mathbf{\Lambda}_{u3}(L) \\ \mathbf{0} \end{bmatrix} \mathbf{y}_t \right\} + \\ l_M\left(\mathbf{s}_x'\mathbf{y}_t\right) \begin{bmatrix} \lambda_{x4} - \mathbf{G}_{12t}\lambda_{u4} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{\Lambda}_{x4}(L) - \mathbf{G}_{12t}\mathbf{\Lambda}_{u4}(L) \\ \mathbf{0} \end{bmatrix} \mathbf{y}_t \end{array} \right\} +
$$

$$
+ \begin{bmatrix} \mathbf{G}_{12t} \\ \mathbf{I} \end{bmatrix} \mathbf{u}_{t+1} + \begin{bmatrix} \mathbf{v}_{xt+1} - \mathbf{G}_{12t}\mathbf{v}_{ut+1} \\ \mathbf{0} \end{bmatrix},
$$

which is equivalently formulated as in (6)-(7). Note that setting in the above all transition functions equal to zero yields a dynamic system equivalent to the structural representation obtained by applying A2 to the VAR model, as described in Section 2.2.

B Solution of the NLQR problem

In any period $t \geq 0$, the value function for the NLQR can be formulated as a time-varying coefficients function: $V\left(\mathbf{y}_t\right) = \mathbf{y}_t'\mathbf{P}_t\mathbf{y}_t + 2\mathbf{y}'\mathbf{p}_t + p_t$. After forming the Bellman equation, replacing the guessed value function on the right side and using the system (9) to form expectations, the NLQR objective becomes:

$$V(\mathbf{y}_t) = \min_{\mathbf{u}_{t+1}} \left\{ \begin{array}{c} (\mathbf{y}_t - \overline{\mathbf{y}})' \mathbf{Q} (\mathbf{y}_t - \overline{\mathbf{y}}) + \\ \beta \left[\begin{array}{c} \mathbf{c}(\mathbf{y}_t) + \mathbf{A}(\mathbf{y}_t)\mathbf{y}_t \\ +\mathbf{B}(\mathbf{y}_t)\mathbf{u}_{t+1} \end{array} \right]' \mathbf{P}_t \left[\begin{array}{c} \mathbf{c}(\mathbf{y}_t) + \mathbf{A}(\mathbf{y}_t)\mathbf{y}_t \\ +\mathbf{B}(\mathbf{y}_t)\mathbf{u}_{t+1} \end{array} \right] + \\ \beta E_t \mathbf{e}'_{t+1} \mathbf{P}_t \mathbf{e}_{t+1} \\ 2\beta \left[\mathbf{c}(\mathbf{y}_t) + \mathbf{A}(\mathbf{y}_t)\mathbf{y}_t + \mathbf{B}(\mathbf{y}_t)\mathbf{u}_{t+1} \right]' \mathbf{p}_t + \beta p_t \end{array} \right\}.$$

Multiplying through gives

$$V(\mathbf{y}_t) = \min_{\mathbf{u}_{t+1}} \left[\begin{array}{c} \mathbf{y}'_t \mathbf{Q} \mathbf{y}_t + \overline{\mathbf{y}}' \mathbf{Q} \overline{\mathbf{y}} - 2\overline{\mathbf{y}}' \mathbf{Q} \mathbf{y}_t + \\ \beta \mathbf{c}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{c}(\mathbf{y}_t) + \beta \mathbf{y}'_t \mathbf{A}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{A}(\mathbf{y}_t) \mathbf{y}_t + \\ \beta \mathbf{u}'_{t+1} \mathbf{B}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{B}(\mathbf{y}_t) \mathbf{u}_{t+1} + 2\beta \mathbf{y}'_t \mathbf{A}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{c}(\mathbf{y}_t) + \\ 2\beta \mathbf{u}'_{t+1} \mathbf{B}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{c}(\mathbf{y}_t) + 2\beta \mathbf{u}'_{t+1} \mathbf{B}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{A}(\mathbf{y}_t) \mathbf{y}_t + \\ \beta tr(\mathbf{\Sigma} \mathbf{P}_t) + 2\beta \mathbf{c}(\mathbf{y}_t)' \mathbf{p}_t + 2\beta \mathbf{x}'_t \mathbf{A}(\mathbf{y}_t)' \mathbf{p}_t + \\ 2\beta \mathbf{u}'_{t+1} \mathbf{B}(\mathbf{y}_t)' \mathbf{p}_t + \beta p_t \end{array} \right].$$

Differentiation of the above w.r.t. $\mathbf{u}_{t+1}$ gives

$$\mathbf{B}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{B}(\mathbf{y}_t) \mathbf{u}_{t+1} + \mathbf{B}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{c}(\mathbf{y}_t) + \mathbf{B}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{A}(\mathbf{y}_t) \mathbf{y}_t + \mathbf{B}(\mathbf{y}_t)' \mathbf{p}_t = \mathbf{0},$$

which yields the solution

$$\mathbf{u}_{t+1} = -\left[\mathbf{B}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{B}(\mathbf{y}_t) \right]^{-1} \left\{ \mathbf{B}(\mathbf{y}_t)' \left[\mathbf{P}_t \mathbf{c}(\mathbf{y}_t) + \mathbf{p}_t \right] + \mathbf{B}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{A}(\mathbf{y}_t) \mathbf{y}_t \right\}.$$

The above is then rewritten in terms of the feedback rule (10) using:

$$\mathbf{k}_t = -\left[\mathbf{B}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{B}(\mathbf{y}_t) \right]^{-1} \mathbf{B}(\mathbf{y}_t)' \left[\mathbf{P}_t \mathbf{c}(\mathbf{y}_t) + \mathbf{p}_t \right], \tag{B.1}$$

$$\mathbf{K}_t = -\left[\mathbf{B}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{B}(\mathbf{y}_t) \right]^{-1} \mathbf{B}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{A}(\mathbf{y}_t). \tag{B.2}$$

Replacing the above solution into the Bellman equation and multiplying through yields:

$$\begin{array}{l} \mathbf{y}'_t \mathbf{P}_t \mathbf{y}_t + 2\mathbf{y}'_t \mathbf{p}_t + p_t \\ = \\ \left[\begin{array}{c} \mathbf{y}'_t \mathbf{Q} \mathbf{y}_t + \overline{\mathbf{y}}' \mathbf{Q} \overline{\mathbf{y}} - 2\overline{\mathbf{y}}' \mathbf{Q} \mathbf{y}_t + \beta \mathbf{c}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{c}(\mathbf{y}_t) + \beta \mathbf{y}'_t \mathbf{A}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{A}(\mathbf{y}_t) \mathbf{y}_t \\ +\beta \mathbf{k}'_t \mathbf{B}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{B}(\mathbf{y}_t) \mathbf{k}_t + \beta \mathbf{y}'_t \mathbf{K}'_t \mathbf{B}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{B}(\mathbf{y}_t) \mathbf{K}_t \mathbf{y}_t + \\ 2\beta \mathbf{y}'_t \mathbf{K}'_t \mathbf{B}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{B}(\mathbf{y}_t) \mathbf{k}_t + 2\beta \mathbf{y}'_t \mathbf{A}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{c}(\mathbf{y}_t) + 2\beta \mathbf{k}'_t \mathbf{B}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{c}(\mathbf{y}_t) \\ +2\beta \mathbf{y}'_t \mathbf{K}'_t \mathbf{B}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{c}(\mathbf{y}_t) + 2\beta \mathbf{k}'_t \mathbf{B}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{A}(\mathbf{y}_t) \mathbf{y}_t + \\ 2\beta \mathbf{y}'_t \mathbf{K}'_t \mathbf{B}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{A}(\mathbf{y}_t) \mathbf{y}_t + \beta tr\left[\mathbf{\Sigma} \mathbf{P}_t \right] + 2\beta \mathbf{c}(\mathbf{y}_t)' \mathbf{p}_t + \\ 2\beta \mathbf{y}'_t \mathbf{A}(\mathbf{y}_t)' \mathbf{p}_t + 2\beta \mathbf{k}'_t \mathbf{B}(\mathbf{y}_t)' \mathbf{p}_t + 2\beta \mathbf{y}'_t \mathbf{K}'_t \mathbf{B}(\mathbf{y}_t)' \mathbf{p}_t + \beta p_t \end{array} \right]. \end{array}$$

Equating the quadratic terms gives:

$$\begin{aligned} \mathbf{P}_t &= \mathbf{Q} + \beta \mathbf{A}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{A}(\mathbf{y}_t) + \beta \mathbf{K}'_t \mathbf{B}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{B}(\mathbf{y}_t) \mathbf{K}_t + \\ &\quad 2\beta \mathbf{K}'_t \mathbf{B}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{A}(\mathbf{y}_t). \end{aligned}$$

Using (B.2), it follows that $\beta \mathbf{K}_t' \mathbf{B}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{B}(\mathbf{y}_t) \mathbf{K}_t = -\beta \mathbf{K}_t' \mathbf{B}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{A}(\mathbf{y}_t)$ and the above simplifies as:

$$\begin{aligned} \mathbf{P}_t &= \mathbf{Q} + \beta \mathbf{A}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{A}(\mathbf{y}_t) - \\ &\quad \beta \mathbf{A}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{B}(\mathbf{y}_t) \left[\mathbf{B}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{B}(\mathbf{y}_t)\right]^{-1} \mathbf{B}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{A}(\mathbf{y}_t). \end{aligned} \tag{B.3}$$

Equating the linear terms gives:

$$\begin{aligned} \mathbf{p}_t &= -\mathbf{Q}\overline{\mathbf{y}} + \beta \mathbf{K}_t' \mathbf{B}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{B}(\mathbf{y}_t) \mathbf{k}_t + \beta \mathbf{A}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{c}(\mathbf{y}_t) + \beta \mathbf{K}_t' \mathbf{B}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{c}(\mathbf{y}_t) + \\ &\quad \beta \mathbf{A}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{B}(\mathbf{y}_t) \mathbf{k}_t + \beta \mathbf{A}(\mathbf{y}_t)' \mathbf{p}_t + \beta \mathbf{K}_t' \mathbf{B}(\mathbf{y}_t)' \mathbf{p}_t. \end{aligned}$$

Using $\beta \mathbf{K}_t' \mathbf{B}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{B}(\mathbf{y}_t) \mathbf{k}_t = -\beta \mathbf{K}_t' \mathbf{B}(\mathbf{y}_t)' [\mathbf{P}_t \mathbf{c}(\mathbf{y}_t) + \mathbf{p}_t]$ the above becomes

$$\begin{aligned} \mathbf{p}_t &= -\mathbf{Q}\overline{\mathbf{y}} - \beta \mathbf{K}_t' \mathbf{B}(\mathbf{y}_t)' [\mathbf{P}_t \mathbf{c}(\mathbf{y}_t) + \mathbf{p}] + \beta \mathbf{A}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{c}(\mathbf{y}_t) + \beta \mathbf{K}_t' \mathbf{B}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{c}(\mathbf{y}_t) + \\ &\quad \beta \mathbf{A}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{B}(\mathbf{y}_t) \mathbf{k}_t + \beta \mathbf{A}(\mathbf{y}_t)' \mathbf{p}_t + \beta \mathbf{K}_t' \mathbf{B}(\mathbf{y}_t)' \mathbf{p}_t, \\ &= -\mathbf{Q}\overline{\mathbf{y}} + \beta \mathbf{A}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{c}(\mathbf{y}_t) + \beta \mathbf{A}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{B}(\mathbf{y}_t) \mathbf{k}_t + \beta \mathbf{A}(\mathbf{y}_t)' \mathbf{p}_t. \end{aligned}$$

Using (B.1) to replace $\mathbf{k}_t$ gives

$$\begin{aligned} \mathbf{p}_t &= -\mathbf{Q}\overline{\mathbf{y}} + \beta \mathbf{A}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{c}(\mathbf{y}_t) - \beta \mathbf{A}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{B}(\mathbf{y}_t) \left[\mathbf{B}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{B}(\mathbf{y}_t)\right]^{-1} \mathbf{B}(\mathbf{y}_t)' [\mathbf{P}_t \mathbf{c}(\mathbf{y}_t) + \mathbf{p}_t] \\ &\quad + \beta \mathbf{A}(\mathbf{y}_t)' \mathbf{p}_t, \\ &= -\mathbf{Q}\overline{\mathbf{y}} + \beta \mathbf{A}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{c}(\mathbf{y}_t) - \beta \mathbf{A}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{B}(\mathbf{y}_t) \left[\mathbf{B}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{B}(\mathbf{y}_t)\right]^{-1} \mathbf{B}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{c}(\mathbf{y}_t) \\ &\quad - \beta \mathbf{A}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{B}(\mathbf{y}_t) \left[\mathbf{B}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{B}(\mathbf{y}_t)\right]^{-1} \mathbf{B}(\mathbf{y}_t)' \mathbf{p}_t + \beta \mathbf{A}(\mathbf{y}_t)' \mathbf{p}_t. \end{aligned}$$

Using (B.2) yields:

$$\mathbf{p}_t = -\mathbf{Q}\overline{\mathbf{y}} + \beta \mathbf{A}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{c}(\mathbf{y}_t) + \beta \mathbf{K}_t' \mathbf{B}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{c}(\mathbf{y}_t) + \beta \mathbf{K}_t \mathbf{B}(\mathbf{y}_t)' \mathbf{p}_t + \beta \mathbf{A}(\mathbf{y}_t)' \mathbf{p}_t,$$

which simplifies as

$$\mathbf{p}_t \left[\mathbf{I} - \beta \mathbf{K}_t' \mathbf{B}(\mathbf{y}_t)' - \beta \mathbf{A}(\mathbf{y}_t)'\right] = -\mathbf{Q}\overline{\mathbf{y}} + \beta \mathbf{A}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{c}(\mathbf{y}_t) + \beta \mathbf{K}_t \mathbf{B}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{c}(\mathbf{y}_t),$$

that is solved as

$$\mathbf{p}_t = \left\{\mathbf{I} - \beta \left[\mathbf{K}_t' \mathbf{B}(\mathbf{y}_t)' - \mathbf{A}(\mathbf{y}_t)'\right]\right\}^{-1} \left\{ \begin{array}{c} \beta \left[\mathbf{A}(\mathbf{y}_t)' + \mathbf{K}_t' \mathbf{B}(\mathbf{y}_t)'\right] \\ \times \mathbf{P}_t \mathbf{c}(\mathbf{y}_t) - \mathbf{Q}\overline{\mathbf{y}} \end{array} \right\}. \tag{B.4}$$

Finally, combining the constant terms gives:

$$\begin{aligned} p_t &= \overline{\mathbf{y}}' \mathbf{Q}\overline{\mathbf{y}} + \beta \mathbf{c}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{c}(\mathbf{y}_t) + \beta \mathbf{k}_t' \mathbf{B}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{B}(\mathbf{y}_t) \mathbf{k}_t + 2\beta \mathbf{k}_t' \mathbf{B}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{c}(\mathbf{y}_t) + \\ &\quad \beta tr\left[\mathbf{\Sigma}_{t+1} \mathbf{P}_t\right] + 2\beta \mathbf{c}(\mathbf{y}_t)' \mathbf{p}_t + 2\beta \mathbf{k}_t' \mathbf{B}(\mathbf{y}_t)' \mathbf{p}_t + \beta p_t, \end{aligned}$$

so that

$$\begin{aligned} p_t(1-\beta) &= \overline{\mathbf{y}}' \mathbf{Q}\overline{\mathbf{y}} + \beta \mathbf{c}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{c}(\mathbf{y}_t) + \beta \mathbf{k}_t' \mathbf{B}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{B}(\mathbf{y}_t) \mathbf{k}_t + \\ &\quad 2\beta \mathbf{k}_t' \mathbf{B}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{c}(\mathbf{y}_t) + \beta tr\left[\mathbf{\Sigma}_{t+1} \mathbf{P}_t\right] + 2\beta \mathbf{c}(\mathbf{y}_t)' \mathbf{p}_t + 2\beta \mathbf{k}_t' \mathbf{B}(\mathbf{y}_t)' \mathbf{p}_t. \end{aligned}$$

Using $\beta \mathbf{k}_t' \mathbf{B}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{B}(\mathbf{y}_t) \mathbf{k}_t = -\beta \mathbf{k}_t' \mathbf{B}(\mathbf{y}_t)' [\mathbf{P}_t \mathbf{c}(\mathbf{y}_t) + \mathbf{p}_t]$ gives:

$$\begin{aligned} p_t(1-\beta) &= \overline{\mathbf{y}}'\mathbf{Q}\overline{\mathbf{y}} + \beta \mathbf{c}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{c}(\mathbf{y}_t) - \beta \mathbf{k}_t' \mathbf{B}(\mathbf{y}_t)' [\mathbf{P}_t \mathbf{c}(\mathbf{y}_t) + \mathbf{p}_t] + \\ & \quad 2\beta \mathbf{k}_t' \mathbf{B}(\mathbf{y}_t)' [\mathbf{P}_t \mathbf{c}(\mathbf{y}_t) + \mathbf{p}_t] + \beta tr[\mathbf{\Sigma}_{t+1}\mathbf{P}_t] + 2\beta \mathbf{c}(\mathbf{y}_t)' \mathbf{p}_t, \end{aligned}$$

which simplifies as

$$\begin{aligned} p_t(1-\beta) &= \overline{\mathbf{y}}'\mathbf{Q}\overline{\mathbf{y}} + \beta \mathbf{c}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{c}(\mathbf{y}_t) + \beta \mathbf{k}_t' \mathbf{B}(\mathbf{y}_t)' [\mathbf{P}_t \mathbf{c}(\mathbf{y}_t) + \mathbf{p}_t] + \\ & \quad \beta tr[\mathbf{\Sigma}_{t+1}\mathbf{P}_t] + 2\beta \mathbf{c}(\mathbf{y}_t)' \mathbf{p}_t \end{aligned}$$

and can then be rewritten as:

$$p_t = (1-\beta)^{-1} \left\{ \begin{array}{c} \overline{\mathbf{y}}'\mathbf{Q}\overline{\mathbf{y}} + \beta \mathbf{c}(\mathbf{y}_t)' \mathbf{P}_t \mathbf{c}(\mathbf{y}_t) + \beta \mathbf{k}_t' \mathbf{B}(\mathbf{y}_t)' \\ [\mathbf{P}_t \mathbf{c}(\mathbf{y}_t) + \mathbf{p}_t] + \beta tr[\mathbf{\Sigma}_{t+1}\mathbf{P}_t] + 2\beta \mathbf{c}(\mathbf{y}_t)' \mathbf{p}_t \end{array} \right\}. \tag{B.5}$$

The optimal feedback rule (10) is a linear function of the state vector with time-varying coefficients, since the Kalman gain $\mathbf{K}_t$ in (B.2) and the intercept term $\mathbf{k}_t$ in (B.1) are both determined by SDC matrices. The feedback rule (10) is derived with the following algorithm in any given $t \geq 0$. First solve the Riccati equation (RE) in (B.3) to obtain $\mathbf{P}_t$. Since the coefficients of $\mathbf{A}(\mathbf{y}_t)$ and $\mathbf{B}(\mathbf{y}_t)$ are predetermined, the RE in (B.3) can be solved, for example, iterating until convergence on $\mathbf{P}_t$. This solution is a function of $\mathbf{y}_t$ and for this reason the RE has to be solved conditional on the value of the state vector $\mathbf{y}_t$. Second, calculate $\mathbf{K}_t$ from equation (B.2). Third, compute $\mathbf{p}_t$ from (B.4). Fourth, use the calculated values for $\mathbf{P}_t$ and $\mathbf{p}_t$ to compute $\mathbf{k}_t$ from (B.1). The optimal feedback policy can then be calculated from equation (10). This gives a time-varying sequence of $\mathbf{u}_{t+1}$ for $t \geq 0$. Finally, compute the sequence of p_t from (B.5) for $t \geq 0$.

The algorithm described above illustrates the simplicity and computational advantages of the SDC method. In any period $t \geq 0$, given $\mathbf{y}_t$, the objects $\mathbf{c}(\mathbf{y}_t)$, $\mathbf{A}(\mathbf{y}_t)$ and $\mathbf{B}(\mathbf{y}_t)$ can be regarded as fixed and the feedback rule coefficients in that period can be computed upon iteration of (B.3), which is effectively a standard algebraic Riccati equation. It is worth observing that in a LQR problem where the constraint of the regulator is a system with fixed coefficients (i.e. $\mathbf{c}(\mathbf{y}_t) = \mathbf{c}$, $\mathbf{A}(\mathbf{y}_t) = \mathbf{A}$ and $\mathbf{B}(\mathbf{y}_t) = \mathbf{B}$), the value function parameters $\mathbf{P}$, $\mathbf{p}$, and p can be computed *offline* before the policy rule in (10) is implemented, since (B.3)-(B.5) depend only on the parameters of the objective function and the time-invariant coefficients $\mathbf{c}$, $\mathbf{A}$ and $\mathbf{B}$. In contrast, the derivation of the SDC solution described above can only be computed *online,* since the parameters of the constraint in (9) depend on the state vector, which in turn changes over time due to the implementation of the feedback rule (10). The *online* solution works as follows. In period $t = 0$, equations (B.1)-(B.5) are solved given $\mathbf{y}_0$ and (10) is used to compute $\mathbf{u}_1$. For $t > 0$, two options are available, depending on whether the state vector is updated with simulated or observed $\mathbf{y}_t$'s. The first consists of replacing the optimal $\mathbf{u}_t$ into (9) to update the state and compute $\mathbf{y}_{t+1}$, which can then be used for the next stage of optimization to derive $\mathbf{u}_{t+1}$, and so on. The second option, which is only feasible for in-sample counterfactual simulation, consists of updating in every period the state vector using its observed value. Clearly, deviations between the observed and the optimal paths of $\mathbf{u}_t$ are smaller when the updating is based on the observed $\mathbf{y}_t$'s. Constraints on some of the policy instruments included in $\mathbf{u}_t$, like for example the nonnegativity of the federal funds rate in Section 5, are implemented in the online solution directly upon computation of the policy vector $\mathbf{u}_t$ in each $t \geq 0$.

C Stability

The VSTAR model under the optimal feedback rule in (12) describes a closed-loop system with time-varying parameters, since the feedback rule coefficients, $\mathbf{k}_t$ and $\mathbf{K}_t$, and the coefficients of the regulator constraint, $\mathbf{c}(\mathbf{y}_t)$, $\mathbf{A}(\mathbf{y}_t)$ and $\mathbf{B}(\mathbf{y}_t)$, are time varying. This system is asymptotically stable if $\mathbf{Q}$ is positive semidefinite and the pair $\{\mathbf{A}(\mathbf{y}_t), \mathbf{B}(\mathbf{y}_t)\}$ is stabilizable for all $\mathbf{y}_t$ over $t \geq 0$ (sufficient conditions). Stabilizability requires that $\mathbf{y}_t$ converges to a fixed point at $t \to \infty$, given any starting value $\mathbf{y}_0$. For any $t \geq 0$, assume the eigenvalues of $\mathbf{\Lambda}_t^*(L)$ to be distinct and consider the time-varying eigenvalue decomposition $\mathbf{\Lambda}_t^*(L) = \mathbf{D}_t \mathbf{R}_t \mathbf{D}_t^{-1}$, where each column of $\mathbf{D}_t$ is an eigenvector of $\mathbf{\Lambda}_t^*(L)$ and $\mathbf{R}_t$ is a diagonal matrix of eigenvalues of $\mathbf{\Lambda}_t^*(L)$. Ignoring the intercept and considering only its deterministic part, the closed-loop system (12) can be written as $\mathbf{y}_{t+1} = \mathbf{D}_t \mathbf{R}_t \mathbf{D}_t^{-1} \mathbf{y}_t$ in any $t \geq 0$. The solution to this difference equation in any given period $t \geq 0$ can be calculated using backward substitution, which yields $\mathbf{y}_j = \mathbf{D}_t \mathbf{R}_t^j \mathbf{D}_t^{-1} \mathbf{y}_0$.[1] This is stable in any given period $t \geq 0$ and for any initial condition $\mathbf{y}_0$ if and only if in that period the eigenvalues in the matrix $\mathbf{A}(\mathbf{y}_t) + \mathbf{B}(\mathbf{y}_t)\mathbf{K}_t$, i.e. the diagonal elements of $\mathbf{R}_t$, are all strictly less than unity in absolute value.

In general, as long as the matrix $\mathbf{Q}$ is positive semidefinite it is always possible for the control solution to stabilize the pair $\{\mathbf{A}(\mathbf{y}_t), \mathbf{B}(\mathbf{y}_t)\}$. Even if the open-loop system (i.e. the VSTAR) is highly unstable in a given period $t \geq 0$, for example dysplaying one or more explosive roots, then the closed-loop system is still stable. Convergence may show poor dynamics in the sense that the control may display an erratic initial path and large swings, thereby requiring the system longer time to settle. If the instruments vector is subject to constraints, these may be binding in some $t \geq 0$ therefore further deteriorating the performance of the control solution. Lewis, Vrabie and Syrmos (2012) suggest that asymptotic stability can be insured more efficiently by employing in any period in which the pair $\{\mathbf{A}(\mathbf{y}_t), \mathbf{B}(\mathbf{y}_t)\}$ is highly unstable another stabilizable pair, $\{\overline{\mathbf{A}}(\mathbf{y}_t), \overline{\mathbf{B}}(\mathbf{y}_t)\}$, whose roots are more stable than the original $\{\mathbf{A}(\mathbf{y}_t), \mathbf{B}(\mathbf{y}_t)\}$. To construct the stabilizable pair $\{\overline{\mathbf{A}}(\mathbf{y}_t), \overline{\mathbf{B}}(\mathbf{y}_t)\}$, one can use the time-invariant pair $\{\mathbf{A}, \mathbf{B}\}$ obtained from the time-invariant version of (1)-(2), i.e. the VAR. If this pair is stabilizable than it can be used as replacement for $\{\mathbf{A}(\mathbf{y}_t), \mathbf{B}(\mathbf{y}_t)\}$ in any period $t \geq 0$ when this is either highly unstable or not stabilizable at all.

Arguably, the choice of the stabilizable pair $\{\overline{\mathbf{A}}(\mathbf{y}_t), \overline{\mathbf{B}}(\mathbf{y}_t)\}$ is not unique and the time-invariant pair $\{\mathbf{A}, \mathbf{B}\}$ is not necessarely the most efficient. For example, the control algorithm could be executed one-period ahead only over a given time horizon, say $t \in (0, T)$ and several stable $\{\mathbf{A}(\mathbf{y}_t), \mathbf{B}(\mathbf{y}_t)\}$ pairs could be identified. Each of these, or some linear combination of them, could be used as stabilizable pair. The preferred pair may be chosen as that yielding less volatility. However, the time-invariant pair $\{\mathbf{A}, \mathbf{B}\}$ has three advantages compared to these alternatives. First, it can be computed offline before knowing the state of the economy and before computing the optimal control rule, as it only requires knowledge of the time invariant matrices $\mathbf{A}$ and $\mathbf{B}$. Second, it is faster to implement as it does not require any preliminary assessment of the many possible stable pairs. Third, the stabilizing

[1] This result is derived in any period $t \geq 0$ keeping $\mathbf{R}_t$ and $\mathbf{D}_t$ fixed and compounding over the upper script j, starting from a given $\mathbf{y}_0$.

pair $\{\mathbf{A},\mathbf{B}\}$ exists as long as a stable VAR can be inferred from the VSTAR, while there is not guarantee that a stable $\{\mathbf{A}(\mathbf{y}_t),\mathbf{B}(\mathbf{y}_t)\}$ pair can be identified over $t \in (0,T)$.

Given the above, stability of the VSTAR under control in the numerical analysis is monitored as follows. First, the numerical algorithm implementing the SDRE solution in (10)-(B.4) is applied. If the resulting loss is lower that that obtained from the simulation of the VSTAR, then the closed-loop solution is kept. Otherwise, the solution is recomputed using the stabilizing pair $\{\mathbf{A},\mathbf{B}\}$ from the VAR in any period $t \geq 0$ in which the VSTAR under control in (2) is unstable. If this still delivers an higher loss, then the the solution is recomputed using the stabilizing pair $\{\mathbf{A},\mathbf{B}\}$ from the VAR in any period $t \geq 0$ in which the open-loop pair $\{\mathbf{A}(\mathbf{y}_t),\mathbf{B}(\mathbf{y}_t)\}$ is unstable.

D Data

Credit risk

- Data sources:
  - Moody's, Moody's Seasoned Baa Corporate Bond Yield [BAA], retrieved from FRED, Federal Reserve Bank of St. Louis;
    https://fred.stlouisfed.org/series/BAA, January 27, 2019.
  - Board of Governors of the Federal Reserve System (US), 10-Year Treasury Constant Maturity Rate [DGS10], retrieved from FRED, Federal Reserve Bank of St. Louis;
    https://fred.stlouisfed.org/series/DGS10, January 27, 2019.
- The credit risk indicator, s, is calculated as the difference between BAA and DGS10.

Inflation

- Data source:
  - U.S. Bureau of Economic Analysis, Personal Consumption Expenditures: Chain-type Price Index, [PCEPI], retrieved from FRED, Federal Reserve Bank of St. Louis;
    https://fred.stlouisfed.org/series/PCEPI, January 27, 2019.
- The inflation rate p is the percent change from year ago of PCEPI, monthly, seasonally adjusted annual rate.

Industrial Production

- Data source:

- Board of Governors of the Federal Reserve System (US), Industrial Production Index [INDPRO], retrieved from FRED, Federal Reserve Bank of St. Louis;

  https://fred.stlouisfed.org/series/INDPRO, January 29, 2019.

- The growth rate of industrial production *ip* is the percent change from year ago of INDPRO, monthly, seasonally adjusted.

Unemployment gap

- Data sources:
  - U.S. Bureau of Labor Statistics, Unemployment Rate [UNRATE], retrieved from FRED, Federal Reserve Bank of St. Louis;

    https://fred.stlouisfed.org/series/UNRATE, January 27, 2019.
  - U.S. Congressional Budget Office, Natural Rate of Unemployment (Short-Term) [NROUST], retrieved from FRED, Federal Reserve Bank of St. Louis;

    https://fred.stlouisfed.org/series/NROUST, January 27, 2019.

- The series UNRATE is in percent, monthly, seasonally adjusted. NROUST is in percent, quarterly, not seasonally adjusted. NROUST is converted in a monthly series using the MATLAB function *interp1.* The unemployment gap *ug* is calculated as the difference between UNRATE and the monthly NROUST.

Federal funds rate

- Data source:
  - Board of Governors of the Federal Reserve System (US), Effective Federal Funds Rate [FEDFUNDS], retrieved from FRED, Federal Reserve Bank of St. Louis;
  - https://fred.stlouisfed.org/series/FEDFUNDS, January 27, 2019.

- The federal funds rate R is the monthly FEDFUNDS.

Fed's Balance Sheet

- Data sources (Total Assets held by the Fed and Total U.S. government securities securities held by the Fed. All data refer to the end of month, are in millions of dollars and not seasonally adjusted):
  - From January 2003 to January 2019, Board of Governors of the Federal Reserve System (US), Assets: Total Assets: Total Assets (Less Eliminations From Consolidation): Wednesday Level [WALCL], retrieved from FRED, Federal Reserve Bank of St. Louis;

    https://fred.stlouisfed.org/series/WALCL, January 27, 2019.

- From June 1996 to December 2002, manually copied from the Consolidated Statement of Condition of All Federal Reserve Banks in the Fed releases of Factors Affecting Reserve Balances - H.4.1;

  https://www.federalreserve.gov/releases/h41/;

- From May 1975 to January 1978, using data downloaded from:

  https://fraser.stlouisfed.org/title/83.

- Data sources (monthly series of nominal GDP):

  - (i) Quarterly data on nominal GDP taken from U.S. Bureau of Economic Analysis, Gross Domestic Product [GDP], retrieved from FRED, Federal Reserve Bank of St. Louis;

    https://fred.stlouisfed.org/series/GDP, January 27, 2019.

    This is monthly interporlated between 1979 and 1992 using the MATLAB function *interp1*;

  - (ii) Monthly data nominal GDP between January 1992 and October 2018 using US Monthly GDP (MGDP) Index from Macroeconomic Advisers by IHS Markit, downloaded from:

    https://ihsmarkit.com/products/us-monthly-gdp-index.html.

- All assets time series are scaled by nominal GDP. TS is the Total U.S. government securities held by the Fed in percent of GDP. PS is the difference between Total Assets held by the Fed in percent of GDP and TS.

E Estimation

The VSTAR in equations (1), (2) and (13) is estimated by adapting the methodology of Auerbach and Gorodnichenko (2012). This works as follows. Consider a draw of the parameters $\mathbf{\Pi}_2$. Conditional on this, it is possible to compute the covariance matrix in (2) and construct the auxiliary vectors $\mathbf{d}_{1t} = [1 - g_M(\mathbf{s}_u'\mathbf{y}_t)][1 - l_M(\mathbf{s}_x'\mathbf{y}_t)]$, $\mathbf{d}_{2t} = [1 - g_M(\mathbf{s}_u'\mathbf{y}_t)]l_M(\mathbf{s}_x'\mathbf{y}_t)$, $\mathbf{d}_{3t} = g_M(\mathbf{s}_u'\mathbf{y}_t)[1 - l_M(\mathbf{s}_x'\mathbf{y}_t)]$, $\mathbf{d}_{4t} = g_M(\mathbf{s}_u'\mathbf{y}_t)l_M(\mathbf{s}_x'\mathbf{y}_t)$. Then the conditional mean of the VSTAR in equation (1) can be written as $\mathbf{y}_{t+1} = \mathbf{\Pi}_1'\mathbf{Y}_t + \mathbf{v}_{t+1}$ where $\mathbf{Y}_t = [\mathbf{d}_{1t}, \mathbf{d}_{1t}\mathbf{y}_t, \mathbf{d}_{2t}, \mathbf{d}_{2t}\mathbf{y}_t, \mathbf{d}_{3t}, \mathbf{d}_{3t}\mathbf{y}_t, \mathbf{d}_{4t}, \mathbf{d}_{4t}\mathbf{y}_t]'$. As a result, the estimation problem reduces to finding the vector $\mathbf{\Pi}_1$ that maximizes $\sum_{t=p^*+1}^{T} \frac{1}{2}(\mathbf{v}_t - \mathbf{\Pi}_1\mathbf{y}_t)'\mathbf{\Sigma}_t^{-1}(\mathbf{v}_t - \mathbf{\Pi}_1\mathbf{y}_t)$. This can be determined analytically via generalized least squares using $\text{vec}(\mathbf{\Pi}_1') = (\sum_{t=p^*+1}^{T} \mathbf{\Sigma}_t^{-1} \otimes \mathbf{y}_t'\mathbf{y}_t)^{-1}\text{vec}(\sum_{t=p^*+1}^{T}(\mathbf{y}_t'\mathbf{v}_t\mathbf{\Sigma}_t^{-1})$. Once $\mathbf{\Pi}_1$ is obtained, the VSTAR log-likelihood can be evaluated from (14). This procedure can be iterated until convergence using standard iterative numerical algorithms.

The algorithm described above is repeated using from one to twelve lags to establish the optimal lag length. The results from this estimation are presented in table E.1. In the table, the first column indicates the number of lags included in the estimated VSTAR; the second the model log-likelihood; the next three columns include the standard AIC,

Table E.1: Statistical fit of the VSTAR under alternative lag lengths.

No. of lags	$\ln L_T(\mathbf{\Pi})$	AIC	HIC	SIC	AIC*	HIC*	SIC*
1	0.89	-0.31	**0.89**	**2.73**	**1.46**	**2.66**	**4.50**
2	1.26	-0.17	1.73	4.66	2.34	4.25	7.18
3	2.32	**-1.38**	1.25	5.30	3.24	5.88	9.93
4	1.02	2.14	5.52	10.72	4.18	7.57	12.76
5	**3.26**	-1.36	2.79	9.15	5.14	9.29	15.65
6	1.41	3.34	8.27	15.83	6.15	11.08	18.64
7	0.64	5.91	11.66	20.45	7.18	12.92	21.71
8	-0.54	9.32	15.89	25.94	8.25	14.82	24.87
9	-0.94	11.23	18.65	29.99	9.36	16.78	28.12
10	0.69	9.11	17.40	30.06	10.49	18.79	31.45
11	0.56	10.55	19.74	33.76	11.67	20.87	34.88
12	0.92	11.05	21.18	36.58	12.89	23.02	38.42

HIC and SIC; the remaining columns report the normalized criteria obtained by scaling the AIC, HIC and SIC by the number of observations according to Lütkepohl and Krätzig (2004). In particular, the formulae used to compute the standard information criteria are AIC $= -2\ln L_T(\mathbf{\Pi}) + 2k/T$, HIC $= -2\ln L_T(\mathbf{\Pi}) + 2\ln(\ln(T))k/T$ and SIC $= -2\ln L_T(\mathbf{\Pi}) + \ln(T)k/T$, where k denotes the number of parameters and T is the number of observations. These scale the parameters penalty (second term on the right side of each formula) by the number of available observations. The normalised criteria are computed by scaling the whole right side in each formula by the number of available observations. The best model fit for each criterion in a given column of Table E.1 is highlighted in bold. According to these results, it is not possible to increase the VSTAR log-likelihood once the model is estimated with six or more lags. The AIC favour a lag length of three. The HIC and SIC, which typically favour more parsimonious specifications, suggest one lag. All normalised criteria in the last three columns favour one lag.

As discussed in the main paper, in the applied macroeconomic literature, there is no agreement on how to specify the mean and the variance of a VSTAR to model nonlinearity in the macroeconomy and in the policy instruments. To validate the proposed VSTAR specification, five alternative specifications are further estimated, each being nested in the unrestricted VSTAR: (i) VSTARC, which constraints the transition parameters in the logistic and gamma functions for the mean and variance to be the same, i.e. $\gamma_{lM} = \gamma_{lV}$ and $\gamma_{gM} = \gamma_{gV}$; (ii) VSTARV, which allows for nonlinearity only in the covariance matrix but keeps a constant mean; (iii) VSTARM, which allows for nonlinearty in the mean but has contant covariance matrix; (iv) VSTARP, which restricts each equation for the QE instruments to follow an AR(1) when the federal funds rate is not at the ZLB, and the equation for the the federal funds rate to follow an AR(1) during the ZLB period; (v) VAR. The VSTARC imposes restrictions on the transition variables similar to those used by Auerbach and Gorodnichenko (2012) and Galvão and Owyang (2018). The VSTARP restricts the dynamics so that QE is exogenous when conventional monetary policy is active, and viceversa, as assumed by Hurn et al. (2022), Sims and Wu (2020) and Sims et al. (2023a). The VSTARM and VSTARV confine nonlinearity to the mean and the covariance matrix of the VSTAR, respectively. The VAR is the linear benchmark.

Table E.2: Statistical fit of the VSTAR and five alternative restricted specifications.

	$\ln L_T(\mathbf{\Pi})$	NoP	AIC	HIC	SIC
VSTAR	0.886	340	-0.306	0.888	2.727
VSTARC	0.751	338	-0.046	1.141	2.970
VSTARP	0.630	304	0.051	1.119	2.763
VSTARV	0.113	170	0.506	1.103	2.023
VSTARM	-18.695	226	38.364	39.158	40.381
VAR	-19.402	56	39.046	39.243	39.546

Notes: $\ln L_T(\mathbf{\Pi})$ is the model log-lkelihood; NoP indicates the number of parameters; AIC, HIC and SIC are standard information criteria.
Source: Author's calculations. See main text for more details.

Table E.2 ranks each estimated model in terms of its log-likelihood, reported in the second column. The other columns report the number of parameters (NoP) and standard information criteria. Overall, the results in Table E.2 validate the use of the specified VSTAR, as this provides the best fit of the data along most criteria. All five restricted specifications are rejected against the unrestricted VSTAR according to a likelihood ratio test with 95 percent confidence. The VSTAR is also the preferred specification under the AIC and the HIC, while it ranks below the VSTARV according to the SIC. Perhaps unsurprisingly given the data dynamics observed in the main paper, the VAR provides the worst fit according to any diagnostic considered in Table E.2. Each model is estimated with full information maximum likelihood (FIML) using standard iterative algorithms.[2]

F Impulse Response Functions

F.1 Algorithm

The algorithm for computing the IRF with sign restrictions includes the following steps:

1. Compute N^Q matrices of random elements, each having the same size of the covariance matrix $\mathbf{\Omega}_t$ in equation (2). Compute N^Q orthogonal matrices $\mathbf{Q}^{nq}$, $nq = 1, ..., N^Q$ each obtained from the QR decomposition of one of these random matrices.

2. Set the sequence of lagged data up to $t-1$ to define the history Γ_{t-1} at date t.

3. Generate a baseline sequence of structural shocks at date t for each variable in $\mathbf{y}_t$ over the time horizon $h = 0, ..., H$. Then generate a sequence of perturbed shocks, which is equal to the baseline except for the shock of interest that is set equal to the value in the baseline plus a pre-specified increase, δ, denoting the magnitude of this shock.

[2] The VSTAR is first estimated using as starting values the coefficients 0.1. The VSTAR coefficients are re-estimated after normalizing the transition variables ug_t and R_t to have zero mean and unit variance, but this does not lead to significant differences in the parameter estimates. The log-likelihood function is maximized using the quasi-Newtown method, the default option in MATLAB *fminunc*.

4. Given Γ_{t-1} and the baseline sequence of shocks in step 3, generate N^Q new paths of realizations of $\mathbf{y}_{t+h}$, $h = 0, ..., H$, by recursively updating the VSTAR model (1)-(2) conditional on Γ_{t-1}, after pre-multiplying the Cholesky decomposition of the covariance matrix (2) by the orthogonal matrices $\mathbf{Q}^{nq}$, $nq = 1, ..., N^Q$.

5. Given Γ_{t-1} and the perturbated sequence of shocks in step 3, generate N^Q new paths of realizations of $\mathbf{y}_{t+h}$, $h = 0, ..., H$, by recursively updating the VSTAR model (1)-(2) conditional on Γ_{t-1}, after pre-multiplying the Cholesky decomposition of the covariance matrix (2) by the orthogonal matrices $\mathbf{Q}^{nq}$, $nq = 1, ..., N^Q$.

6. Subtract each path for $\mathbf{y}_{t+h}$ in step 4 from the corresponding path for $\mathbf{y}_{t+h}$ in step 5, $h = 0, ..., H$. This gives N^Q estimates of the IRFs conditional on Γ_{t-1}.

7. Set aside the IRFs in step 6 that satisfy the required sign restrictions.

8. Since the IRFs in step 7 depends on the particular random draw for the structural shocks in 3, repeat steps 3 to 7 N^R times and compute the median of the resulting IRF estimates. By the law of large numbers, this median converges to the conditional IRFs of $\mathbf{y}_{t+h}$ at horizon $h = 0, 1, ..., H$, to a given shock conditional on Γ_{t-1}.

9. The unconditional IRFs of $\mathbf{y}_{t+h}$ at horizon $h = 0, 1, ..., H$, can be computed by conditioning on the average of the subset of all histories of interest. Alternatively, unconditional IRFs of $\mathbf{y}_{t+h}$ at horizon $h = 0, 1, ..., H$, can also be computed by repeating steps 2-8 over many histories Γ_{t-1}, each of which is randomly drawn with replacement from the original data, and then averaging the values of the resulting conditional IRFs. This second procedure is however more time consuming and computationally intensive than the first.

The IRFs calculated in the paper are based on $H = 24$, $N^Q = N^Q = 1000$ and $\delta = 1$. To simulate the effect of a (negative) credit shock the first element in each sequence of the perturbed shocks in step 3 is set equal to the baseline plus δ. Effectively, this is equivalent to consider increase in the credit spread as a result of a credit shock. Confidence bands are computed using two standard deviations of the median IRF in step 8. The calculation of the IRF under the optimal policy uses the same algorithm described above except that the N^Q new paths of realizations of $\mathbf{y}_{t+h}$, $h = 0, ..., H$, are obtained by recursively updating the VSTAR model in (12) rather than the VSTAR in (1)-(2). The averages of the histories used to compute the unconditional IRFs reported in Table F.1.

F.2 Robustness results

For robustness, Figure F.1 shows how the responses of the policy instruments change once evaluated against four alternatives, considering inclusion of additional variables that might have a bearing on the analysis and different types of sign restrictions. Specifically, column (a) shows the responses to a supply-type shock, such that both the unemployment

Times	Start	End	s	p	ip	ug	TS	PS	R
Normal	Jun-94	May-01	1.91	1.86	4.80	-0.48	4.93	0.79	5.44
Pre-ZLB	Jan-02	Dec-08	2.48	2.45	1.30	0.33	5.28	1.28	2.75
ZLB	Jan-09	Dec-15	3.00	1.29	0.63	2.50	9.81	9.57	0.13
Post-ZLB	Jan-16	Oct-18	2.30	1.64	1.05	-0.22	12.5	10.18	1.01

Table F.1: Histories used for computing IRFs during normal times, pre-ZLB period, ZLB period and post-ZLB period.

gap and inflation increase at the one month horizon; column (b) shows the responses to a demand-type shock, such that the unemployment gap increases while inflation decreases at the one month horizon; column (c) shows the responses to the credit-type shock once the VSTAR is re-estimated after replacing industrial production with the growth rate of real GDP; column (d) the responses to the credit-type shock once the VSTAR is re-estimated after replacing industrial production with the percentage change in the federal debt-to-GDP ratio. Each column reports the confidence bands of the policy instrument responses under a different simulation with black dotted lines for normal times and red dotted lines for the ZLB. Confidence bands of the responses to the credit-type shock from Figure **??** are also included in the background for reference.

As for the credit-type shock, all the responses in these four simulations correctly show how different policy instruments operate at different time periods (active conventional monetary policy and passive QE during normal times, passive conventional monetary policy and active QE at the ZLB). The monetary policy response to the supply shock in column (a) is contractionary during normal times while it is expansionary during the ZLB period. This is because the spread increases with inflation during this simulation. As the reduction in inflation puts downward pressure on the spread, the opposite is observed in column (b), where the monetary policy response to the demand shock is expansionary during normal times and contractionary during the ZLB. The responses of the policy instruments in column (c) show the correct sign on impact, but are way more muted compared to the benchmark reflecting the lower persistence of GDP growth relative to industrial production. Finally, the responses to the credit-type shock obtained once controlling for public debt in column (d) are very similar to those from the benchmark VSTAR. This latter result provides some reassurance on the fact that the observed responses during the ZLB period are the outcome of QE intervention rather than being a mere reflection of fiscal actions.

G Micro-founded Welfare Loss

In this appendix I derive a micro-founded welfare loss function from a structural model that includes a central bank that undertakes unconventional monetary policy, through large-scale asset purchases (quantitative easing, QE), while conventional monetary policy is constrained by the zero lower bound (ZLB). I work with the four-equation New Keynesian model of Sims et al. (2023b). For brevity, I will not reproduce the log-linear equilibrium conditions, these are available in the Appendix of the original paper. I will instead start with a brief outline of the economic environment and of the key aggregate equations, to

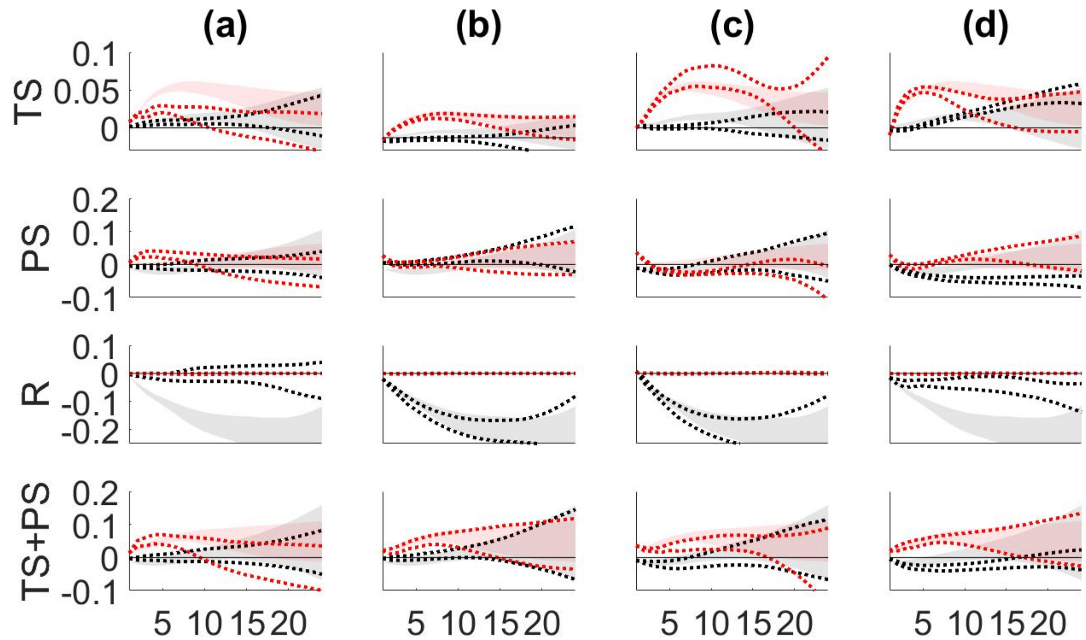

Figure F.1: Dotted lines: Two standard deviation confidence bands of IRFs of policy instruments to a supply shock (a), demand shock (b), credit shock when the VSTAR includes either GDP (c) or debt (d) instead of industrial production; during normal times (black) and the ZLB (red). Shaded areas are confidence bands for the benchmark specification. All variables are in percentage.

then focus on the derivation of the micro-founded welfare loss function.

The model features two types of households, savers and borrowers, who differ in their degree of patience and access to financial markets; producers, who set prices in a staggered fashion; financial intermediaries; and the central bank. Monetary policy includes the short term nominal interest rate and quantitative easing (QE), which consists of long term bonds purchases financed with the creation of reserves. Prices are subject to nominal frictions as in Calvo (1983), with $\theta \in (0,1)$, the share of firms unable to reset prices each period. Final output is a Dixit and Stiglitz (1977) composite good, $Y = \left(\int_0^1 y(\nu)^{1/\epsilon} d\nu\right)^{\epsilon}$, with $\epsilon > 1$, the elasticity of substitution across differentiated intermediate varieties. Savers have separable flow utility functions over their consumption, C, and employment, N, given by $u(C) - v(N) = C^{1-1/\sigma}/(1-1/\sigma) - N^{1+\eta}/(1+\eta)$, with $\sigma > 0$, the elasticity of intertemporal substitution, and $\eta > 0$, the inverse Frisch elasticity of labor supply. Their discount factor is $\beta \in (0,1)$. Borrowers have a higher discount factor, do not earn labor income and can only finance consumption by issuing long-term nominal bonds.

The equilibrium conditions can be approximated around the zero-inflation steady state to derive the Phillips and IS curves that determine the aggregate dynamics of inflation, π_t, and of the output deviation from its steady state, $\widehat{Y}_t \equiv \left(Y_t - \overline{Y}\right)/\overline{Y}$. These curves capture the transmission mechanisms of conventional monetary policy, which controls the nominal interest rate, i_t; and unconventional monetary policy, measured as the deviation of QE from its steady state, $\widehat{\mathtt{QE}}_t \equiv \left(\mathtt{QE}_t - \overline{\mathtt{QE}}\right)/\overline{\mathtt{QE}}$. In particular, the equations describing the Phillips and IS curves are given by

$$\pi_t = \kappa \widehat{Y}_t + \beta E_t \pi_{t+1} - \varsigma \widehat{\mathtt{QE}}_t, \tag{G.1}$$

$$\widehat{Y}_t - \xi \widehat{\mathtt{QE}}_t = E_t \left(\widehat{Y}_{t+1} - \xi \widehat{\mathtt{QE}}_{t+1} \right) - \varphi \left(i_t - E_t \pi_{t+1} - r^n \right). \tag{G.2}$$

The parameter $\kappa \equiv \lambda \left(\frac{1}{\varphi} + \eta \right) > 0$ stands for the elasticity of inflation to changes in the output gap, where $\lambda \equiv (1-\theta)(1-\theta\beta)/\theta$ is the elasticity of inflation with respect to marginal cost, and $\varphi \equiv (1-z)\sigma > 0$ is the intertemporal elasticity of substitution of total household expenditure, with $z \equiv \overline{C^b}/\left(\overline{C} + \overline{C^b}\right)$, the steady-state share of borrower consumption. The parameter $\varsigma \equiv (\lambda/\varphi)\xi$ stands for the elasticity of inflation to changes in QE, with $\xi \equiv z\left(\overline{\mathtt{b}}^{\mathrm{cb}}/\overline{\mathtt{b}}\right) \in (0,1)$ the elasticity of private expenditure to changes in QE, which depends on the size of the central bank's balance sheet in steady state, $\overline{\mathtt{b}}^{\mathrm{cb}}$, as a proportion of the total the steady-state real value of long-term bonds, $\overline{\mathtt{b}}$ in the economy. The parameter $r^n \equiv -\log\beta$ stands for the long-run natural rate of interest.

Deriving an objective function that serves as an approximation of households' utility is not straightforward, as the model of the economy features two types of households with different discount factors. This heterogeneity complicates the formulation of a utilitarian welfare function, see Gollier and Zeckhauser (2005). Specifically, if the central bank maximizes the average discounted sum of the flow utility for each group while adhering to each household's discount factor, this would result in a non-stationary solution where the consumption of the most impatient individuals (borrowers) diminishes over time. To avoid this issue, I assume that the central bank cares equally about both households and uses a common factor, β, to discount the sum of their flow utilities. Further, to evaluate the loss function at the ZLB, I follow Woodford (2011) and assume a two-state Markov environment, with the economy starting at the ZLB with the natural rate of interest $r^n = \overline{r}_L < 0$, and returning to $r^n > 0$, with probability $1-\mu$ each period.

The central bank wishes to maximise the total expected discounted utility of the two agents, as follows

$$E_t \sum_{j=0}^{\infty} \beta^j \left[u\left(C_{t+j}\right) - v\left(N_{t+j}\right) + u\left(C^b_{t+j}\right) \right], \tag{G.3}$$

where $u(C)$, $u\left(C^b\right)$ and $v(N)$ represent, respectively, the per-period utility from consumption for savers, the utility from consumption for borrowers, and the disutility from labor. This can be formulated as:

$$\mathbf{W} = E_t \sum_{j=0}^{\infty} \beta^j \left[u\left(Y_{t+j} - C^b_{t+j}\right) - v\left(N_{t+j}\right) + u\left(C^b_{t+j}\right) \right] = E_t \sum_{j=0}^{\infty} \beta^j \mathbf{w}_{t+j}. \tag{G.4}$$

The central bank is assumed to have enough instruments to implement an efficient (first-best optimal) steady-state equilibrium, satisfying the following conditions: $u'\left(\overline{C}\right) = u'\left(\overline{C}^b\right)$

(which implies $\overline{C} = \overline{C}^b$ and thus $z = 1/2$), $v'\left(\overline{N}\right) = u'\left(\overline{C}\right)$, and $\overline{Y} = \overline{N} = \overline{C} + \overline{C}^b$. Considering small disturbances around the efficient steady state, the loss function of the central bank can be expressed in consumption equivalent, as follows

$$\mathbf{L} = \left\{E_t\left[-\sum_{j=0}^{\infty}\beta^j\left(\frac{\mathbf{w}_{t+j} - \overline{\mathbf{w}}}{u'\left(\overline{C}\right)\overline{C}}\right)\right]\right\} = \left\{E_t\left[\sum_{j=0}^{\infty}\beta^j \mathtt{L}_{t+j}\right]\right\}, \tag{G.5}$$

with $\mathtt{L}_t = -\left(\mathbf{w}_t - \overline{\mathbf{w}}\right) / \left(u'\left(\overline{C}\right)\overline{C}\right)$, the flow loss function of the central bank. The second-order Taylor expansion of $\mathbf{w}_t$ around steady state is given by

$$\begin{aligned} \mathbf{w}_t - \overline{\mathbf{w}} = u'\left(Y_t - \overline{Y}\right) + \frac{u''}{2}\left[\left(Y_t - \overline{Y}\right)^2 + 2\left(C_t^b - \overline{C}^b\right)^2\right] \\ - u''\left[\left(Y_t - \overline{Y}\right) + \left(Y_t - \overline{Y}\right)\left(C_t^b - \overline{C}^b\right)\right] - v'\left(N_t - \overline{N}\right) - \frac{v''}{2}\left(N_t - \overline{N}\right)^2, \end{aligned} \tag{G.6}$$

where all derivatives are evaluated at the steady-state equilibrium, I use the fact that $\overline{C} = \overline{C}^b$, and denote $u'\left(\overline{C}\right) = u'$, $u''\left(\overline{C}\right) = u''$, $v'\left(\overline{N}\right) = v'$, $v''\left(\overline{N}\right) = v''$. Making use of the resource constraint, $Y_t = C_t + C_t^b$, yields

$$\begin{aligned} \mathbf{w}_t - \overline{\mathbf{w}} &= u'\left(Y_t - \overline{Y}\right) + \frac{u''}{2}\left[\left(Y_t - \overline{Y}\right)^2 + 2\left(C_t^b - \overline{C}^b\right)^2\right] \\ &\quad - u''\left[\left(Y_t - \overline{Y}\right) + \left(C_t + C_t^b - \overline{C} - \overline{C}^b\right)\left(C_t^b - \overline{C}^b\right)\right] \\ &\quad - v'\left(N_t - \overline{N}\right) - \frac{v''}{2}\left(N_t - \overline{N}\right)^2, \\ &= u'\left(Y_t - \overline{Y}\right) + \frac{u''}{2}\left[\left(Y_t - \overline{Y}\right)^2\right] - u''\left[\left(Y_t - \overline{Y}\right) + \left(C_t - \overline{C}\right)\left(C_t^b - \overline{C}^b\right)\right] \\ &\quad - v'\left(N_t - \overline{N}\right) - \frac{v''}{2}\left(N_t - \overline{N}\right)^2. \end{aligned} \tag{G.7}$$

Next, applying the approximation $\left(\frac{X_t - \overline{X}}{\overline{X}}\right) \simeq \widehat{X}_t + \frac{\widehat{X}_t^2}{2}$ to each term in (G.7), yields

$$\frac{\mathbf{w}_t - \overline{\mathbf{w}}}{u'\,\overline{Y}} = \widehat{Y}_t + \frac{1}{2}\left(\widehat{Y}_t^2 - \frac{\widehat{Y}_t^2 - 2\widehat{Y}}{\varphi} + \frac{1}{\sigma}\widehat{C}_t\widehat{C}_t^b\right) - \widehat{N}_t - \frac{(1+\eta)\,\widehat{N}_t^2}{2} + \mathcal{O}\left(3\right), \tag{G.8}$$

with $\mathcal{O}(3)$ collecting all the terms of third order or higher, and where I used the fact that $\overline{Y} = 2\overline{C}^b$, and with $1/\varphi = -\left(u''/u'\right)\overline{Y} = 2/\sigma$.

Next, from the equilibrium condition $\widehat{N}_t = \widehat{v}_t^p + \widehat{Y}_t$, it follows that up to a second-order approximation we have

$$\widehat{v}_t^p = \ln\int_0^1\left(\frac{P_t\left(\nu\right)}{P_t}\right)^{-\epsilon}d\nu \simeq \left[\frac{\epsilon \mathrm{var}\left(P_t\left(\nu\right)\right)}{2}\right], \tag{G.9}$$

with $\mathrm{var}\left(\bullet\right)$ that denotes the cross-sectional variance of prices. Substituting $\widehat{N}_t = \widehat{v}_t^p + \widehat{Y}_t$ and (G.9) in (G.8), yields

$$\frac{\mathbf{w}_t - \overline{\mathbf{w}}}{u'\,\overline{Y}} = -\frac{1}{2}\left(\frac{1}{\varphi} + \eta\right)\left(\widehat{Y}_t\right)^2 + \left(\frac{\xi/\sigma}{\overline{\mathtt{b}}^{\mathrm{cb}}/\overline{\mathtt{b}}}\right)\widehat{C}_t\widehat{C}_t^b - \frac{\epsilon \mathrm{var}\left(P_t\left(j\right)\right)}{2} + \mathcal{O}\left(3\right), \tag{G.10}$$

with $z = 1/2$ in the efficient steady state.

Finally, the quadratic approximation to the central bank's welfare function is obtained by making use of the result in Woodford (2003), that

$$\sum_{t=0}^{\infty} \beta^t \text{var}\left(P_t\left(j\right)\right) \simeq \frac{\theta}{\left(1-\beta\theta\right)\left(1-\theta\right)} \sum_{t=0}^{\infty} \beta^t \pi_t^2. \tag{G.11}$$

After combining (G.10) and (G.11), assuming small disturbances around the steady state and the two-state Markov equilibrium structure described above, a quadratic approximation to the central bank's loss function (G.3), valid for small disturbances around the efficient (first-best) steady state, yields

$$\mathtt{L} = \frac{1}{2}\left[\overline{\pi}_L^2 + \frac{\kappa}{\epsilon}\left(\overline{Y}_L\right)^2 + \varsigma \mathtt{I}_L\right], \tag{G.12}$$

where, from now on, the subscript L denotes the value of the corresponding variable at the ZLB. The first two terms in the loss function (G.12) are standard. They capture the *stabilization* motive of the central bank, as measured by welfare losses associated with, in turn, the volatility of inflation and the output gap. The weights attached to each of these two objectives depend on the private's sector parameters. The last term of the loss function (G.12) corresponds to

$$\mathtt{I}_L = -\frac{1}{2}\left(\frac{1-g}{b^{cb}/b}\right)\overline{C}_L\overline{C}_L^b. \tag{G.13}$$

This captures welfare losses accruing due to consumption inequality between borrowers and savers: for a given average consumption, $\left(\overline{C}_L + \overline{C}_L^b\right)/2$, the inequality term $\mathtt{I}_L$ is minimized the more equal are $\overline{C}_L$ and $\overline{C}_L^b$. Consumption inequality affects welfare. This is because a benevolent planner wants to spread the welfare costs of business cycle fluctuations equally among households. Thus, the term $\mathtt{I}_L$ should be seen as capturing a *redistributive* motive by the central bank.

Making use of the economy's resource constraint (in log-linear form), $\widehat{Y}_t = (1-z)\,\widehat{C}_t + z\widehat{C}_t^b$, the borrowers' budget constraint $\widehat{C}_t^b = \left(\frac{\bar{\mathtt{b}}^{\text{cb}}}{\bar{\mathtt{b}}}\right)\widehat{\mathtt{QE}}_t$, alongside the definitions of the parameters φ and ξ, the inequality term (G.13) can be written as

$$\mathtt{I}_L = -\left(\overline{Y}_L - \xi\overline{\mathtt{QE}}_L\right)\overline{\mathtt{QE}}_L. \tag{G.14}$$

As a result the loss function in equation (G.12) can be written as

$$\mathtt{L} = \frac{1}{2}\left[\overline{\pi}_L^2 + \frac{\kappa}{\epsilon}\left(\overline{Y}_L\right)^2 - \varsigma\overline{Y}_L\overline{\mathtt{QE}}_L + \varsigma\xi\overline{\mathtt{QE}}_L^2\right], \tag{G.15}$$

which shows that both the level and volatility of QE matter for welfare.

H Alternative Optimization Approaches

The optimization gains obtained with the SDC method are compared with those from three alternatives.

LIN is based on the VAR estimates and identification under A2. Thus the structural model is that from the VAR under A2 reported in Section 2.2. Since this is linear and homoscedastic, optimization is amenable to dynamic programming. The optimal policy is a linear feedback rule $\mathbf{u}_t = \mathbf{k}_t^{(LIN)} + \mathbf{K}_t^{(LIN)}\mathbf{y}_{t-1}$, whose coefficients can be computed using Riccati equation iteration as in Ljungqvist and Sargent (2018).

MPC is based on the estimated VSTAR and identification under A1. Thus the structural model takes the form of (4). The optimal policy feedback rule $\mathbf{u}_t = \mathbf{k}_t^{(MPC)} + \mathbf{K}_t^{(MPC)}\mathbf{y}_{t-1}$ can be combined in any period t with the system in (4) to form the reduced-form model:

$$\begin{bmatrix} \mathbf{x}_{t+1} \\ \mathbf{u}_{t+1} \end{bmatrix} = \begin{bmatrix} \mathbf{k}_t^{(VSTARA1)} \\ \mathbf{k}_t^{(MPC)} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_t^{(VSTARA1)} \\ \mathbf{K}_t^{(MPC)} \end{bmatrix} \mathbf{y}_t + \begin{bmatrix} \mathbf{e}_{t+1} \\ \mathbf{0} \end{bmatrix}$$

where

$$\begin{aligned} \mathbf{k}_t^{(VSTARA1)} &= [1 - g_M(\mathbf{s}_u'\mathbf{y}_t)][1 - l_M(\mathbf{s}_x'\mathbf{y}_t)][\boldsymbol{\phi}_1 + l_M(\mathbf{s}_x'\mathbf{y}_t)\boldsymbol{\phi}_2] + \\ &\quad g_M(\mathbf{s}_u'\mathbf{y}_t)[1 - l_M(\mathbf{s}_x'\mathbf{y}_t][\boldsymbol{\phi}_3 + l_M(\mathbf{s}_x'\mathbf{y}_t)\boldsymbol{\phi}_4] \\ \mathbf{K}_t^{(VSTARA1)} &= [1 - g_M(\mathbf{s}_u'\mathbf{y}_t)][1 - l_M(\mathbf{s}_x'\mathbf{y}_t)][\boldsymbol{\Phi}_{1*} + l_M(\mathbf{s}_x'\mathbf{y}_t)\boldsymbol{\Phi}_{2*}] + \\ &\quad g_M(\mathbf{s}_u'\mathbf{y}_t)[1 - l_M(\mathbf{s}_x'\mathbf{y}_t][\boldsymbol{\Phi}_{3*} + l_M(\mathbf{s}_x'\mathbf{y}_t)\boldsymbol{\Phi}_{4*}] \\ \boldsymbol{\Phi}_{j*} &= \begin{bmatrix} \boldsymbol{\Phi}_j \\ \boldsymbol{\Gamma}_j \end{bmatrix}, j = 1, 2, 3, 4. \end{aligned}$$

In the above, the coefficients $\mathbf{k}_t^{(VSTARA1)}$ and $\mathbf{K}_t^{(VSTARA1)}$ are those obtained from the estimated VSTAR, whereas the coefficients $\mathbf{k}_t^{(MPC)}$ and $\mathbf{K}_t^{(MPC)}$ are optimized selecting the one period ahead forecast of $\mathbf{y}_t$ minimizes the loss (15).

DNS is also based on the estimated VSTAR and the structural model under A1 in (4). As above, policy is set as the time-varying coefficients feedback rule $\mathbf{u}_t = \mathbf{k}_t^{(DNS)} + \mathbf{K}_t^{(DNS)}\mathbf{x}_{t-1}$ to derive estimates of $\widehat{\mathbf{u}}_t$, and then forming $\mathbf{y}_t = \begin{bmatrix} \mathbf{x}_t' & \widehat{\mathbf{u}}_t' \end{bmatrix}'$. Again the quasi-Newton method is employed to find the coefficients $\mathbf{k}_t^{(DNS)}$ and $\mathbf{K}_t^{(DNS)}$ that minimize the loss (15).

The results for the optimization gains under different methods and samples are presented below. In particular, Table H.1 refers to the SDC method over the whole sample; Tables H.2 and H.3 to the LIN method for the post 2008 and the whole sample, respectively; Tables H.4 and H.5 to the MPC method for the post 2008 and the whole sample, respectively; Tables H.6 and H.7 to the DNS method for the post 2008 and the whole sample, respectively. Table H.8 presents the results about the computational speed from each of the ten estimation trials carried out under different optimization methods and samples. The computations are carried out in MATLAB2021 with a Dell PowerEdge T630 server with 2 x Intel Xeon E5-2643v3 processors, with a 3.4GHz base frequency/3.7GHz Turbo frequency, totalling 12 cores (24 threads), and 320GB of 2133MHz DDR4 RAM, running Windows Server 2019.

Table H.1: Optimization gain, SDC method: 1979-2018

	Volatilities					Loss	Stabilization	
	$(p_t - \overline{p})^2$	ug_t^2	ΔTS_t^2	ΔPS_t^2	ΔR_t^2	V	G	$\widehat{u}$
Policy	**Baseline**: $\upsilon_p = \upsilon_{ug} = \upsilon_{\Delta TS} = \upsilon_{\Delta PS} = \upsilon_{\Delta R} = 1$							
Actual	4.75	2.66	0.32	0.02	0.11	7.85		
Optimal	3.47	2.40	0.35	0.12	0.18	6.53	16.88	1.15
	Weights II: $\upsilon_p = 0.5$, $\upsilon_{ug} = \upsilon_{\Delta TS} = \upsilon_{\Delta PS} = \upsilon_{\Delta R} = 1$							
Actual	4.75	2.66	0.32	0.02	0.11	5.48		
Optimal	3.81	2.35	0.35	0.09	0.15	4.84	11.64	0.80
	Weights III: $\upsilon_{ug} = 0.5$, $\upsilon_p = \upsilon_{\Delta TS} = \upsilon_{\Delta PS} = \upsilon_{\Delta R} = 1$							
Actual	4.75	2.66	0.32	0.02	0.11	6.52		
Optimal	3.44	2.52	0.35	0.11	0.18	5.34	18.07	1.54
	Weights IV: $\upsilon_{ug} = \upsilon_p = 1$, $\upsilon_{\Delta TS} = \upsilon_{\Delta PS} = \upsilon_{\Delta R} = 0.5$							
Actual	4.75	2.66	0.32	0.02	0.11	7.63		
Optimal	3.11	2.31	0.36	0.19	0.26	5.83	23.56	1.34
	Weights V: $\upsilon_{ug} = \upsilon_p = \upsilon_{\Delta R} = 1$, $\upsilon_{\Delta TS} = \upsilon_{\Delta PS} = 0.5$							
Actual	4.75	2.66	0.32	0.02	0.11	7.79		
Optimal	3.12	2.32	0.35	0.19	0.26	6.02	22.74	1.33
	Weights VI: $\upsilon_{ug} = \upsilon_p = \upsilon_{\Delta TS} = \upsilon_{\Delta PS} = 1$, $\upsilon_{\Delta R} = 0.5$							
Actual	4.75	2.66	0.32	0.02	0.11	7.69		
Optimal	3.45	2.40	0.37	0.12	0.19	6.34	17.61	1.16

Table H.2: Optimization gain, LIN method: 2008-2017

	Terms in the loss function					Loss	Stabilization	
	$(p_t - \overline{p})^2$	ug_t^2	ΔTS_t^2	ΔPS_t^2	ΔR_t^2	V	G	$\widehat{u}$
Policy	**Baseline**: $\upsilon_p = \upsilon_{ug} = \upsilon_{\Delta TS} = \upsilon_{\Delta PS} = \upsilon_{\Delta R} = 1$							
Actual	1.24	5.46	0.02	0.05	0.38	7.15		
Optimal	1.03	5.13	0.05	0.04	0.31	6.57	8.12	0.76
	Weights II: $\upsilon_p = 0.5$, $\upsilon_{ug} = \upsilon_{\Delta TS} = \upsilon_{\Delta PS} = \upsilon_{\Delta R} = 1$							
Actual	1.24	5.46	0.02	0.05	0.38	6.53		
Optimal	1.11	5.15	0.04	0.04	0.28	6.07	6.97	0.67
	Weights III: $\upsilon_{ug} = 0.5$, $\upsilon_p = \upsilon_{\Delta TS} = \upsilon_{\Delta PS} = \upsilon_{\Delta R} = 1$							
Actual	1.24	5.46	0.02	0.05	0.38	4.42		
Optimal	1.02	5.24	0.04	0.04	0.31	4.04	8.64	0.87
	Weights IV: $\upsilon_{ug} = \upsilon_p = 1$, $\upsilon_{\Delta TS} = \upsilon_{\Delta PS} = \upsilon_{\Delta R} = 0.5$							
Actual	1.24	5.46	0.02	0.05	0.38	6.92		
Optimal	0.96	4.97	0.07	0.05	0.36	6.17	10.94	0.87
	Weights V: $\upsilon_{ug} = \upsilon_p = \upsilon_{\Delta R} = 1$, $\upsilon_{\Delta TS} = \upsilon_{\Delta PS} = 0.5$							
Actual	1.24	5.46	0.02	0.05	0.38	6.93		
Optimal	0.96	5.12	0.04	0.05	0.36	6.32	8.83	0.78
	Weights VI: $\upsilon_{ug} = \upsilon_p = \upsilon_{\Delta TS} = \upsilon_{\Delta PS} = 1$, $\upsilon_{\Delta R} = 0.5$							
Actual	1.24	5.46	0.02	0.05	0.38	7.14		
Optimal	1.04	4.99	0.08	0.05	0.30	6.41	10.13	0.85

Table H.3: Optimization gain, LIN method: 1979-2018

	Terms in the loss function					Loss	Stabilization	
	$(p_t - \overline{p})^2$	ug_t^2	ΔTS_t^2	ΔPS_t^2	ΔR_t^2	V	G	$\widehat{u}$
Policy	**Baseline**: $\upsilon_p = \upsilon_{ug} = \upsilon_{\Delta TS} = \upsilon_{\Delta PS} = \upsilon_{\Delta R} = 1$							
Actual	4.75	2.66	0.32	0.02	0.11	7.85		
Optimal	3.98	2.50	0.30	0.02	0.17	6.97	11.24	0.94
	Weights II: $\upsilon_p = 0.5$, $\upsilon_{ug} = \upsilon_{\Delta TS} = \upsilon_{\Delta PS} = \upsilon_{\Delta R} = 1$							
Actual	4.75	2.66	0.32	0.02	0.11	5.48		
Optimal	4.31	2.50	0.30	0.02	0.12	5.10	6.86	0.61
	Weights III: $\upsilon_{ug} = 0.5$, $\upsilon_p = \upsilon_{\Delta TS} = \upsilon_{\Delta PS} = \upsilon_{\Delta R} = 1$							
Actual	4.75	2.66	0.32	0.02	0.11	6.52		
Optimal	3.88	2.55	0.30	0.02	0.17	5.65	13.46	1.33
	Weights IV: $\upsilon_{ug} = \upsilon_p = 1$, $\upsilon_{\Delta TS} = \upsilon_{\Delta PS} = \upsilon_{\Delta R} = 0.5$							
Actual	4.75	2.66	0.32	0.02	0.11	7.63		
Optimal	3.66	2.41	0.32	0.03	0.26	6.37	16.50	1.12
	Weights V: $\upsilon_{ug} = \upsilon_p = \upsilon_{\Delta R} = 1$, $\upsilon_{\Delta TS} = \upsilon_{\Delta PS} = 0.5$							
Actual	4.75	2.66	0.32	0.02	0.11	7.79		
Optimal	3.65	2.49	0.30	0.03	0.26	6.59	15.41	1.10
	Weights VI: $\upsilon_{ug} = \upsilon_p = \upsilon_{\Delta TS} = \upsilon_{\Delta PS} = 1$, $\upsilon_{\Delta R} = 0.5$							
Actual	4.75	2.66	0.32	0.02	0.11	7.69		
Optimal	4.00	2.42	0.32	0.02	0.16	6.77	11.99	0.96

Table H.4: Optimization Gains, MPC method: 2008-2018

	Terms in the loss function					Loss	Stabilization	
	$(p_t - \overline{p})^2$	ug_t^2	ΔTS_t^2	ΔPS_t^2	ΔR_t^2	V	G	$\widehat{u}$
Policy	**Baseline**: $\upsilon_p = \upsilon_{ug} = \upsilon_{\Delta TS} = \upsilon_{\Delta PS} = \upsilon_{\Delta R} = 1$							
Actual	1.24	5.46	0.02	0.05	0.38	7.15		
Optimal	1.14	5.44	0.08	0.05	0.38	7.09	0.83	0.24
	Weights II: $\upsilon_p = 0.5$, $\upsilon_{ug} = \upsilon_{\Delta TS} = \upsilon_{\Delta PS} = \upsilon_{\Delta R} = 1$							
Actual	1.24	5.46	0.02	0.05	0.38	6.53		
Optimal	1.14	5.44	0.08	0.05	0.38	6.52	0.14	0.10
	Weights III: $\upsilon_{ug} = 0.5$, $\upsilon_p = \upsilon_{\Delta TS} = \upsilon_{\Delta PS} = \upsilon_{\Delta R} = 1$							
Actual	1.24	5.46	0.02	0.05	0.38	4.42		
Optimal	1.14	5.44	0.08	0.05	0.38	4.37	1.14	0.32
	Weights IV: $\upsilon_{ug} = \upsilon_p = 1$, $\upsilon_{\Delta TS} = \upsilon_{\Delta PS} = \upsilon_{\Delta R} = 0.5$							
Actual	1.24	5.46	0.02	0.05	0.38	6.92		
Optimal	1.14	5.44	0.08	0.05	0.38	6.83	1.29	0.30
	Weights V: $\upsilon_{ug} = \upsilon_p = \upsilon_{\Delta R} = 1$, $\upsilon_{\Delta TS} = \upsilon_{\Delta PS} = 0.5$							
Actual	1.24	5.46	0.02	0.05	0.38	6.93		
Optimal	1.14	5.44	0.08	0.05	0.38	6.87	0.87	0.25
	Weights VI: $\upsilon_{ug} = \upsilon_p = \upsilon_{\Delta TS} = \upsilon_{\Delta PS} = 1$, $\upsilon_{\Delta R} = 0.5$							
Actual	1.24	5.46	0.02	0.05	0.38	7.14		
Optimal	1.14	5.44	0.08	0.05	0.38	7.05	1.27	0.30

Table H.5: Optimization Gains, MPC method: 1979-2018

	Terms in the loss function					Loss	Stabilization	
	$(p_t - \overline{p})^2$	ug_t^2	ΔTS_t^2	ΔPS_t^2	ΔR_t^2	V	G	$\widehat{u}$
Policy	**Baseline**: $\upsilon_p = \upsilon_{ug} = \upsilon_{\Delta TS} = \upsilon_{\Delta PS} = \upsilon_{\Delta R} = 1$							
Actual	4.75	2.66	0.32	0.02	0.11	7.85		
Optimal	4.62	2.64	0.30	0.02	0.11	7.69	2.09	0.41
	Weights II: $\upsilon_p = 0.5$, $\upsilon_{ug} = \upsilon_{\Delta TS} = \upsilon_{\Delta PS} = \upsilon_{\Delta R} = 1$							
Actual	4.75	2.66	0.32	0.02	0.11	5.48		
Optimal	4.62	2.64	0.30	0.02	0.11	5.38	1.81	0.32
	Weights III: $\upsilon_{ug} = 0.5$, $\upsilon_p = \upsilon_{\Delta TS} = \upsilon_{\Delta PS} = \upsilon_{\Delta R} = 1$							
Actual	4.75	2.66	0.32	0.02	0.11	6.52		
Optimal	4.62	2.64	0.30	0.02	0.11	6.37	2.41	0.56
	Weights IV: $\upsilon_{ug} = \upsilon_p = 1$, $\upsilon_{\Delta TS} = \upsilon_{\Delta PS} = \upsilon_{\Delta R} = 0.5$							
Actual	4.75	2.66	0.32	0.02	0.11	7.63		
Optimal	4.62	2.64	0.30	0.02	0.11	7.47	2.03	0.39
	Weights V: $\upsilon_{ug} = \upsilon_p = \upsilon_{\Delta R} = 1$, $\upsilon_{\Delta TS} = \upsilon_{\Delta PS} = 0.5$							
Actual	4.75	2.66	0.32	0.02	0.11	7.79		
Optimal	4.62	2.64	0.31	0.02	0.11	7.63	1.99	0.39
	Weights VI: $\upsilon_{ug} = \upsilon_p = \upsilon_{\Delta TS} = \upsilon_{\Delta PS} = 1$, $\upsilon_{\Delta R} = 0.5$							
Actual	4.75	2.66	0.32	0.02	0.11	7.69		
Optimal	4.62	2.64	0.30	0.02	0.11	7.54	2.02	0.39

Table H.6: Optimization Gains, DNS method: 2008-2018

	Terms in the loss function					Loss	Stabilization	
	$(p_t - \overline{p})^2$	ug_t^2	ΔTS_t^2	ΔPS_t^2	ΔR_t^2	V	G	$\widehat{u}$
Policy	**Baseline**: $\upsilon_p = \upsilon_{ug} = \upsilon_{\Delta TS} = \upsilon_{\Delta PS} = \upsilon_{\Delta R} = 1$							
Actual	1.24	5.46	0.02	0.05	0.38	7.15		
Optimal	1.14	5.44	0.08	0.05	0.38	7.08	0.93	0.26
	Weights II: $\upsilon_p = 0.5$, $\upsilon_{ug} = \upsilon_{\Delta TS} = \upsilon_{\Delta PS} = \upsilon_{\Delta R} = 1$							
Actual	1.24	5.46	0.02	0.05	0.38	6.53		
Optimal	1.14	5.44	0.08	0.05	0.38	6.51	0.25	0.13
	Weights III: $\upsilon_{ug} = 0.5$, $\upsilon_p = \upsilon_{\Delta TS} = \upsilon_{\Delta PS} = \upsilon_{\Delta R} = 1$							
Actual	1.24	5.46	0.02	0.05	0.38	4.42		
Optimal	1.14	5.44	0.08	0.05	0.38	4.36	1.30	0.34
	Weights IV: $\upsilon_{ug} = \upsilon_p = 1$, $\upsilon_{\Delta TS} = \upsilon_{\Delta PS} = \upsilon_{\Delta R} = 0.5$							
Actual	1.24	5.46	0.02	0.05	0.38	6.92		
Optimal	1.14	5.44	0.08	0.05	0.38	6.83	1.34	0.30
	Weights V: $\upsilon_{ug} = \upsilon_p = \upsilon_{\Delta R} = 1$, $\upsilon_{\Delta TS} = \upsilon_{\Delta PS} = 0.5$							
Actual	1.24	5.46	0.02	0.05	0.38	6.93		
Optimal	1.14	5.44	0.08	0.05	0.38	6.87	0.95	0.26
	Weights VI: $\upsilon_{ug} = \upsilon_p = \upsilon_{\Delta TS} = \upsilon_{\Delta PS} = 1$, $\upsilon_{\Delta R} = 0.5$							
Actual	1.24	5.46	0.02	0.05	0.38	7.14		
Optimal	1.14	5.44	0.08	0.05	0.38	7.04	1.30	0.31

Table H.7: Optimization Gains, DNS method: 1979-2018

	Terms in the loss function					Loss	Stabilization	
	$(p_t - \overline{p})^2$	ug_t^2	ΔTS_t^2	ΔPS_t^2	ΔR_t^2	V	G	$\widehat{u}$
Policy	**Baseline**: $\upsilon_p = \upsilon_{ug} = \upsilon_{\Delta TS} = \upsilon_{\Delta PS} = \upsilon_{\Delta R} = 1$							
Actual	4.75	2.66	0.32	0.02	0.11	7.85		
Optimal	4.62	2.64	0.46	0.02	0.11	7.84	0.10	0.09
	Weights II: $\upsilon_p = 0.5$, $\upsilon_{ug} = \upsilon_{\Delta TS} = \upsilon_{\Delta PS} = \upsilon_{\Delta R} = 1$							
Actual	4.75	2.66	0.32	0.02	0.11	5.48		
Optimal	4.62	2.64	0.46	0.02	0.11	5.53	-1.05	n.a.
	Weights III: $\upsilon_{ug} = 0.5$, $\upsilon_p = \upsilon_{\Delta TS} = \upsilon_{\Delta PS} = \upsilon_{\Delta R} = 1$							
Actual	4.75	2.66	0.32	0.02	0.11	6.52		
Optimal	4.62	2.64	0.46	0.02	0.11	6.52	0.00	0.02
	Weights IV: $\upsilon_{ug} = \upsilon_p = 1$, $\upsilon_{\Delta TS} = \upsilon_{\Delta PS} = \upsilon_{\Delta R} = 0.5$							
Actual	4.75	2.66	0.32	0.02	0.11	7.63		
Optimal	4.62	2.64	0.46	0.02	0.11	7.55	1.00	0.28
	Weights V: $\upsilon_{ug} = \upsilon_p = \upsilon_{\Delta R} = 1$, $\upsilon_{\Delta TS} = \upsilon_{\Delta PS} = 0.5$							
Actual	4.75	2.66	0.32	0.02	0.11	7.79		
Optimal	4.62	2.64	0.46	0.02	0.11	7.78	0.10	0.09
	Weights VI: $\upsilon_{ug} = \upsilon_p = \upsilon_{\Delta TS} = \upsilon_{\Delta PS} = 1$, $\upsilon_{\Delta R} = 0.5$							
Actual	4.75	2.66	0.32	0.02	0.11	7.69		
Optimal	4.62	2.64	0.46	0.02	0.11	7.61	0.99	0.28

Table H.8: Computational speed trials

	2008-2018				1978-2018			
Trial	SDC	LIN	MPC	NS	SDC	LIN	MPC	NS
1	0.574	0.559	6.680	4.436	1.499	1.670	21.360	15.394
2	0.463	0.498	6.452	4.378	1.470	1.774	21.746	15.450
3	0.528	0.492	6.498	4.288	1.626	1.872	21.653	15.364
4	0.462	0.490	6.877	4.409	1.501	1.851	21.889	15.590
5	0.524	0.485	6.582	4.380	1.637	1.730	21.667	15.123
6	0.486	0.520	6.567	4.528	1.484	1.778	21.822	15.564
7	0.460	0.495	6.520	4.421	1.518	1.690	21.390	15.517
8	0.467	0.491	6.574	4.494	1.494	1.856	21.241	15.263
9	0.466	0.520	6.531	4.298	1.531	1.701	21.534	15.326
10	0.508	0.483	6.524	4.362	1.630	1.763	21.612	14.918
Average	0.494	0.503	6.580	4.399	1.539	1.768	21.592	15.351

References

Auerbach, A. J. and Y. Gorodnichenko (2012). Measuring the output responses to fiscal policy. *American Economic Journal: Economic Policy 4*(2), 1–27.

Calvo, G. A. (1983). Staggered prices in a utility-maximizing framework. *Journal of Monetary Economics 12*(3), 383–398.

Dixit, A. K. and J. E. Stiglitz (1977). Monopolistic competition and optimum product diversity. *The American economic review 67*(3), 297–308.

Galvão, A. B. and M. T. Owyang (2018). Financial stress regimes and the macroeconomy. *Journal of Money, Credit and Banking 50*(7), 1479–1505.

Gollier, C. and R. Zeckhauser (2005). Aggregation of heterogeneous time preferences. *Journal of Political Economy 113*(4), 878–896.

Hurn, S., N. Johnson, A. Silvennoinen, and T. Teräsvirta (2022). Transition from the Taylor rule to the zero lower bound. *Studies in Nonlinear Dynamics & Econometrics 26*(5), 635–647.

Ljungqvist, L. and T. J. Sargent (2018). *Recursive Macroeconomic Theory*. MIT press.

Lütkepohl, H. and M. Krätzig (2004). *Applied Time Series Econometrics*. Cambridge university press.

Sims, E., J. C. Wu, and J. Zhang (2023a). The four-equation New Keynesian model. *Review of Economics and Statistics 105*(4), 931–947.

Sims, E., J. C. Wu, and J. Zhang (2023b). The four-equation New Keynesian model. *Review of Economics and Statistics 105*(4), 931–947.

Sims, E. R. and J. C. Wu (2020). Are QE and conventional monetary policy substitutable? *International Journal of Central Banking 16*(1), 195–230.

Woodford, M. (2003). *Interest and prices*. Manuscript, Princeton University.

Woodford, M. (2011). Simple analytics of the government expenditure multiplier. *American Economic Journal: Macroeconomics 3*(1), 1–35.