Vasily A. Ogorodnikov* and Olga V. Sereseva

Probabilistic properties of non-Gaussian piecewise-linear processes on Poisson flows with independent random values at points of flow

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Abstract: The paper is focused on study of non-stationary piecewise-linear processes on Poisson point flows with independent identically distributed random variables at support points. An approach to calculate the correlation function of the process on the base of the total probability formula is considered. A general expression for the correlation function of a non-stationary process is obtained. Particular cases are considered. Using the method of direct simulation, it is shown numerically that the correlation function of the process has a point of inflection.

Keywords: Random process, Poisson flow, correlation function.

MSC 2010: 60G55, 65C05

Some approaches to modelling the piecewise-linear non-Gaussian processes on point flows were considered in [1] relative to simulation of price series and the study of various trading algorithms based on these models. The corresponding models use intervals between points of flow and the distributions of process values at support points as input characteristics. Various approaches to construction of processes were considered, in particular, the alternation of distributions for ascending and descending sections of the polyline was used with the help of a special Markov chain.

Special piecewise-linear processes were used in [9] for construction of models for climate prediction. The typical peculiarity of such processes is that the segments of a piecewise-linear function form a discontinuous function. These researches have shown a prospect of using simulation algorithms for piecewise-linear processes to solve practical problems.

The issues related to the study of different types of piecewise-linear processes on point flows were considered in [3–8]. Those processes are modifications of piecewise-constant processes on point flows proposed and studied previously in [2]. One has to specify probabilistic characteristics of point flows and distribution of random variables at those points to apply numerical stochastic modelling of piecewise-linear processes which, in their turn, may be used for description of some real processes, for example, for modelling solar radiation scattering processes in stochastic cloudy media, modelling climate series, etc. The way of specification of random values in Poisson processes essentially determines the properties of the process. Thus, for example, specifying additive random variables [3] with successively increasing number of summands at flow points, we get a non-stationary process, and specifying them as independent identically distributed random variables [4, 5], we get an asymptotically stationary one. In particular, one-point characteristics were considered for such types of processes and their asymptotic properties were studied. An approach to the study of correlation structure of piecewise-linear processes on Poisson flows was proposed in [5].

In this paper we present the results of study of a process on Poisson flows whose values are IIDRV with finite variance at Poisson support points. Considering processes of such type, we obtained exact expressions

***Corresponding author: Vasily A. Ogorodnikov,** Institute of Computational Mathematics and Mathematical Geophysics,, Novosibirsk 630090, Russia; Novosibirsk State University, Novosibirsk 630090, Russia. E-mail: ova@osmf.sscc.ru
Olga V. Sereseva, Institute of Computational Mathematics and Mathematical Geophysics, Novosibirsk 630090, Russia

for various its characteristics determining the correlation structure of the process. The properties of those characteristics were studied numerically.

1 Statistical characteristics of a piecewise-linear process on Poisson flows with independent identically distributed variables at support points

In this paper we study an approach to calculation of the correlation function and characteristics of the random process related to it [4]:

$$Y(t) = (Y_{k+1} - Y_k)\frac{t - S_k}{S_{k+1} - S_k} + Y_k = (Y_{k+1} - Y_k)\, Q_k(t) + Y_k, \qquad S_k \leqslant t < S_{k+1}, \quad k = 0, 1, \ldots \tag{1.1}$$

Here $S_0 = 0$, $S_k = \sum_{i=1}^{k} X_i$, while X_i are independent positive random variables with the density $f(x) = \lambda \exp(-\lambda x)$, $\lambda > 0$. Note that $k = k(t)$ is an integer-valued random variable,

$$k(t) = \mathrm{Min}\{k \geqslant 1 : S_k \geqslant t\} \in [0, \infty), \quad t > 0$$

used to denote the number of a random interval $S_k \leqslant t < S_{k+1}$ covering the point t [3]. As was indicated above, we consider the random variables Y_k as IID with an arbitrary one-dimensional probability distribution and finite variance.

The formal representation of the correlation function of this process (1.1) has the form

$$\mathrm{corr}(Y(t), Y(t+h)) = r(t, h) = \frac{E[Y(t)Y(t+h)] - E[Y(t)]E[Y(t+h)]}{\sqrt{D[Y(t)]}\sqrt{D[Y(t+h)]}}. \tag{1.2}$$

Represent the points of process (1.1) in the form

$$\begin{gathered} S_0 = 0, \qquad S_n = S_0 + X_1 + X_2 + \ldots + X_n \\ S_{n+1} = S_n + X_{n+1} \\ S_m = S_{n+1} + X_{n+2} + \ldots + X_{n+m+1} \\ S_{m+1} = S_m + X_{n+m+2}, \quad n \geqslant 1, \quad m \geqslant 1 \end{gathered} \tag{1.3}$$

and the independent variables Y_n, Y_{n+1}, Y_m, and Y_{m+1} corresponding to these points are identically distributed and do not depend on $Q(t) = Q_n(t) = Q_{n(t)}(t)$ and $Q(t+h) = Q_m(t+h) = Q_{m(t+h)}(t+h)$. The flow points S_n, S_{n+1} and S_m, S_{m+1} and also the values Y_n, Y_{n+1} and Y_m, Y_{m+1} determine two different segments of piecewise-linear function (1.1).

In order to calculate $E[Y(t)Y(t+h)]$ in (1.2), we consider the following events:

$$\text{1a.} \quad B_{1,nm}(t, t+h) = \{S_n < t,\ t < S_{n+1} < t+h,\ S_{n+1} < S_m < t+h,\ S_{m+1} > t+h\}$$
$$n = 1, 2, \ldots, \qquad m = 1, 2, \ldots$$

The events of this group of events (see Fig. 2) consist in the fact that the points t and $t+h$, where $t, h > 0$, belong to two nonadjacent intervals and $S_n \leqslant t < S_{n+1}$ and $S_{n+1+m} \leqslant t+h < S_{n+1+m+1}$, and the beginning of the first interval is at the point $S_n < t$, $n = 1, 2, \ldots$, and the beginning of the second interval is $S_{n+1+m} > S_{n+1}$, $m = 1, 2, \ldots$

$$\text{2a.} \quad B_{2,nm}(t, t+h) = \{S_n \leqslant t,\ S_{n+m} > t+h\}$$
$$n = 1, 2, \ldots, \qquad m = 1.$$

The events from this group of events (see Fig. 3) consist in the fact that the points t and $t+h$ belong to the same interval and $S_n \leqslant t < S_{n+1}$ and $S_n \leqslant t+h < S_{n+1}$, and its beginning is at the point $S_n < t$, $n = 1, 2, \ldots$

$$\text{3a.} \quad B_{3,nm}(t, t+h) = \{S_n \leqslant t,\ t < S_{n+m} \leqslant t+h,\ S_{n+m+1} > t+h\}$$
$$n = 1, 2, \ldots, \quad m = 1.$$

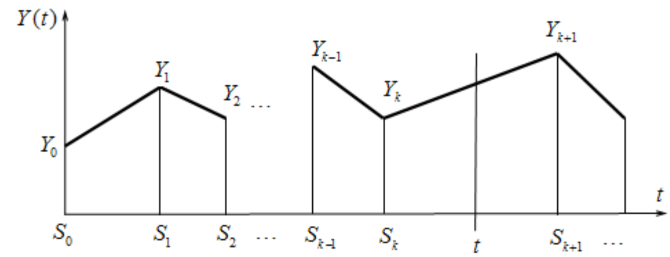

Fig. 1: Trajectory of the process $Y(t)$.

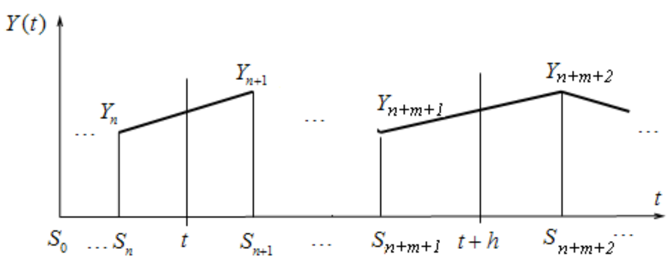

Fig. 2: Events $B_{1,nm}(t, t+h)$.

Fig. 3: Events $B_{2,nm}(t, t+h)$.

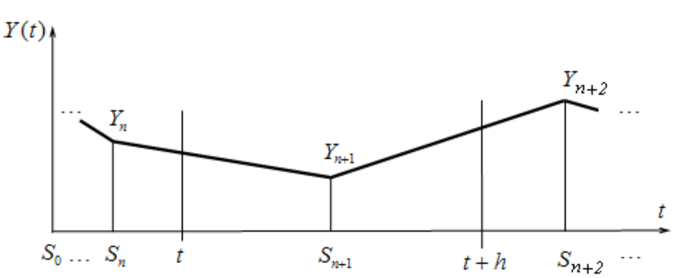

Fig. 4: Events $B_{3,nm}(t, t+h)$.

The events of this group of events (see Fig. 4) consist in the fact that the points t and $t+h$, where $t, h > 0$, belong to two adjacent intervals and $S_n \leqslant t < S_{n+1}$, $S_{n+1} \leqslant t+h < S_{n+2}$, and the beginning of the first interval is at the point $S_n < t$, $n = 1, 2, \ldots$

$$4a. \quad B_{4,nm}(t, t+h) = \{S_{n+m-1} > t+h\}, \quad n = 1, \quad m = 1.$$

This event (see Fig. 5) consists in the fact that the points t and $t+h$ belong to the same interval $S_0 \leqslant t < S_1$

$$5a. \quad B_{5,nm}(t, t+h) = \{t < S_n \leqslant t+h,\ S_{n+m} > t+h\}, \quad n = 1, \quad m = 1.$$

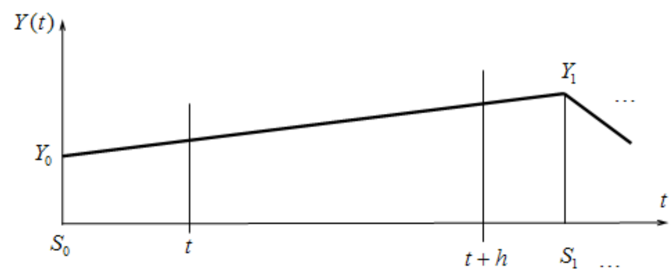

Fig. 5: Events $B_{4,nm}(t, t+h)$.

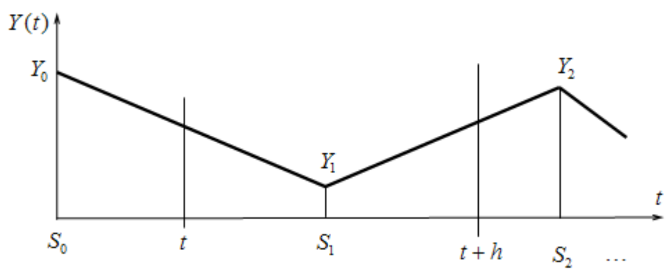

Fig. 6: Events $B_{5,nm}(t, t+h)$.

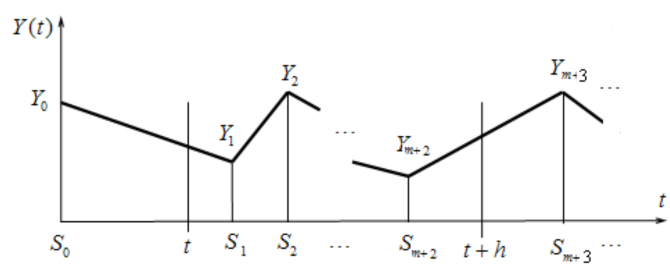

Fig. 7: Events $B_{6,nm}(t, t+h)$.

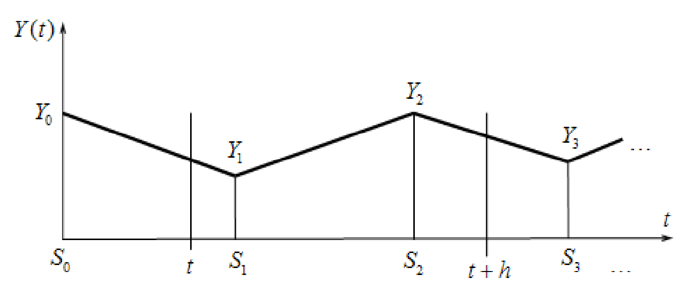

Fig. 8: Events $B_{7,nm}(t, t+h)$.

This event (see Fig. 6) consists in the fact that the points t and $t+h$ belong to two adjacent intervals and $S_0 \leqslant t < S_1$, $S_1 \leqslant t+h < S_2$

6a. $B_{6,nm}(t, t+h) = \{t < S_n \leqslant t+h,\ S_n < S_{n+1} \leqslant t+h,\ S_{n+1} < S_{n+m+1} \leqslant t+h,\ S_{n+m+2} > t+h\}$
$$n = 1, \quad m = 1, 2, \ldots$$

The events of this group of events (see Fig. 7) consist in the fact that the points t and $t+h$ belong to two nonadjacent intervals and $S_0 \leqslant t < S_1$, $S_{m+2} \leqslant t+h < S_{m+3}$, and the beginning of the second interval satisfies the inequality $S_{1+m} > S_1$, $m = 1, 2, \ldots$

$$\text{7a.}\quad B_{7,nm}(t, t+h) = \{(t < S_n \leqslant t+h,\ S_n < S_{n+1} \leqslant t+h,\ S_{n+m+1} > t+h,\ S_{m+1} > S_m\}$$
$$n = 1, \quad m = 1.$$

This event (see Fig. 8) consists in the fact that the points t and $t+h$ belong to two intervals $S_0 \leqslant t < S_1$ and $S_2 \leqslant t+h < S_3$, and one interval (S_1, S_2) lies between these intervals.

Taking into account events 1a–7a, the formula of total probability for calculation of the mean $E[Y(t)Y(t+h)]$ takes the form

$$\begin{aligned} E[Y(t)Y(t+h)] &= s(t,h) = \sum_{k=1}^{7} s_k(t,h) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} P[B_{1,nm}(t,t+h)]E[Y(t)Y(t+h)|B_{1,nm}(t,t+h)] \\ &+ \sum_{n=1}^{\infty} P[B_{2,n1}(t,t+h)]E[Y(t)Y(t+h)|B_{2,n1}(t,t+h)] + \sum_{n=1}^{\infty} P[B_{3,n1}(t,t+h)]E[Y(t)Y(t+h)|B_{3,n1}(t,t+h)] \\ &+ P[B_{4,11}(t,t+h)]E[Y(t)Y(t+h)|B_{4,11}(t,t+h) + P[B_{5,11}(t,t+h)]E[Y(t)Y(t+h)|B_{5,11}(t,t+h) \\ &+ \sum_{m=1}^{\infty} P[B_{6,1m}(t,t+h)]E[Y(t)Y(t+h)|B_{6,1m}(t,t+h) + P[B_{7,11}(t,t+h)]E[Y(t)Y(t+h)|B_{7,11}(t,t+h)]. \end{aligned} \tag{1.4}$$

In order to get the probabilities of events 1a–7a, we calculate the joint distribution density of the variables S_n, S_{n+1}, S_m, and S_{m+1}, $n, m = 1, 2, \ldots$ Assuming (1.3), this density has the form

$$f_{S_n S_{n+1} S_m S_{m+1}}(y_1, y_2, y_3, y_4) = \lambda^4 \frac{(\lambda y_1)^{n-1}}{(n-1)!} \frac{\lambda^{m-1}(y_3 - y_2)^{m-1}}{(m-1)!} \mathrm{e}^{-\lambda y_4}.$$

In this case the probabilities of events 1a–7a are determined by the relations

$$\text{1b.}\quad P(B_{1,nm}(t,t+h)) = \lambda^{n+m+1}\mathrm{e}^{-\lambda(t+h)} \frac{t^n}{n!}\frac{h^{m+1}}{(m+1)!}, \quad n = 1, 2, \ldots, \quad m = 1, 2, \ldots$$

$$\text{2b.}\quad P(B_{2,nm}(t,t+h)) = \lambda^n \frac{t^n}{n!}\mathrm{e}^{-\lambda(t+h)}, \quad n = 1, 2, \ldots$$

$$\text{3b.}\quad P(B_{3,nm}(t,t+h)) = \mathrm{e}^{-\lambda(t+h)} h \frac{\lambda^{n+1} t^n}{n!}, \quad n = 1, 2, \ldots, \quad m = 1$$

$$\text{4b.}\quad P(B_{4,nm}(t,t+h)) = \mathrm{e}^{-\lambda(t+h)}, \quad n = 1$$

$$\text{5b.}\quad P(B_{5,nm}(t,t+h)) = \lambda h \mathrm{e}^{-\lambda(t+h)}, \quad n = 1$$

$$\text{6b.}\quad P(B_{6,nm}(t,t+h)) = \frac{\lambda^{m+2} h^{m+2}}{(m+2)!}\mathrm{e}^{-\lambda(t+h)}, \quad n = 1, \quad m = 1, 2, \ldots$$

$$\text{7b.}\quad P(B_{7,nm}(t,t+h)) = \lambda^2 \mathrm{e}^{-\lambda(t+h)} \frac{h^2}{2}, \quad n = 1, \quad m = 1.$$

It is easy to show that the sum of these probabilities equals one. Let us write down the conditional distribution densities of the variables S_n, S_{n+1}, S_m, S_{m+1} under fulfillment of conditions 1a–7a.

$$\text{1c.}\quad f(y_1, y_2, y_3, y_4 \,|\, B_{1,nm}(t,t+h)) = \frac{\lambda y_1^{n-1}(y_3 - y_2)^{m-1}\mathrm{e}^{-\lambda y_4} nm(m+1)}{\mathrm{e}^{-\lambda(t+h)} t^n h^{m+1}}$$

$$\text{2c.}\quad f(y_1, y_2 \,|\, B_{2,nm}(t,t+h)) = \frac{\lambda n y_1^{n-1}\mathrm{e}^{-\lambda y_2}}{t^n \mathrm{e}^{-\lambda(t+h)}}$$

$$\text{3c.}\quad f(y_1, y_2, y_3 \,|\, B_{3,nm}(t,t+h)) = \frac{\lambda y_1^{n-1}\mathrm{e}^{-\lambda y_3} n}{h t^n}\mathrm{e}^{\lambda(t+h)}$$

$$\text{4c.}\quad f(y_1 \,|\, B_{4,nm}(t,t+h)) = \frac{\lambda \mathrm{e}^{-\lambda y_1}}{\mathrm{e}^{-\lambda(t+h)}}$$

$$\text{5c.}\quad f(y_1, y_2 \,|\, B_{5,nm}(t,t+h)) = \frac{\lambda \mathrm{e}^{-\lambda y_2}}{h \mathrm{e}^{-\lambda(t+h)}}$$

$$\text{6c.}\quad f(y_1, y_2, y_3, y_4 \,|\, B_{6,nm}(t,t+h)) = \lambda \frac{(m+2)(m+1)m(y_3 - y_2)^{m-1}}{h^{m+2}\mathrm{e}^{-\lambda(t+h)}}\mathrm{e}^{-\lambda y_4}$$

$$\text{7c.}\quad f(y_1, y_2, y_3 \,|\, B_{7,nm}(t,t+h)) = \frac{2\lambda \mathrm{e}^{-\lambda y_3}}{\mathrm{e}^{-\lambda(t+h)} h^2}.$$

Taking into account 1b–7b, 1c–7c, and the relations

$$Q_n(t) = \frac{t - S_n}{S_{n+1} - S_n}, \qquad Q_m(t+h) = \frac{t + h - S_m}{S_{m+1} - S_m}$$

and also that Y_k and Y_l, $k \neq l$, are IID variables with $EY_k = \mu$ and $DY_k = \sigma^2$, in total probability formula (1.4) for $E[Y(t)Y(t+h)]$ for the conditional means we have

1d. $E[Y(t)Y(t+h) \,|\, B_{1,nm}(t, t+h)]$

$$= E\bigg[\Big((Y_{n+1} - Y_n)\frac{t - S_n}{S_{n+1} - S_n} + Y_n\Big)\Big((Y_{m+1} - Y_m)\frac{t + h - S_m}{S_{m+1} - S_m} + Y_m\Big) \,\Big|\, B_{1,nm}(t, t+h)\bigg]$$

$$= E\bigg[\Big(\Big[(Y_{n+1} - Y_n)(Y_{m+1} - Y_m)\frac{t - S_n}{S_{n+1} - S_n}\frac{t + h - S_m}{S_{m+1} - S_m}\Big] + \Big[(Y_{n+1} - Y_n)Y_m\frac{t - S_n}{S_{n+1} - S_n}\Big]$$

$$+ \Big[Y_n(Y_{m+1} - Y_m)\frac{t + h - S_m}{S_{m+1} - S_m}\Big] + [Y_n Y_m]\Big) \,\Big|\, B_{1,nm}(t, t+h)\bigg] = E[Y_n]E[Y_m] = \mu^2$$

$$s_1(t,h) = \mu^2 \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \lambda^{n+m+1} e^{-\lambda(t+h)} \frac{t^n}{n!}\frac{h^{m+1}}{(m+1)!} = \mu^2 e^{-\lambda(t+h)}(1 - e^{\lambda t})(1 - e^{\lambda h} + \lambda h)$$

2d. $E[Y(t)Y(t+h) \,|\, B_{2,nm}(t, t+h)]$

$$= E\bigg[\Big((Y_{n+1} - Y_n)\frac{t - S_n}{S_{n+1} - S_n} + Y_n\Big)\Big((Y_{n+1} - Y_n)\frac{t + h - S_n}{S_{n+1} - S_n} + Y_n\Big) \,\Big|\, B_{2,nm}(t, t+h)\bigg]$$

$$= 2\sigma^2 E\Big[\frac{t - S_n}{S_{n+1} - S_n}\frac{t + h - S_n}{S_{n+1} - S_n} \,\Big|\, B_n^{(2)}(t, t+h)\Big] - \sigma^2 E\Big[\frac{t - S_n}{S_{n+1} - S_n} \,\Big|\, B_n^{(2)}(t, t+h)\Big]$$

$$- \sigma^2 E\Big[\frac{t + h - S_n}{S_{n+1} - S_n} \,\Big|\, B_{2,n1}(t, t+h)\Big] + \sigma^2 + \mu^2$$

$$s_2(t,h) = \sum_{n=1}^{\infty} \lambda^n \frac{t^n}{n!} e^{-\lambda(t+h)} E[Y(t)Y(t+h) \,|\, B_n^{(2)}(t, t+h)]$$

$$= 2\sigma^2 \sum_{n=1}^{\infty} \lambda^n \frac{t^n}{n!} e^{-\lambda(t+h)} \int_0^t \int_{t+h}^{\infty} \frac{t - y_1}{y_2 - y_1}\frac{t + h - y_1}{y_2 - y_1}\frac{\lambda n y_1^{n-1} e^{-\lambda y_2}}{t^n e^{-\lambda(t+h)}} dy_2 dy_1$$

$$- \sigma^2 \sum_{n=1}^{\infty} \lambda^n \frac{t^n}{n!} e^{-\lambda(t+h)} \int_0^t \int_{t+h}^{\infty} \frac{t - y_1}{y_2 - y_1}\frac{\lambda n y_1^{n-1} e^{-\lambda y_2}}{t^n e^{-\lambda(t+h)}} dy_2 dy_1$$

$$- \sigma^2 \sum_{n=1}^{\infty} \lambda^n \frac{t^n}{n!} e^{-\lambda(t+h)} \int_0^t \int_{t+h}^{\infty} \frac{t + h - y_1}{y_2 - y_1}\frac{\lambda n y_1^{n-1} e^{-\lambda y_2}}{t^n e^{-\lambda(t+h)}} dy_2 dy_1 + (\sigma^2 + \mu^2)(e^{-\lambda h} - e^{-\lambda(t+h)})$$

$$= 2\sigma^2 \int_0^t \int_{t+h}^{\infty} \frac{t - y_1}{y_2 - y_1}\frac{t + h - y_1}{y_2 - y_1}\lambda^2 e^{\lambda y_1} e^{-\lambda y_2} dy_2 dy_1 - \sigma^2 \int_0^t \int_{t+h}^{\infty} \frac{t - y_1}{y_2 - y_1}\lambda^2 e^{\lambda y_1} e^{-\lambda y_2} dy_2 dy_1$$

$$- \sigma^2 \int_0^t \int_{t+h}^{\infty} \frac{t + h - y_1}{y_2 - y_1}\lambda^2 e^{\lambda y_1} e^{-\lambda y_2} dy_2 dy_1 + (\sigma^2 + \mu^2)(e^{-\lambda h} - e^{-\lambda(t+h)})$$

3d. $E[Y(t)Y(t+h) \,|\, B_{3,n1}(t, t+h)]$

$$= E\bigg[\Big((Y_{n+1} - Y_n)\frac{t - S_n}{S_{n+1} - S_n} + Y_n\Big)\Big((Y_{n+2} - Y_{n+1})\frac{t + h - S_{n+1}}{S_{n+2} - S_{n+1}} + Y_{n+1}\Big) \,\Big|\, B_{3,n1}(t, t+h)\bigg]$$

$$= -\sigma^2 E\Big[\frac{t - S_n}{S_{n+1} - S_n}\frac{t + h - S_{n+1}}{S_{n+2} - S_{n+1}} \,\Big|\, B_{3,n1}(t, t+h)\Big] + \sigma^2 E\Big[\frac{t - S_n}{S_{n+1} - S_n} \,\Big|\, B_{3,n1}(t, t+h)\Big] + \mu^2$$

$$s_3(t,h) = \sum_{n=1}^{\infty} \mathrm{e}^{-\lambda(t+h)} h \frac{\lambda^{n+1} t^n}{n!} E[Y(t)Y(t+h) \,|\, B_{3,n1}(t,t+h)]$$
$$= \sigma^2 \sum_{n=1}^{\infty} \mathrm{e}^{-\lambda(t+h)} h \frac{\lambda^{n+1} t^n}{n!} \left(E\left[\frac{t-S_n}{S_{n+1}-S_n} \,\Big|\, B_{3,n1}(t,t+h) \right] - E\left[\frac{t-S_n}{S_{n+1}-S_n} \frac{t+h-S_{n+1}}{S_{n+2}-S_{n+1}} \,\Big|\, B_{3,n1}(t,t+h) \right] \right)$$
$$+ \mu^2 \lambda h (\mathrm{e}^{-\lambda h} - \mathrm{e}^{-\lambda(t+h)})$$
$$= \lambda^3 \sigma^2 \int_0^t \left(\int_t^{t+h} \left(\int_{t+h}^{\infty} \frac{t-y_1}{y_2-y_1} \frac{y_3-t-h}{y_3-y_2} \mathrm{e}^{\lambda y_1} \mathrm{e}^{-\lambda y_3} \mathrm{d}y_3 \right) \mathrm{d}y_2 \right) \mathrm{d}y_1 + \mu^2 \lambda h (\mathrm{e}^{-\lambda h} - \mathrm{e}^{-\lambda(t+h)})$$

4d. $E[Y(t)Y(t+h) \,|\, B_{4,11}(t,t+h)] = E\left[\left((Y_1 - Y_0)\frac{t}{S_1} + Y_0 \right) \left((Y_1 - Y_0)\frac{t+h}{S_1} + Y_0 \right) \,\Big|\, B_{4,11}(t,t+h) \right]$

$$= 2\sigma^2 E\left[\frac{t}{S_1} \frac{t+h}{S_1} \,\Big|\, B_{4,11}(t,t+h) \right] - \sigma^2 E\left[\frac{t}{S_1} \,\Big|\, B_{4,11}(t,t+h) \right] - \sigma^2 E\left[\frac{t+h}{S_1} \,\Big|\, B_{4,11}(t,t+h) \right] + \sigma^2 + \mu^2$$
$$s_4(t,h) = 2\sigma^2 \lambda t \mathrm{e}^{-\lambda(t+h)} - 2\sigma^2 \lambda^2 t(t+h)\Gamma[0,\lambda(t+h)] - 2\sigma^2 \lambda t \Gamma[0,\lambda(t+h)] - \sigma^2 \lambda h \Gamma[0,\lambda(t+h)]$$
$$+ (\sigma^2 + \mu^2)\mathrm{e}^{-\lambda(t+h)}$$

5d. $E[Y(t)Y(t+h) \,|\, B_{5,11}(t,t+h)]$

$$= E\left[\left((Y_1 - Y_0)\frac{t-S_0}{S_1-S_0} + Y_0 \right) \left((Y_2 - Y_1)\frac{t+h-S_1}{S_2-S_1} + Y_1 \right) \,\Big|\, B_{5,11}(t,t+h) \right]$$
$$= -\sigma^2 E\left[\frac{t}{S_1} \frac{t+h-S_1}{S_2-S_1} \,\Big|\, B_{5,11}(t,t+h) \right] + \sigma^2 E\left[\frac{t}{S_1} \,\Big|\, B_{5,11}(t,t+h) \right] + \mu^2$$
$$s_5(t,h) = \lambda h \mathrm{e}^{-\lambda(t+h)} \left(-\sigma^2 E\left[\frac{t}{S_1} \frac{t+h-S_1}{S_2-S_1} \,\Big|\, B_{5,11}(t,t+h) \right] + \sigma^2 E\left[\frac{t}{S_1} \,\Big|\, B_{5,11}(t,t+h) \right] + \mu^2 \right)$$
$$= -\sigma^2 \int_t^{t+h} \int_{t+h}^{\infty} \frac{t}{y_1} \frac{t+h-y_1}{y_2-y_1} \lambda^2 \mathrm{e}^{-\lambda y_2} \mathrm{d}y_2 \mathrm{d}y_1 + \sigma^2 \int_t^{t+h} \frac{\lambda t}{y_1} \mathrm{e}^{-\lambda(t+h)} \mathrm{d}y_1 + \mu^2 \lambda h \mathrm{e}^{-\lambda(t+h)}$$

6d. $E[Y(t)Y(t+h) \,|\, B_{6,1m}(t,t+h)]$

$$= E\left[\left((Y_1 - Y_0)\frac{t-S_0}{S_1-S_0} + Y_0 \right) \left((Y_{m+1} - Y_m)\frac{t+h-S_m}{S_{m+1}-S_m} + Y_m \right) \,\Big|\, B_{6,1m}(t,t+h) \right] = \mu^2$$
$$s_6(t,h) = \mathrm{e}^{-\lambda(t+h)} (\mathrm{e}^{\lambda h} - 1 - \lambda h - \frac{1}{2}\lambda^2 h^2)\mu^2$$

7d. $E[Y(t)Y(t+h) \,|\, B_{7,11}(t,t+h)]$

$$= E\left[\left((Y_1 - Y_0)\frac{t-S_0}{S_1-S_0} + Y_0 \right) \left((Y_3 - Y_2)\frac{t+h-S_2}{S_3-S_2} + Y_2 \right) \,\Big|\, B_{7,11}(t,t+h) \right] = \mu^2$$
$$s_7(t,h) = \lambda^2 \mathrm{e}^{-\lambda(t+h)} \frac{h^2}{2} \mu^2.$$

The final expression for the correlation function of process $Y(t)$ has form (1.2), where $E[Y(t)Y(t+h)]$ is determined by expression (1.4) and $E[Y(t)]$ and $D[Y(t)]$ have the form [8]:

$$E[Y(t)] = \mu, \qquad D[Y(t)] = \sigma^2 \left(2(E[Q^2(t)] - E[Q(t)]) + 1 \right) \tag{1.5}$$

where $\mu = EY_n,\ \sigma^2 = D[Y_n],\ n = 1, 2, \ldots,$

$$E[Q^k(t)] = \frac{1}{k+1}\left(1 - \mathrm{e}^{-\lambda t}(\lambda t + 1) + (\lambda t)^k (1 + k + \lambda t)\Gamma[1-k; \lambda t]\right), \quad k = 1, 2, \ldots$$

and

$$\Gamma[a,\ z] = \int_z^{\infty} x^{a-1} \mathrm{e}^{-x} \mathrm{d}x.$$

Tab. 1: Dependence of the functions $s_1 = s_1(t,h), \ldots, s_7 = s_7(t,h)$, and $s = s(t,h)$ on h for $t = 7.5$, $\lambda = 1$.

h	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s
0	0	0.6662	0	4.559×10^{-4}	0	0	0	0.6667
0.0125	0	0.6539	0.0117	4.497×10^{-4}	6.636×10^{-6}	0	0	0.6660
0.025	0	0.6418	0.0222	4.436×10^{-4}	1.284×10^{-5}	0	0	0.6644
0.05	0	0.6186	0.0405	4.316×10^{-4}	2.422×10^{-5}	0	0	0.6595
0.075	0	0.5963	0.0561	4.199×10^{-4}	3.443×10^{-5}	0	0	0.6529
0.1	0	0.5751	0.0695	4.086×10^{-4}	4.364×10^{-5}	0	0	0.6450
0.125	0	0.5550	0.0810	3.975×10^{-4}	5.199×10^{-5}	0	0	0.6361
0.150	0	0.5352	0.0910	3.869×10^{-4}	5.952×10^{-5}	0	0	0.6266
0.175	0	0.5165	0.0996	3.763×10^{-4}	6.636×10^{-5}	0	0	0.6165
0.2	0	0.4985	0.1070	3.662×10^{-4}	7.255×10^{-5}	0	0	0.6060
0.4	0	0.3780	0.1377	2.942×10^{-4}	1.044×10^{-4}	0	0	0.5160
0.6	0	0.2890	0.1310	2.364×10^{-4}	1.156×10^{-4}	0	0	0.4294
0.8	0	0.2224	0.1304	1.901×10^{-4}	1.155×10^{-4}	0	0	0.3531
1	0	0.1720	0.1161	1.529×10^{-4}	1.093×10^{-4}	0	0	0.2884
2	0	0.0450	0.0505	5.166×10^{-5}	5.899×10^{-5}	0	0	0.1006
3	0	0.0153	0.0191	1.757×10^{-5}	2.570×10^{-4}	0	0	0.0344
4	0	0.0048	0.0069	6.012×10^{-6}	1.035×10^{-5}	0	0	0.0118
5	0	0.0016	0.0025	2.067×10^{-6}	4.015×10^{-6}	0	0	0.0040
6	0	0.0005	0.0009	7.138×10^{-7}	1.523×10^{-6}	0	0	0.0014
7	0	0.0002	0.0003	2.474×10^{-7}	5.698×10^{-7}	0	0	0.0005

For $t = 0$ the expression for correlation function (1.2) is

$$\operatorname{corr}(Y(0), Y(h)) = \frac{e^{-\lambda h} - h\lambda\,\Gamma[0, \lambda h]}{\sqrt{2(E[Q^2(h)] - E[Q(h)]) + 1}}. \tag{1.6}$$

Based on direct simulation of trajectories of process (1.1), we estimated functions of form (1.6) for various values of the parameter λ. The calculations have shown that these estimates coincide with function (1.6) up to statistical error.

2 Numerical experiments

Based on estimates of the correlation function $r(t, t+h)$ of process (1.1) obtained from model samples for the case $F(x) = 1 - \exp(-\lambda x)$, $\lambda = 0.25$ with different values of t, it was shown in [4] that for $t > 7.5$ the process becomes close to a stationary one in its correlations, a similar behaviour of the process is observed relative to means and variances. Examples of correlation functions $r(t,\tau)$ of the process $Y(t)$ for $t = 20$ were also presented in that paper for different values of the parameter λ. The presence of an inflection point in these correlation functions is typical for this process, which differs them essentially from correlation functions of piecewise-constant processes on Poisson point flows and on Palm's flows whose characteristic feature is the convexity downwards [2].

We also studied the function $E[Y(t)Y(t+h)]$ numerically on the base of total probability formula (1.4). Table 1 presents the results of calculations of the dependence of the functions $s_1(t,h), \ldots, s_7(t,h)$ on h for $t = 7.5$, $\lambda = 1$. The one-dimensional distribution of random variables Y_k at Poisson points was specified by the standard normal distribution with the mathematical expectation $\mu = 0$ and variance $\sigma^2 = 1$. As is seen from the table, the main contribution into the covariance $E[Y(t)Y(t+h)]$ is introduced by the functions $s_2(t,h)$ and $s_3(t,h)$, and the second central moment (which coincides with the variance in this case) is mainly determined by the function $s_2(t,h)$ and, as is seen from the table, for $t = 7.5$ it is close to the asymptotic value of the variance of the process defined by formula (1.5) and equal to $\frac{2}{3}\sigma^2$ [8]. For given $t = 7.5$ the contribution of all functions $s_1(t,h)$, $s_6(t,h)$, $s_7(t,h)$ is zero for all h (this is confirmed by theoretical calculations) and

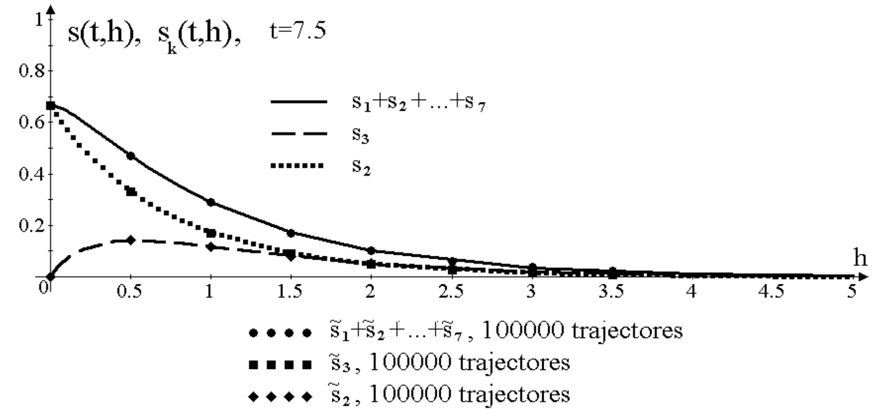

Fig. 9: Dependence of the functions $s_2 = s_2(t, h)$, $s_3 = s_3(t, h)$ and the covariance $s(t, h)$ on h.

the contribution of the functions $s_4(t, h)$, $s_5(t, h)$ is rather small. The last column of the table presents the dependence of $s(t, h) = E[Y(t)Y(t + h)]$ on h as the sum of the functions $s_1(t, h), \dots, s_7(t, h)$.

Figure 9 shows the dependence of the functions $s_2(t, h)$ and $s_3(t, h)$ and also the covariances $E[Y(t)Y(t+h)]$ on h for the same t, λ, μ, σ^2. It is seen from Fig. 9 that the inflection of the function $E[Y(t)Y(t + h)]$ is determined by the function $s_3(t, h)$ which, in its turn, is determined by the character of the probabilities $P(B_{3,nm}(t, t{+}h))$. Similar calculations were performed on the base of model samples. In this case we simulated 100000 trajectories of the process and, using these trajectories, estimated the contribution into the estimate of the covariance $E[Y(t)Y(t + h)]$ under fulfillment of conditions 1a–7a. In Fig. 9 these sample values $\tilde{s}_2(t, h)$, $\tilde{s}_3(t, h)$, and $\tilde{s}(t, h)$ of the functions $s_2(t, h)$,$s_3(t, h)$ and $s(t, h)$ are indicated by the corresponding points. The results of calculations with model samples are consistent with the results shown in Table 1 up to statistical error.

3 Conclusion

In this paper we have obtained exact expressions for the covariance function of the process $Y(t)$ (1.1). Numerical experiments show that for $t > 7.5$ the process is close to a stationary one. Further we assume to apply theoretical studies to asymptotic properties of correlation functions and one-dimensional distributions of the considered process.

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