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Research Article

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Berry–Esseen bounds and almost sure CLT for the quadratic variation of the bifractional Brownian motion

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Abstract: Let B be a bifractional Brownian motion with parameters $H \in (0,1)$ and $K \in (0,1]$. For any $n \geq 1$, set $Z_n = \sum_{i=0}^{n-1}[n^{2HK}(B_{(i+1)/n} - B_{i/n})^2 - \mathbb{E}((B_{i+1} - B_i)^2)]$. We use Malliavin calculus and the so-called Stein's method on Wiener chaos introduced by Nourdin and Peccati [11] to derive, in the case when $0 < HK \leq 3/4$, Berry–Esseen-type bounds for the Kolmogorov distance between the law of the correct renormalization V_n of Z_n and the standard normal law. Finally, we study almost sure central limit theorems for the sequence V_n.

Keywords: Kolmogorov distance, central limit theorem, almost sure central limit theorem, bifractional Brownian motion, multiple stochastic integrals, quadratic variation

MSC 2010: 60F05, 60G15, 60H05, 60H07

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1 Introduction

Let $B = (B_t, t \geq 0)$ be a bifractional Brownian motion (bifBm) with parameters $H \in (0,1)$ and $K \in (0,1]$, defined on some probability space $(\Omega, \mathcal{F}, P)$. (Here and in the following, we assume that $\mathcal{F}$ is the sigma-field generated by B.) This means that B is a centered Gaussian process with the covariance function $E[B_s B_t] = R_{H,K}(s,t)$, where

$$R_{H,K}(s,t) = \frac{1}{2^K}((t^{2H} + s^{2H})^K - |t-s|^{2HK}). \tag{1.1}$$

The case $K = 1$ corresponds to the fractional Brownian motion (fBm) with Hurst parameter H. The process B has no stationary increments, but it has the quasi-helix property (in the sense of J. P. Kahane)

$$2^{-K}|t-s|^{2HK} \leq \mathbb{E}(|B_t - B_s|^2) \leq 2^{1-K}|t-s|^{2HK}, \tag{1.2}$$

so B has γ-Hölder continuous paths for any $\gamma \in (0, HK)$ thanks to the Kolmogorov–Centsov theorem and it is a self-similar process, that is, for any constant $a > 0$, the processes $(B_{at}, t \geq 0)$ and $(a^{HK}B_t, t \geq 0)$ have the same distribution. The bifBm B can be extended for $1 < K < 2$ with $H \in (0,1)$ and $HK \in (0,1)$ (see [1]). We refer to [6, 8, 9, 13] for further details on the subject.

An example of an interesting problem related to B is the study of the asymptotic behavior of the quadratic variation of B on $[0,1]$, defined as

$$Z_n = \sum_{i=0}^{n-1}[n^{2HK}(B_{(i+1)/n} - B_{i/n})^2 - \mathbb{E}((B_{i+1} - B_i)^2)], \quad n \geq 1.$$

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Let us consider the correct renormalization V_n of Z_n, given as

$$V_n = \frac{Z_n}{\sqrt{\mathrm{Var}(Z_n)}}. \tag{1.3}$$

Recall that if Y, Z are two real-valued random variables, then the Kolmogorov distance between the law of Y and the law of Z is given by

$$d_{\mathrm{Kol}}(Y, Z) = \sup_{-\infty<z<\infty} |P(Y \le z) - P(Z \le z)|.$$

In the particular case of the fBm (that is, when $K = 1$) and thanks to the seminal works of Breuer and Major [4], Dobrushin and Major [5], Giraitis and Surgailis [7] and Taqqu [14], it is well known that we have the following as $n \to \infty$.

- If $0 < H < 3/4$, then
$$\frac{V_n}{\sigma_H \sqrt{n}} \xrightarrow{\text{law}} \mathcal{N}(0, 1).$$
- If $H = 3/4$, then
$$\frac{V_n}{\sigma_H \sqrt{n \log n}} \xrightarrow{\text{law}} \mathcal{N}(0, 1).$$
- If $H > 3/4$, then
$$\frac{V_n}{n^{2H-1}} \xrightarrow{\text{law}} Z \sim \text{“Hermite random variable”}.$$

Here, $\sigma_H > 0$ denotes an explicit constant depending only on H. Moreover, explicit bounds for the Kolmogorov distance between the law of V_n and the standard normal law are obtained by [11, Theorem 4.1], [3, Theorem 1.2] and [10, Theorem 5.6]. The following cases hold true. For some constant c_H depending only on H, we have the bounds

$$d_{\mathrm{Kol}}(V_n, \mathcal{N}(0, 1)) \le c_H \times \begin{cases} \frac{1}{\sqrt{n}} & \text{if } H \in (0, 5/8), \\ \frac{(\log n)^{3/2}}{\sqrt{n}} & \text{if } H = 5/8, \\ n^{4H-3} & \text{if } H \in (5/8, 3/4), \\ \frac{1}{\sqrt{\log n}} & \text{if } H = 3/4. \end{cases}$$

On other hand, Bercu, Nourdin and Taqqu [2] proved the almost sure central limit theorem (ASCLT) for V_n. Recently, Tudor [15] studied the subfractional Brownian motion case.

Let us now describe the results that we will prove in the present paper. First, in Theorem 3.3, we use Malliavin calculus and Stein's method to derive, in the case when $HK \in (0, 3/4]$, explicit bounds for the Kolmogorov distance between the law of V_n and the standard normal law. More precisely, according to the value of HK, we consider the cases

$$d_{\mathrm{Kol}}(V_n, \mathcal{N}(0, 1)) \le c_{H,K} \times \begin{cases} n^{-1/2} & \text{if } HK \in (0, 1/2], \\ n^{2HK-3/2} & \text{if } HK \in [1/2, 3/4), \\ \frac{1}{\sqrt{\log n}} & \text{if } HK = 3/4, \end{cases}$$

where $c_{H,K}$ is a constant depending only on H and K. In Theorem 4.2, we prove the almost sure central limit theorem for V_n.

The rest of the paper is organized as follows. Section 2 deals with preliminaries concerning Malliavin calculus, Stein's method and related topics needed throughout the paper. Section 3 and Section 4 contain our main results, concerning Berry–Esseen bounds and the ASCLT for the quadratic variation of the bifractional Brownian motion.

2 Preliminaries

In this section, we briefly recall some basic facts concerning Gaussian analysis and Malliavin calculus that are used in this paper; we refer to [12] for further details. Let $\mathfrak{H}$ be a real separable Hilbert space. For any $q \geq 1$, we denote by $\mathfrak{H}^{\otimes q}$ (resp. $\mathfrak{H}^{\odot q}$) the qth tensor product (resp. qth symmetric tensor product) of $\mathfrak{H}$. We write $X = \{X(h), h \in \mathfrak{H}\}$ to indicate a centered isonormal Gaussian process on $\mathfrak{H}$. This means that X is a centered Gaussian family, defined on some probability space $(\Omega, \mathcal{F}, P)$ and such that $E[X(g)X(h)] = \langle g, h\rangle_{\mathfrak{H}}$ for every $g, h \in \mathfrak{H}$. (Here and in the following, we assume that $\mathcal{F}$ is the sigma-field generated by X.)

For every $q \geq 1$, let $\mathcal{H}_q$ be the qth Wiener chaos of X, that is, the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_q(X(h)), h \in \mathfrak{H}, \|h\|_{\mathfrak{H}} = 1\}$, where H_q is the qth Hermite polynomial defined as

$$H_q(x) = (-1)^q e^{x^2/2} \frac{d^q}{dx^q}(e^{-x^2/2}).$$

The mapping $I_q(h^{\otimes q}) = H_q(X(h))$ provides a linear isometry between the symmetric tensor product $\mathfrak{H}^{\odot q}$ (equipped with the modified norm $\|\cdot\|_{\mathfrak{H}^{\odot q}} = \sqrt{q!}\|\cdot\|_{\mathfrak{H}^{\otimes q}}$) and $\mathcal{H}_q$. Specifically, for all $f, g \in \mathfrak{H}^{\odot q}$ and $q \geq 1$, one has

$$E[I_q(f)I_q(g)] = q!\langle f, g\rangle_{\mathfrak{H}^{\otimes q}}. \tag{2.1}$$

On the other hand, it is well known that any random variable Z belonging to $L^2(\Omega)$ admits the chaotic expansion

$$Z = E[Z] + \sum_{q=1}^{\infty} I_q(f_q), \tag{2.2}$$

where the series converges in $L^2(\Omega)$ and the kernels f_q, belonging to $\mathfrak{H}^{\odot q}$, are uniquely determined by Z.

Let $\{e_k, k \geq 1\}$ be a complete orthonormal system in $\mathfrak{H}$. Given $f \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$, for every $r = 0, \dots, p \wedge q$, the rth contraction of f and g is the element of $\mathfrak{H}^{\otimes(p+q-2r)}$ defined as

$$f \otimes_r g = \sum_{i_1=1,\dots,i_r=1}^{\infty} \langle f, e_{i_1} \otimes \cdots \otimes e_{i_r}\rangle_{\mathfrak{H}^{\otimes r}} \otimes \langle g, e_{i_1} \otimes \cdots \otimes e_{i_r}\rangle_{\mathfrak{H}^{\otimes r}}.$$

In particular, note that $f \otimes_0 g = f \otimes g$ and, moreover, $f \otimes_p g = \langle f, g\rangle_{\mathfrak{H}^{\otimes p}}$ when $p = q$. Since, in general, the contraction $f \otimes_r g$ is not necessarily symmetric, we denote its symmetrization by $f \widetilde{\otimes}_r g \in \mathfrak{H}^{\odot(p+q-2r)}$. When $f \in \mathfrak{H}^{\odot q}$, we write $I_q(f)$ to indicate its qth multiple integral with respect to X. If $f \in \mathfrak{H}^{\odot p}$ and $f \in \mathfrak{H}^{\odot q}$, then the formula

$$I_p(f)I_q(g) = \sum_{r=0}^{p\wedge q} r!\binom{p}{r}\binom{q}{r} I_{p+q-2r}(f \widetilde{\otimes}_r g) \tag{2.3}$$

is useful in computing the product of such multiple integrals.

Let $\mathcal{S}$ be the set of all smooth cylindrical random variables, that is, ones that can be expressed as $F = f(X(\phi_1), \dots, X(\phi_n))$ for $n \geq 1$, where $f : \mathbb{R}^n \to \mathbb{R}$ is a $\mathcal{C}^\infty$-function such that f and all its derivatives have at most polynomial growth and $\phi_i \in \mathfrak{H}$. The Malliavin derivative of F with respect to X is the square integrable $\mathfrak{H}$-valued random variable defined as

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(X(\phi_1), \dots, X(\phi_n))\phi_i.$$

In particular, $DX(h) = h$ for every $h \in \mathfrak{H}$. As usual, $\mathbb{D}^{1,2}$ denotes the closure of the set of smooth random variables with respect to the norm

$$\|F\|_{1,2}^2 = E[F^2] + E[\|DF\|_{\mathfrak{H}}^2].$$

The Malliavin derivative D verifies the chain rule, that is, if $\varphi : \mathbb{R}^n \to \mathbb{R}$ is $\mathcal{C}_b^1$ and if $(F_i)_{i=1,\dots,n}$ is a sequence of elements of $\mathbb{D}^{1,2}$, then $\varphi(F_1, \dots, G_n) \in \mathbb{D}^{1,2}$ and we have

$$D\varphi(F_1, \dots, G_n) = \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i}(F_1, \dots, G_n)DF_i.$$

Recall the following results concerning CLT and ASCLT for multiple stochastic integrals.

Theorem 2.1 (Nourdin and Peccati [11]). *Let $q \geq 2$ be an integer and let $F = I_q(f)$ with $f \in \mathfrak{H}^{\odot q}$. Then,*

$$d_{\mathrm{Kol}}(F, N) \leq \sqrt{E\Big[\Big(1 - \frac{1}{q}\|DF\|^2_{\mathcal{H}}\Big)^2\Big]}, \tag{2.4}$$

where $N \sim \mathcal{N}(0, 1)$.

Theorem 2.2 (Bercu, Nourdin and Taqqu [2]). *Let $q \geq 2$ be an integer and let $\{G_n\}_{n\geq 1}$ be a sequence of the form $G_n = I_q(f_n)$ with $f_n \in \mathfrak{H}^{\odot q}$. Assume that $E[G_n^2] = q!\|f_n\|^2_{\mathfrak{H}^{\otimes q}} = 1$ for all n and that*

$$G_n \xrightarrow{\text{law}} N \sim \mathcal{N}(0, 1)$$

as $n \to \infty$. If the conditions

(i) $\sum_{n=2}^{\infty} \frac{1}{n \log^2 n} \sum_{k=1}^{n} \frac{1}{k} \|f_k \otimes_r f_k\|_{\mathfrak{H}^{\otimes 2(q-r)}} < \infty$ *for every* $1 \leq r \leq q - 1$,

(ii) $\sum_{n=2}^{\infty} \frac{1}{n \log^3 n} \sum_{k,l=1}^{n} \frac{|\langle f_k, f_l\rangle_{\mathfrak{H}^{\otimes q}}|}{kl} < \infty$

are satisfied, then $\{G_n\}_{n\geq 1}$ satisfies an ASCLT. In other words, almost surely, for any bounded and continuous function $\varphi : \mathbb{R} \to \mathbb{R}$,

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \varphi(G_k) \to \mathbb{E}\varphi(N)$$

as $n \to \infty$.

From now, assume on one hand that $X = B$ is a bifBm with parameters $H \in (0, 1)$ and $K \in (0, 1]$ and on the other hand that $\mathfrak{H}$ is a real separable Hilbert space defined by denoting the set of all $\mathbb{R}$-valued step functions on $[0, \infty)$ by $\mathcal{E}$ and by defining $\mathfrak{H}$ as the Hilbert space obtained by closing $\mathcal{E}$ with respect to the scalar product

$$\langle 1_{[0,s]}, 1_{[0,t]}\rangle_{\mathfrak{H}} = R_{H,K}(s, t) = \frac{1}{2^K}((t^{2H} + s^{2H})^K - |t - s|^{2HK}).$$

In particular, one has $B_t = B(1_{[0,t]})$.

3 Berry–Esseen bounds in the CLT for the quadratic variation of the bifBm

In this section, we prove that a CLT holds for every $HK \in (0, 3/4]$, where V_n was defined in (1.3). Using Stein's method, we also derive the Berry–Esseen bounds for this convergence.

3.1 General setup

Let us define

$$\theta(i, j) = 2^{-K}(\gamma(i, j) + \rho(i - j)), \quad i, j \in \mathbb{N},$$

where

$$\gamma(i, j) = ((i + 1)^{2H} + (j + 1)^{2H})^K - (i^{2H} + (j + 1)^{2H})^K - ((i + 1)^{2H} + j^{2H})^K + (i^{2H} + j^{2H})^K \tag{3.1}$$

and

$$\rho(r) = |r + 1|^{2HK} + |r - 1|^{2HK} - 2|r|^{2HK}, \quad r \in \mathbb{Z}. \tag{3.2}$$

Observe that the function γ is symmetric and that $\rho(0) = 2$, $\rho(x) = \rho(-x)$ and ρ behaves asymptotically as

$$\rho(r) = 2HK(2HK - 1)|r|^{2HK-2}, \quad |r| \to \infty. \tag{3.3}$$

In particular, $\sum_{r\in\mathbb{Z}} \rho^2(r) < \infty$ if and only if $HK \in (0, 3/4)$.

We will use the notation

$$\delta_{k/n} = 1_{[k/n,(k+1)/n]} \quad \text{and} \quad \sigma = \sqrt{\frac{1}{8}\sum_{r\in\mathbb{Z}} \rho^2(r)}. \tag{3.4}$$

Using the self-similarity property of B and (1.1), we deduce that

$$n^{2HK}\langle \delta_{i/n}, \delta_{j/n}\rangle_{\mathfrak{H}} = n^{2HK}\mathbb{E}((B_{(i+1)/n} - B_{i/n})(B_{(j+1)/n} - B_{j/n})) = \mathbb{E}((B_{i+1} - B_i)(B_{j+1} - B_j)) = \theta(i,j).$$

Hence, we can write the quadratic variation of B with respect to a subdivision

$$\pi_n = \left\{0 < \frac{1}{n} < \frac{2}{n} < \cdots < 1\right\}$$

of $[0,1]$ as

$$\begin{aligned} Z_n &= \sum_{k=0}^{n-1} [n^{2HK}(B_{(k+1)/n} - B_{k/n})^2 - \theta(k,k)] \\ &= \sum_{k=0}^{n-1} [n^{2HK}(I_1(\delta_{k/n}))^2 - \theta(k,k)] \\ &= I_2\Bigg(\underbrace{n^{2HK}\sum_{k=0}^{n-1}\delta_{k/n}^{\otimes 2}}_{g_n}\Bigg) \\ &= I_2(g_n). \end{aligned} \tag{3.5}$$

Thus, we can also write the correct renormalization V_n of Z_n, defined in (1.3), as

$$V_n = \frac{Z_n}{\sqrt{\operatorname{Var}(Z_n)}} = \frac{I_2(g_n)}{\sqrt{\operatorname{Var}(Z_n)}}. \tag{3.6}$$

Before computing the Kolmogorov distance, we start with the following results which are used throughout the paper. Here and in the following, the notation $a_n \trianglelefteq b_n$ means that $\sup_{n\geq 1} |a_n|/|b_n| < \infty$.

Lemma 3.1. *The following assertions hold true.*

(i) *Fixing $y \geq 0$ (resp. $x \geq 0$), the function $x \to \gamma(x,y)$ (resp. $y \to \gamma(x,y)$), defined in* (3.1), *is increasing for $H \in (0, 1/2]$.*

(ii) *For any $H \in (0,1)$ and $K \in (0,1]$, the function γ is negative and, for j large, we have*

$$\gamma(0,j) \sim c_{H,K} j^{2HK-2}, \tag{3.7}$$

$$\gamma(j,j) \sim c_{H,K} j^{2HK-2}. \tag{3.8}$$

If $j \leq l$, then

$$|\gamma(j,l)| \leq c_{H,K} l^{2HK-2}, \tag{3.9}$$

where $c_{H,K}$ is an explicit constant depending only on H and K.

Proof. For (i), fixing $y \geq 0$ gives

$$\begin{aligned} \frac{\partial\gamma}{\partial x}(x,y) &= 2HK(x+1)^{2H-1}[((x+1)^{2H} + (y+1)^{2H})^{K-1} - ((x+1)^{2H} + y^{2H})^{K-1}] \\ &\quad - 2HKx^{2H-1}[(x^{2H} + (y+1)^{2H})^{K-1} - (x^{2H} + y^{2H})^{K-1}] \\ &= 2HK[g(1+x) - g(x)], \end{aligned} \tag{3.10}$$

where

$$g(x) = x^{2H-1}[(x^{2H} + (y+1)^{2H})^{K-1} - (x^{2H} + y^{2H})^{K-1}].$$

If $H \in (0, 1/2]$ and $K \in (0, 1]$, then γ is increasing since the function g is increasing in $(0, \infty)$. Indeed,

$$\begin{aligned} g'(x) &= (2H-1)x^{2H-2}[(x^{2H} + (y+1)^{2H})^{K-1} - (x^{2H} + y^{2H})^{K-1}] \\ &\quad + 2H(K-1)x^{4H-2}[(x^{2H} + (y+1)^{2H})^{K-2} - (x^{2H} + y^{2H})^{K-2}] \\ &\geq 0. \end{aligned}$$

For (ii), in order to show that γ is negative, it suffices to remark the decreasing property of the function

$$p : x \in [0, \infty) \to (a+x)^K - (b+x)^K.$$

By a straightforward expansion of the function γ, we can easily prove (3.7) and (3.8).

If $H \leq 1/2$, by (i), the function $x \to |\gamma(x, y)|$ is decreasing. Thus, we deduce that

$$|\gamma(k, l)| \leq |\gamma(0, l)| \sim c_{H,K} l^{2HK-2}.$$

If $H > 1/2$, we rewrite γ as $\gamma(k, l) = g_k(1+l) - g_k(l)$, where $g_k(x) := ((k+1)^{2H} + x^{2H})^K - (k^{2H} + x^{2H})^K$. Applying the mean value theorem, for some $x_{k,l} \in [l, l+1]$, we obtain

$$\begin{aligned} |\gamma(k, l)| &= 2HKx_{k,l}^{2H-1}[(x_{k,l}^{2H} + k^{2H})^{K-1} - (x_{k,l}^{2H} + (k+1)^{2H})^{K-1}] \\ &\leq 2HK(l+1)^{2H-1}[(l^{2H} + k^{2H})^{K-1} - (l^{2H} + (k+1)^{2H})^{K-1}]. \end{aligned}$$

Again, by the mean value theorem on $y \to (l^{2H} + y^{2H})^{K-1}$, for some $y_{k,l} \in [k, k+1]$, we have

$$[(l^{2H} + k^{2H})^{K-1} - (l^{2H} + (k+1)^{2H})^{K-1}] = 2H(K-1)y_{k,l}^{2H-1}[l^{2H} + y_{k,l}^{2H}]^{K-2}.$$

Consequently, for $k \leq l$, we have

$$|\gamma(k, l)| \leq 4H^2K(1-K)(l+1)^{2H-1}(k+1)^{2H-1}[l^{2H} + k^{2H}]^{K-2} \leq c_{H,K} l^{2HK-2}$$

and (ii) follows. □

Proposition 3.2. *Let Z_n be the sequence defined in* (3.5) *and let σ be the constant given by* (3.4).

(i) *Assume that* $0 < HK < 3/4$. *Then,*

$$\frac{\operatorname{Var}(Z_n)}{4^{2-K} n\sigma^2} \to 1 \tag{3.11}$$

as $n \to \infty$.

(ii) *Assume that* $HK = 3/4$. *Then,*

$$\frac{\operatorname{Var}(Z_n)}{4^{2-K}\sigma^2 n \log n} \to 1 \tag{3.12}$$

as $n \to \infty$.

Proof. To show (3.11), we write

$$\begin{aligned} \frac{\operatorname{Var}(Z_n)}{4^{2-K} n\sigma^2} - 1 &= \frac{n^{-1}}{4^{2-K}\sigma^2}\mathbb{E}[I_2^2(g_n)] - 1 \\ &= \frac{n^{-1}}{2^{3-2K}\sigma^2}\|g_n\|_{\mathfrak{H}^{\otimes 2}}^2 - 1 \\ &= \frac{n^{4HK-1}}{2^{3-2K}\sigma^2}\sum_{k,l=0}^{n-1}\langle \delta_{k/n}^{\otimes 2}, \delta_{l/n}^{\otimes 2}\rangle_{\mathfrak{H}^{\otimes 2}} - 1 \\ &= \frac{n^{4HK-1}}{2^{3-2K}\sigma^2}\sum_{k,l=0}^{n-1}\langle \delta_{k/n}, \delta_{l/n}\rangle_{\mathfrak{H}}^2 - 1 \\ &= \frac{n^{-1}}{2^{3-2K}\sigma^2}\sum_{k,l=0}^{n-1}\theta^2(k, l) - 1 \\ &= \frac{n^{-1}}{8\sigma^2}\sum_{k,l=0}^{n-1}\gamma^2(k, l) + \bigg(\frac{n^{-1}}{8\sigma^2}\sum_{k,l=0}^{n-1}\rho^2(k-l) - 1\bigg) + \frac{n^{-1}}{4\sigma^2}\sum_{k,l=0}^{n-1}\gamma(k, l)\rho(k-l) \\ &=: J_1(n) + J_2(n) + J_3(n). \end{aligned}$$

As in the proof of [11, Theorem 4.1], we have

$$J_2(n) \to 0 \tag{3.13}$$

as $n \to \infty$. On the other hand,

$$J_1(n) = \frac{n^{-1}}{8\sigma^2}\sum_{k,l=0}^{n-1}\gamma^2(k,l) = \frac{n^{-1}}{8\sigma^2}\sum_{k=0}^{n-1}\gamma^2(k,k) + \frac{n^{-1}}{4\sigma^2}\sum_{0\le k<l\le n-1}\gamma^2(k,l) =: J_{1,1}(n) + J_{1,2}(n).$$

By (3.8), the sum

$$J_{1,1}(n) = \frac{n^{-1}}{8\sigma^2}\sum_{k=0}^{n-1}\gamma^2(k,k)$$

behaves as

$$\frac{n^{-1}}{8\sigma^2}\sum_{k=0}^{n-1}k^{4HK-4},$$

which goes to zero as $n \to \infty$, since $HK < 3/4$. Thus,

$$J_{1,1}(n) \to 0 \tag{3.14}$$

as $n \to \infty$. Now, we study the convergence of $J_{1,2}(n)$. We first fix two positive constants α and β such that $\alpha + \beta = 1$ and $4HK - 2 < \beta < 1$. We deduce from (3.9) that

$$J_{1,2}(n) = \frac{n^{-1}}{4\sigma^2}\sum_{0\le k<l\le n-1}\gamma^2(k,l) \le C_{H,K}\frac{n^{-1}}{4\sigma^2}\sum_{0\le l\le n-1}l^{4HK-3} \le C_{H,K}\frac{n^{-\alpha}}{4\sigma^2}\sum_{0\le l\le n-1}l^{4HK-3-\beta} \to 0$$

as $n \to \infty$. Hence,

$$J_{1,2}(n) \to 0 \tag{3.15}$$

as $n \to \infty$. Combining (3.14) and (3.15) leads to

$$J_1(n) \to 0 \tag{3.16}$$

as $n \to \infty$. Finally, from (3.16) and (3.13), together with the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
|J_3(n)| &\le \frac{n^{-1}}{4\sigma^2}\sum_{k,l=0}^{n-1}|\gamma(k,l)\rho(k-l)| \\
&\le \bigg(\frac{n^{-1}}{4\sigma^2}\sum_{k,l=0}^{n-1}\gamma^2(k,l)\bigg)^{1/2}\bigg(\frac{n^{-1}}{4\sigma^2}\sum_{k,l=0}^{n-1}\rho^2(k-l)\bigg)^{1/2} \\
&= 2\sqrt{J_1(n)(J_2(n)+1)},
\end{aligned} \tag{3.17}$$

which goes to zero as $n \to \infty$ and the convergence (3.11) follows.

We now prove (3.12). Following similar arguments as in the proof of (3.11), we have

$$\begin{aligned}
\frac{\operatorname{Var}(Z_n)}{4^{2-K}\sigma^2 n\log n} - 1 &= \frac{n^{-1}}{8\sigma^2\log n}\sum_{k,l=0}^{n-1}\gamma^2(k,l) + \bigg(\frac{n^{-1}}{8\sigma^2\log n}\sum_{k,l=0}^{n-1}\rho^2(k-l) - 1\bigg) \\
&\quad + \frac{n^{-1}}{4\sigma^2\log n}\sum_{k,l=0}^{n-1}\gamma(k,l)\rho(k-l) \\
&= \frac{1}{\log n}J_1(n) + \frac{1}{\log n}J_2(n) + \frac{1}{\log n}J_3(n).
\end{aligned}$$

From [3, p. 490] we have

$$\frac{1}{\log n}J_2(n) \to 0 \tag{3.18}$$

as $n \to \infty$. On the other hand, since $HK = 3/4$ and from the fact that

$$\log n \sim \sum_{1}^{n-1} \frac{1}{k},$$

we deduce easily from (3.16) and (3.17) that

$$\frac{1}{\log n} J_1(n) + \frac{1}{\log n} J_3(n) \to 0$$

as $n \to \infty$. □

3.2 A Berry–Esseen bound for $0 < HK \leq 3/4$

Our first main result is summarized in the following theorem.

Theorem 3.3. *Let $N \sim \mathcal{N}(0,1)$ and let V_n be defined by* (3.6)*. Then, V_n converges in distribution to N. In addition, for some constant $c_{H,K}$ depending uniquely on H and K and for every $n \geq 1$, we have*

$$d_{\mathrm{Kol}}(V_n, N) \leq c_{H,K} \times \begin{cases} \dfrac{1}{\sqrt{n}} & \text{if } HK \in (0, 1/2], \\ n^{2HK-3/2} & \text{if } HK \in [1/2, 3/4), \\ \dfrac{1}{\sqrt{\log n}} & \text{if } HK = 3/4. \end{cases}$$

Proof. From (3.5) we have

$$DZ_n = 2n^{2HK} \sum_{k=0}^{n-1} I_1(\delta_{k/n})\delta_{k/n}$$

and

$$\|DZ_n\|_{\mathfrak{H}}^2 = 4n^{4HK} \sum_{k,l=0}^{n-1} I_1(\delta_{k/n}) I_1(\delta_{l/n}) \langle \delta_{k/n}, \delta_{l/n} \rangle_{\mathfrak{H}},$$

and by the multiplication formula (2.3) we get

$$\begin{aligned} \|DZ_n\|_{\mathfrak{H}}^2 &= 4n^{4HK} \sum_{k,l=0}^{n-1} I_2(\delta_{k/n} \tilde{\otimes} \delta_{l/n}) \langle \delta_{k/n}, \delta_{l/n} \rangle_{\mathfrak{H}} + 4n^{4HK} \sum_{k,l=0}^{n-1} \langle \delta_{k/n}, \delta_{l/n} \rangle_{\mathfrak{H}}^2 \\ &= 4n^{4HK} \sum_{k,l=0}^{n-1} I_2(\delta_{k/n} \tilde{\otimes} \delta_{l/n}) \langle \delta_{k/n}, \delta_{l/n} \rangle_{\mathfrak{H}} + \mathbb{E}\|DZ_n\|_{\mathfrak{H}}^2. \end{aligned}$$

Combining this with the fact that $\mathbb{E}\|DZ_n\|_{\mathfrak{H}}^2 = 2\operatorname{Var}(Z_n)$, we obtain that

$$\frac{1}{2}\|DV_n\|_{\mathfrak{H}}^2 - 1 = \frac{2n^{4HK}}{\operatorname{Var}(Z_n)} \sum_{k,l=0}^{n-1} I_2(\delta_{k/n} \tilde{\otimes} \delta_{l/n}) \langle \delta_{k/n}, \delta_{l/n} \rangle_{\mathfrak{H}}.$$

It follows that

$$\begin{aligned} \mathbb{E}\bigg[\bigg(\frac{1}{2}\|DV_n\|_{\mathfrak{H}}^2 - 1\bigg)^2\bigg] &= \frac{4n^{8HK}}{\operatorname{Var}^2(Z_n)} \mathbb{E}\bigg[\bigg(\sum_{k,l=0}^{n-1} I_2(\delta_{k/n} \tilde{\otimes} \delta_{l/n}) \langle \delta_{k/n}, \delta_{l/n} \rangle_{\mathfrak{H}}\bigg)^2\bigg] \\ &= \frac{8n^{8HK}}{\operatorname{Var}^2(Z_n)} \sum_{i,j,k,l=0}^{n-1} \langle \delta_{i/n}, \delta_{j/n} \rangle_{\mathfrak{H}} \langle \delta_{k/n}, \delta_{l/n} \rangle_{\mathfrak{H}} \langle \delta_{i/n} \tilde{\otimes} \delta_{j/n}, \delta_{k/n} \tilde{\otimes} \delta_{l/n} \rangle_{\mathfrak{H}^{\otimes 2}} \\ &= \frac{8n^2}{\operatorname{Var}^2(Z_n)} A(n), \end{aligned} \tag{3.19}$$

where

$$\begin{aligned}
A(n) &= n^{8HK-2} \sum_{i,j,k,l=0}^{n-1} \langle \delta_{i/n}, \delta_{j/n} \rangle_{\mathfrak{H}} \langle \delta_{k/n}, \delta_{l/n} \rangle_{\mathfrak{H}} \langle \delta_{i/n} \widetilde{\otimes} \delta_{j/n}, \delta_{k/n} \widetilde{\otimes} \delta_{l/n} \rangle_{\mathfrak{H}^{\otimes 2}} \\
&= \frac{n^{8HK-2}}{2} \sum_{i,j,k,l=0}^{n-1} \langle \delta_{i/n}, \delta_{j/n} \rangle_{\mathfrak{H}} \langle \delta_{k/n}, \delta_{l/n} \rangle_{\mathfrak{H}} (\langle \delta_{i/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \langle \delta_{j/n}, \delta_{l/n} \rangle_{\mathfrak{H}} + \langle \delta_{i/n}, \delta_{l/n} \rangle_{\mathfrak{H}} \langle \delta_{j/n}, \delta_{k/n} \rangle_{\mathfrak{H}}) \\
&= n^{8HK-2} \sum_{i,j,k,l=0}^{n-1} \langle \delta_{i/n}, \delta_{j/n} \rangle_{\mathfrak{H}} \langle \delta_{i/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \langle \delta_{k/n}, \delta_{l/n} \rangle_{\mathfrak{H}} \langle \delta_{j/n}, \delta_{l/n} \rangle_{\mathfrak{H}}.
\end{aligned}$$

Hence, using that fact that

$$|ab| \le \frac{1}{2}(a^2 + b^2), \quad a, b \in \mathbb{R},$$

we have

$$\begin{aligned}
|A(n)| &\le \frac{n^{8HK-2}}{2} \sum_{i,j,k=0}^{n-1} |\langle \delta_{i/n}, \delta_{j/n} \rangle_{\mathfrak{H}} \langle \delta_{i/n}, \delta_{k/n} \rangle_{\mathfrak{H}}| \bigg(\sum_{l=0}^{n-1} \langle \delta_{k/n}, \delta_{l/n} \rangle_{\mathfrak{H}}^2 \bigg) \\
&\quad + \frac{n^{8HK-2}}{2} \sum_{i,j,k=0}^{n-1} |\langle \delta_{i/n}, \delta_{j/n} \rangle_{\mathfrak{H}} \langle \delta_{i/n}, \delta_{k/n} \rangle_{\mathfrak{H}}| \bigg(\sum_{l=0}^{n-1} \langle \delta_{j/n}, \delta_{l/n} \rangle_{\mathfrak{H}}^2 \bigg) \\
&= n^{8HK-2} \sum_{i,j,k=0}^{n-1} |\langle \delta_{i/n}, \delta_{j/n} \rangle_{\mathfrak{H}} \langle \delta_{i/n}, \delta_{k/n} \rangle_{\mathfrak{H}}| \bigg(\sum_{l=0}^{n-1} \langle \delta_{k/n}, \delta_{l/n} \rangle_{\mathfrak{H}}^2 \bigg). \qquad (3.20)
\end{aligned}$$

By (3.9) and (3.3) we obtain

$$\begin{aligned}
n^{4HK} \sum_{l=0}^{n-1} \langle \delta_{k/n}, \delta_{l/n} \rangle_{\mathfrak{H}}^2 &= \sum_{l=0}^{n-1} \theta^2(k,l) \\
&\le 2^{1-2K} \bigg(\sum_{l=0}^{n-1} \gamma^2(k,l) + \sum_{l=0}^{n-1} \rho^2(k-l) \bigg) \\
&= 2^{1-2K} \bigg(\sum_{l=0}^{k} \gamma^2(k,l) + \sum_{l=k+1}^{n-1} \gamma^2(k,l) + \sum_{r=-k}^{n-1-k} \rho^2(r) \bigg) \\
&\le 2^{1-2K} \bigg(\sum_{l=0}^{k} k^{4HK-4} + \sum_{l=1}^{n-1} l^{4HK-4} + 2 \sum_{r=0}^{n-1} \rho^2(r) \bigg) \\
&\trianglelefteq 1 + \sum_{l=0}^{n-1} l^{4HK-4}. \qquad (3.21)
\end{aligned}$$

On the other hand, by using (3.3) we get

$$\begin{aligned}
n^{4HK-2} \sum_{i,j,k=0}^{n-1} |\langle \delta_{i/n}, \delta_{j/n} \rangle_{\mathfrak{H}} \langle \delta_{i/n}, \delta_{k/n} \rangle_{\mathfrak{H}}| &= \frac{1}{n^2} \sum_{i,j,k=0}^{n-1} |\theta(i,j)\theta(i,k)| \\
&= \frac{1}{n^2} \sum_{i=0}^{n-1} \bigg(\sum_{j=0}^{n-1} |\theta(i,j)| \bigg)^2 \\
&\le \frac{1}{n^2} \sum_{i=0}^{n-1} \bigg(\sum_{j=0}^{n-1} |\gamma(i,j)| + \sum_{j=0}^{n-1} |\rho(i-j)| \bigg)^2 \\
&= 2^{-2K} \frac{1}{n^2} \sum_{i=0}^{n-1} \bigg(\sum_{j=0}^{i} |\gamma(i,j)| + \sum_{j=i+1}^{n-1} |\gamma(i,j)| + \sum_{r=-i}^{n-1-i} |\rho(r)| \bigg)^2 \\
&\le 2^{-2K} \frac{1}{n^2} \sum_{i=1}^{n-1} \bigg(i^{2HK-1} + \sum_{j=1}^{n-1} j^{2HK-2} + 2 \sum_{r=0}^{n-1} |\rho(r)| \bigg)^2 \\
&\trianglelefteq \frac{1}{n^2} \sum_{i=1}^{n-1} i^{4HK-2} + \frac{1}{n} \bigg(\sum_{j=1}^{n-1} j^{2HK-2} \bigg)^2. \qquad (3.22)
\end{aligned}$$

By (3.20), (3.21) and (3.22) we have

$$|A(n)| \trianglelefteq \frac{1}{n^2}\sum_{i=1}^{n-1} i^{4HK-2} + \frac{1}{n}\Big(\sum_{j=1}^{n-1} j^{2HK-2}\Big)^2 := D(n). \tag{3.23}$$

If $0 < HK < 1/2$, then

$$D(n) = \frac{1}{n^2}\sum_{i=1}^{n-1} i^{4HK-2} + \frac{1}{n}\Big(\sum_{j=1}^{n-1} j^{2HK-2}\Big)^2 \leq \frac{1}{n}\sum_{i=1}^{\infty} i^{4HK-3} + \frac{1}{n}\Big(\sum_{j=1}^{\infty} j^{2HK-2}\Big)^2 \trianglelefteq \frac{1}{n}. \tag{3.24}$$

If $1/2 \leq HK < 3/4$, then, by using the fact that for $\alpha > -1$ we have

$$\sum_{k=1}^{n-1} r^{\alpha} \sim \frac{n^{\alpha+1}}{\alpha+1}$$

as $n \to \infty$, we get

$$D(n) = \frac{1}{n^2}\sum_{i=1}^{n-1} i^{4HK-2} + \frac{1}{n}\Big(\sum_{j=1}^{n-1} j^{2HK-2}\Big)^2 \leq \sum_{i=1}^{n-1} i^{4HK-4} + \Big(\sum_{j=1}^{n-1} j^{2HK-5/2}\Big)^2 \trianglelefteq n^{4HK-3}. \tag{3.25}$$

Combining (2.4), (3.19), (3.11), (3.24) and (3.25), we deduce that

$$d_{\mathrm{Kol}}(V_n, N) \trianglelefteq \begin{cases} \frac{1}{\sqrt{n}} & \text{if } HK \in (0, 1/2], \\ n^{2HK-3/2} & \text{if } HK \in [1/2, 3/4). \end{cases}$$

Assume now that $HK = 3/4$. From (3.20), (3.21) and (3.22), together with the fact that

$$\sum_{r=1}^{n-1} r^{-1} \sim \log n$$

as $n \to \infty$, we have

$$\frac{|A(n)|}{\log^2 n} \trianglelefteq \frac{1}{\log n}\bigg(\frac{1}{n^2}\sum_{i=1}^{n-1} i^{-1} + \frac{1}{n}\Big(\sum_{j=1}^{n-1} j^{-1/2}\Big)^2\bigg) \trianglelefteq \frac{1}{\log n} \tag{3.26}$$

and this completes the proof of the theorem. □

4 The almost sure central limit theorem

We are going now to prove the second main result of this paper, which states the ASCLT of the bifractional Brownian motion and its quadratic variation.

Proposition 4.1. *For all $H \in (0, 1)$ and $K \in (0, 1]$, and for any bounded and continuous function $\varphi : \mathbb{R} \to \mathbb{R}$, we have, almost surely,*

$$\frac{1}{\log n}\sum_{k=1}^{n} \frac{1}{k}\varphi(k^{-HK}B_k) \to \mathbb{E}\varphi(N)$$

as $n \to \infty$, where $N \sim \mathcal{N}(0, 1)$.

Proof. The proof is straightforward by applying [2, Thorem 4.1 and Corollary 3.7] and the fact that

$$|E[B_j B_l]| = 2^{-K}((j^{2H} + l^{2H})^K - |j - l|^{2HK}) \leq 2^{-K}(j^{2HK} + l^{2HK} - |j - l|^{2HK}) = 2^{1-K}|E[B_j^{HK} B_l^{HK}]|,$$

where B^{HK} is a fractional Brownian motion with Hurst parameter HK. □

Theorem 4.2. *If $HK \in (0, 3/4]$, then the sequence $(V_n)_{n\geq 0}$ satisfies the ASCLT. In other words, for any bounded and continuous function $\varphi : \mathbb{R} \to \mathbb{R}$, we have, almost surely,*

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\varphi(V_k) \to \mathbb{E}\varphi(N)$$

as $n \to \infty$, where $N \sim \mathcal{N}(0, 1)$.

Proof. We shall make use of Theorem 2.2. From Theorem 3.3, $(V_n)_n$ satisfies the CLT, so it remains to check conditions (i) and (ii). The cases $HK \in (0, 3/4)$ and $H = 3/4$ are treated separately. By (3.6), we can write $V_n = I_2(g_n)$, where

$$g_n = \frac{n^{2HK}}{\sqrt{\operatorname{Var}(Z_n)}}\sum_{k=1}^{n}\delta_{k/n}^{\otimes 2},$$

which implies that

$$g_n \otimes_1 g_n = \frac{n^{4HK}}{\operatorname{Var}(Z_n)}\sum_{k,l=1}^{n}\langle\delta_{k/n}, \delta_{l/n}\rangle_{\mathfrak{H}}\delta_{k/n}\otimes\delta_{l/n}.$$

We deduce that

$$\|g_n \otimes_1 g_n\|^2_{\mathfrak{H}^{\otimes 2}} = \frac{n^2}{\operatorname{Var}^2(Z_n)}A(n). \tag{4.1}$$

Assume that $HK \in (0, 3/4)$. Combining (3.11), (3.23), (3.24) and (3.25), we have

$$\|g_n \otimes_1 g_n\|^2_{\mathfrak{H}^{\otimes 2}} \trianglelefteq (n^{-1} + n^{4HK-3}) \trianglelefteq \begin{cases} n^{-1} & \text{if } HK \in (0, 1/2), \\ n^{4HK-3} & \text{if } HK \in [1/2, 3/4). \end{cases}$$

Consequently, condition (i) in Theorem 2.2 is satisfied.

On the other hand, by (3.11), for $k < l$, we have

$$\begin{aligned}\langle g_k, g_l\rangle_{\mathfrak{H}^{\otimes 2}} &= \frac{(kl)^{2HK}}{\sqrt{\operatorname{Var}(Z_k)}\sqrt{\operatorname{Var}(Z_l)}}\sum_{i=0}^{k-1}\sum_{j=0}^{l-1}\langle\delta_{i/k}, \delta_{j/l}\rangle^2_{\mathfrak{H}} \\ &\leq c_{H,K}\frac{1}{\sqrt{kl}}\sum_{i=0}^{k-1}\sum_{j=0}^{l-1}\theta^2(i, j) \\ &\leq c_{H,K}\frac{1}{\sqrt{kl}}\Bigg[\sum_{i=0}^{k-1}\sum_{j=0}^{l-1}\rho^2(i-j) + \Bigg(\sum_{0\leq i\leq j\leq k-1} + \sum_{i=0}^{k-1}\sum_{j=k}^{l-1}\Bigg)\gamma^2(i, j)\Bigg].\end{aligned}$$

As in the proof of [2, Theorem 5.1], we obtain that

$$\frac{1}{\sqrt{kl}}\sum_{i=0}^{k-1}\sum_{j=0}^{l-1}\rho^2(i-j) \leq c_{H,K}\sqrt{\frac{k}{l}}.$$

Using Lemma 3.1, we obtain

$$\frac{1}{\sqrt{kl}}\sum_{0\leq i\leq j\leq k-1}\gamma^2(i, j) \leq c_{H,K}\sqrt{\frac{k}{l}}\sum_{0\leq i\leq k-1} i^{4HK-4} \leq c_{H,K}\sqrt{\frac{k}{l}}$$

and, again from Lemma 3.1, we have

$$\frac{1}{\sqrt{kl}}\sum_{i=0}^{k-1}\sum_{j=k}^{l}\gamma^2(i, j) \leq \frac{1}{\sqrt{kl}}\sum_{i=0}^{k-1}\sum_{j=1}^{l} j^{4HK-4} \leq c_{H,K}\sqrt{\frac{k}{l}}.$$

Combining all the above bounds, we obtain

$$\langle g_k, g_l\rangle_{\mathfrak{H}^{\otimes 2}} \leq c_{H,K}\sqrt{\frac{k}{l}}.$$

Finally, condition (ii) in Theorem 2.2 is satisfied.

Now, suppose that $HK = 3/4$. It follows from (4.1), (3.12) and (3.26) that

$$\|g_k \otimes_1 g_k\|^2_{\mathfrak{H}^{\otimes 2}} = \frac{k^2 \log^2 k}{\mathrm{Var}^2(Z_k)} \frac{A(k)}{\log^2 k} \leq c_{H,K} \log^{-1} k,$$

which leads to

$$\sum_{n=2}^{\infty} \frac{1}{n \log^2 n} \sum_{k=1}^{n} \frac{1}{k} \|g_k \otimes g_k\|_{\mathfrak{H}^{\otimes 2}} \leq c_{H,K} \sum_{n=2}^{\infty} \frac{1}{n \log^2 n} \sum_{k=1}^{n} \frac{1}{k\sqrt{\log k}} \leq c_{H,K} \sum_{n=2}^{\infty} \frac{1}{n \log^{3/2} n} < \infty.$$

To finish the proof, it suffices to show that

$$\langle g_k, g_l \rangle_{\mathfrak{H}^{\otimes 2}} \leq c_{H,K} \sqrt{\frac{k \log l}{l \log k}} \tag{4.2}$$

for all $k > l$. According to (3.12), we have

$$\begin{aligned}
\langle g_k, g_l \rangle_{\mathfrak{H}^{\otimes 2}} &= \frac{(kl)^{2HK}}{\sqrt{\mathrm{Var}(Z_k)}\sqrt{\mathrm{Var}(Z_l)}} \sum_{i=0}^{k-1} \sum_{j=0}^{l-1} \langle \delta_{i/k}, \delta_{j/l} \rangle^2_{\mathfrak{H}} \\
&\leq \frac{c_{H,K}}{\sqrt{l \log k}\sqrt{k \log l}} \sum_{i=0}^{k-1} \sum_{j=0}^{l-1} \theta^2(i,j) \\
&\leq \frac{c_{H,K}}{\sqrt{l \log k}\sqrt{k \log l}} \Bigg[\sum_{i=0}^{k-1} \sum_{j=0}^{l-1} \rho^2(i-j) + \Bigg(\sum_{0 \leq i \leq j \leq k-1} + \sum_{i=0}^{k-1} \sum_{j=k}^{l-1} \Bigg) \gamma^2(i,j) \Bigg].
\end{aligned}$$

As in the proof of [2, Proposition 6.4], we have

$$\frac{1}{\sqrt{l \log k}\sqrt{k \log l}} \sum_{i=0}^{k-1} \sum_{j=0}^{l-1} \rho^2(i-j) \leq c_{H,K} \sqrt{\frac{k \log l}{l \log k}}$$

for all $1 \leq k \leq l$. Using Lemma 3.1 and the fact that

$$\sum_{r=1}^{n-1} r^{-1} \sim \log n$$

as $n \to \infty$, we deduce that

$$\frac{1}{\sqrt{l \log k}\sqrt{k \log l}} \sum_{0 \leq i \leq j \leq k-1} \gamma^2(i,j) \leq c_{H,K} \frac{k \log l}{\sqrt{l \log k}\sqrt{k \log l}} \leq c_{H,K} \sqrt{\frac{k \log l}{l \log k}}. \tag{4.3}$$

Again from Lemma 3.1, we obtain that

$$\frac{1}{\sqrt{l \log k}\sqrt{k \log l}} \sum_{i=0}^{k-1} \sum_{j=k}^{l-1} \gamma^2(i,j) \leq c_{H,K} \sqrt{\frac{k \log l}{l \log k}}, \tag{4.4}$$

which completes the proof of the theorem. □

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