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A block Newton's method for computing invariant pairs of nonlinear matrix pencils

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Abstract: The block inverse iteration and block Newton's methods proposed by the authors of this paper for computing invariant pairs of regular linear matrix pencils are generalized to the case of regular nonlinear matrix pencils. Numerical properties of the proposed methods are demonstrated with a typical quadratic eigenproblem.

Keywords: Newton's method, inverse subspace iteration, regular nonlinear matrix pencil, invariant pair, Sylvester equation.

MSC 2010: 65F15, 65F10, 65F08

1 Introduction

A block Newton method has been proposed in [7] for ordinary partial linear eigenvalue problems. The theory of its convergence was constructed in terms of the integral performance criteria for dichotomy. This method occurred to be quite efficient and has got further development by various authors (see, e.g., [3, 4, 9]). In particular, a variant of this method was proposed in [4] for computing the invariant pairs of regular linear matrix pencils of general form. An efficient variant of the inverse subspace iteration with tuning was proposed for computing the initial guess. It was proposed in [5] to use methods from [4] for computing the minimal invariant pairs of regular nonlinear matrix pencils in the framework of the method of successive linear problems supplemented with the deflation procedure from [10]. Though the proposed combination of four algorithms to be quite efficient, the question about a direct generalization of the algorithms from [4] to the case of nonlinear matrix pencils has remained open. It would allow one to decrease essentially the logical complexity and the number of parameters comparing to the method from [5]. The aim of this paper is to propose and justify such generalization.

Recall that nonlinear matrix pencils in the matrix analysis usually mean pencils of the form

$$T(\lambda) = \sum_{i=1}^{d} T_i f_i(\lambda) \tag{1.1}$$

where $T_1, \ldots, T_d$ are square complex matrices of order n and $f_1, \ldots, f_d$ are scalar functions of a complex variable analytic in some domain Ω of the complex plane (see, e.g., [10]). Pencil (1.1) is called regular if there exists $\lambda_0 \in \Omega$ such that $\det(T(\lambda_0)) \neq 0$. The set of eigenvalues of a regular pencil of form (1.1), i.e., the set of solutions to the equation $\det(T(\lambda)) = 0$ is not more than countable. If λ_* is one of those roots, then nonzero vectors belonging to the kernel of the matrix $T(\lambda_*)$ are called eigenvectors of pencil (1.1) corresponding to the eigenvalue λ_*.

The pair (X, Λ), where $X \in \mathbb{C}^{n\times p}$, $\Lambda \in \mathbb{C}^{p\times p}$, is called the minimal invariant pair of pencil (1.1) if, first,

$$T(X, \Lambda) = \sum_{i=1}^{d} T_i X f_i(\Lambda) = 0$$

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where

$$f(\Lambda) = \frac{1}{2\pi \mathrm{i}} \oint_{\gamma} f(z)(zI - \Lambda)^{-1} \mathrm{d}z$$

and γ is a sufficiently smooth simple positively oriented closed contour in Ω enveloping all eigenvalues of the matrix Λ; second, there exists a positive integer number l such that the matrix

$$\left[X^T, (X\Lambda)^T, \ldots, (X\Lambda^{l-1})^T\right]$$

has full rank. The smallest l satisfying this condition is called the minimality index of the pair (X, Λ). In particular, if $l = 1$, then the matrix X has full rank. If (X, Λ) is a minimal invariant pair of regular pencil (1.1), then the spectrum of Λ is a subset of the set of finite eigenvalues of this pencil and the eigenvectors u of the matrix Λ and the eigenvectors x of pencil (1.1) are connected in the following way: $x = Xu$.

In this paper we consider the problem of computation of the minimal invariant pair of index $l = 1$ corresponding to finite eigenvalues of pencil (1.1) which are the closest ones to a given point λ_0. Applying the change of variables $\lambda \to \lambda + \lambda_0$ and introducing new notations, we can reduce this problem to computing the minimal invariant pair of index 1 corresponding to minimal in magnitude finite eigenvalues of the pencil

$$A(\lambda) = A_0 + \lambda A_1 + \lambda^2 N(\lambda). \tag{1.2}$$

Here

$$A_0 = T(\lambda_0), \quad A_1 = T'(\lambda_0), \quad N(\lambda) = \sum_{i=1}^{d} T_i g_i(\lambda) \tag{1.3}$$

where $g_i(\lambda) = f_i(\lambda + \lambda_0) - f_i(\lambda_0) - f_i'(\lambda_0)(\lambda + \lambda_0)$. Without loss of generality, we assume that λ_0 is not an eigenvalue of pencil (1.1) and, hence, the matrix A_0 is nonsingular. Sections 2 and 3 describe the proposed variant of the inverse subspace iteration and block Newton's method, respectively. Section 4 presents the combination of these two methods where the first method is used to obtain a sufficiently good initial guess and the second one is applied to its fast refinement. Section 5 demonstrates the numerical properties of these methods on the example of a typical square eigenvalue problem appearing in the study of spatial stability of hydrodynamic flows.

Describing the proposed algorithms, we use an additional procedure for orthonormalization of the columns of a given full-rank matrix $W \in \mathbb{C}^{n\times p}$ by computing its QR-decomposition [8], i.e., $W = QR$, where $Q \in \mathbb{C}^{n\times p}$ is a unitary rectangular matrix, $Q^*Q = I$, and $R \in \mathbb{C}^{p\times p}$ is an upper triangular one. Hereinafter $*$ denotes the symbol of conjugate transposition and I is the identity matrix of the corresponding order. We write down the result of this procedure as $(Q, R) = \mathrm{ort}(W)$, or $Q = \mathrm{ort}(W)$ when the matrix R is not needed. In addition, $\|B\|_2$ denotes below the second norm of a vector or matrix B.

We normalize the matrix X of the required minimal invariant pair so that its columns were $A_0^*A_0$-orthonormalized, i.e., $Y^*Y = I$, where $Y = A_0X$.

2 Inverse subspace iteration

In this section we describe the proposed variant of the inverse subspace iteration method for computing the minimal invariant pair (X, Λ) of index 1 corresponding to the p smallest in magnitude finite eigenvalues of regular nonlinear matrix pencil (1.2). Up to normalization, this method is a direct generalization of the variant of inverse subspace iteration which is proposed in [4] for the linear matrix pencils.

Algorithm 1.
Given $\varepsilon > 0$, $X_0 \in \mathbb{C}^{n\times p}$ with $X_0A_0^*A_0^*X_0 = 1$, and a square nonzero matrix Λ_0 of order p.
For $k = 1, 2, \ldots$
1. Solve for X_k the system

$$A_0X_k = -A_1X_{k-1} - N(X_{k-1}, \Lambda_{k-1})\Lambda_{k-1}. \tag{2.1}$$

2. Compute $(Y_k, G_k) = \mathrm{ort}(A_0 X_k)$. Set $X_k := X_k G_k^{-1}$, $\widetilde{A}_1 = Y_k^* A_1 X_k$, $\widetilde{T}_i = Y_k^* T_i X_k$, $i = 1, \ldots, d$.
3. Solve for Λ_k the matrix equation
$$I + \tilde{A}_1 \Lambda_k + \widetilde{N}(\Lambda_k)\Lambda_k^2 = 0 \tag{2.2}$$
where
$$\widetilde{N}(\lambda) = \sum_{i=1}^{d} \widetilde{T}_i g_i(\lambda).$$
4. Compute the residual
$$R_k = Y_k + A_1 X_k \Lambda_k + N(X_k, \Lambda_k)\Lambda_k^2. \tag{2.3}$$
Test the convergence: if $\|R_k\|_2 \leqslant \varepsilon$, then set $X_{\mathrm{out}} = X_k$, $\Lambda_{\mathrm{out}} = \Lambda_k$, $Y_{\mathrm{out}} = Y_k$, $R_{\mathrm{out}} = R_k$ and stop.

At the first step of Algorithm 1 we have to solve block system of linear equations (2.1). If the order n of the matrix A_0 is not too large, then this system may be solved on the basis of LU-decomposition [8] of the matrix A_0, i.e.,
$$A_0 = D\Pi_L^{-1} L U \Pi_U^{-1} \tag{2.4}$$
where Π_L and Π_U are the permutation matrices, D is a left diagonal scaling matrix, and L and U are the lower and upper triangular matrices, respectively. If A_0 is of large order, then this system can be solved, for example, by GMRES [11] using the right preconditioning based on the incomplete LU-decomposition of the matrix A_0 and tuning to accelerate the convergence as it was proposed in [4].

At the third step of Algorithm 1 we have to solve for Λ_k matrix equation (2.2). To do that, we use the following iterative algorithm.

Algorithm 2.
Given $\widetilde{\varepsilon} > 0$ and nonzero matrix $\widetilde{\Lambda}_0$ of order p.
For $j = 0, 1, \ldots$
1. Compute the residual $\widetilde{R}_j = \widetilde{A}_0 + \widetilde{A}_1 \widetilde{\Lambda}_j + \widetilde{N}(\widetilde{\Lambda}_j)\widetilde{\Lambda}_j^2$.
Test the convergence: if $\|\widetilde{R}_j\|_2 \leqslant \widetilde{\varepsilon}$, then set $\Lambda_k = \widetilde{\Lambda}_j$ and stop.
2. Compute $\widetilde{\Lambda}_{j+1} = \widetilde{\Lambda}_j - \widetilde{A}_1^{-1} \widetilde{R}_j$.

If we have no *a priori* information concerning the matrix X_0 which has to be specified in the initialization of Algorithm 1, then X_0 can be taken as a random rectangular matrix with $A_0^* A_0$-orthonormalized columns.

It should be noted that in the case $N(\lambda) \equiv 0$ (i.e., when considered matrix pencil (1.2) is linear) Algorithm 1 coincides with the variant of inverse subspace iteration proposed in [4] up to normalization of the matrices X_k, In this case the matrix Λ_k is computed exactly in one step of Algorithm 2.

3 Block Newton's method

A block Newton method was also proposed in [4] for computing the invariant pair corresponding to the smallest in magnitude finite eigenvalues of a regular linear matrix pencil. In this section we generalize this method to nonlinear matrix pencils for computing the minimal invariant pair of index 1 corresponding to minimal in magnitude finite eigenvalues of pencil (1.2).

Let (X_k, Λ_k) be an approximate minimal invariant pair and $X_k \in \mathbb{C}^{n\times p}$ have $A_0^* A_0$-orthonormalized columns and the matrix $X_k^* A_0^* A_1 X_k$ be nonsingular. Concerning the matrix Λ_k, we assume that it is a solution to equation (2.2).

Note that the left-hand side of equation (2.2) is the projection of residual (2.3) associated with the approximate minimal invariant pair (X_k, Λ_k) onto the span of columns of the matrix Y_k.

Under these assumptions, the kth iteration of the described method consists in the following. First we compute residual (2.3). Then we solve for Φ_k the generalized Sylvester equation
$$A_0 \Phi_k + (I - Y_k Y_k^*) A_1 \Phi_k \Lambda_k = R_k \tag{3.1}$$

and apply the Newton step

$$X_{k+1} = X_k - \Phi_k.$$

After that we normalize the obtained matrix X_{k+1} and calculate the corresponding matrices Y_{k+1}, Λ_{k+1}, and R_{k+1} in a similar way to Steps 2–4 of Algorithm 1.

Since $Y_k^* R_k = 0$, equation (3.1) implies $Y_k^* A_0 \Phi_k = 0$. Therefore, equality (3.1) can be rewritten in the form

$$(I - Y_k Y_k^*)\,(A_0\Phi_k + A_1\Phi_k\Lambda_k) = (I - Y_k Y_k^*)R_k, \qquad Z_k^*\Phi_k = 0 \tag{3.2}$$

where $Z_k = \mathrm{ort}(A_0^* Y_k)$.

In the case $N(\lambda) \equiv 0$ the algorithm described above coincides with the block Newton method proposed in [4] for linear pencils. Moreover, the generalized Sylvester equation has the same form (3.2) in the cases of linear or nonlinear pencil (1.2). This allows us to use the following algorithm proposed in [4] for solving equation (3.2).

Let us compute the Schur decomposition [8]:

$$\Lambda_k = Q_k C_k Q_k^* \tag{3.3}$$

where Q_k is a unitary matrix and C_k is an upper triangular one and make the change of variables $\Phi_k := \Phi_k Q_k$ and $R_k := R_k Q_k$. By Φ_{kj} and R_{kj} we denote the jth columns of the matrices Φ_k and R_k, respectively, and by $c_{ij}^{(k)}$ we denote the ijth entry of matrix C_k. In this case system (3.2) can be rewritten in the form

$$(I - Y_k Y_k^*)\left(A_0 + c_{jj}^{(k)} A_1\right)\Phi_{kj} = \Omega_{kj},\ Z_k^*\Phi_{kj} = 0, \quad j = 1, 2, \ldots, p \tag{3.4}$$

where

$$\Omega_{k1} = (I - Y_k Y_k^*)R_{k1}, \quad \Omega_{kj} = (I - Y_k Y_k^*)\left(R_{kj} - \sum_{i=1}^{j-1} c_{ij}^{(k)} A_1 \Phi_{ki}\right), \quad j > 1.$$

We solve each system in (3.4) with the use of the right preconditioning, i.e.,

$$H_k \Gamma_{kj} = \Omega_{kj}, \qquad \Phi_{kj} = L_k \Gamma_{kj} \tag{3.5}$$

where

$$H_k = (I - Y_k Y_k^*)\left(A_0 + c_{jj}^{(k)} A_1\right) L_k \tag{3.6}$$

and L_k is the preconditioning matrix. Applying GMRES to preconditioned system (3.5), we obtain the approximate solution $\hat{\Gamma}_{kj}$ satisfying the inequality

$$\left\|\Omega_{kj} - H_k \hat{\Gamma}_{kj}\right\|_2 \leqslant \delta \|R_k\|_2 \tag{3.7}$$

where δ is a given tolerance. The second equalities in (3.4) and (3.5) show that the matrix L_k must satisfy the equality $L_k = (I - Z_k Z_k^*)L_k$. On the other hand, at each step of GMRES, the approximate solution belongs to Krylov subspace generated by the matrix H_k, which satisfies the equality $H_k = (I - Y_k Y_k^*)H_k$, and the initial residual $r^0 = \Omega_{kj} - H_k \hat{\Gamma}_{kj}^0$, which satisfies the equality $r^0 = (I - Y_k Y_k^*)r^0$. This means that the approximate solution belongs to the subspace $(I - Y_k Y_k^*)\mathbb{C}^n$ and, therefore, the result does not change (in the exact arithmetic) if we use the matrix $L_k(I - Y_k Y_k^*)$ instead of L_k. Thus, the most general form of the preconditioning matrix L_k is

$$L_k = (I - Z_k Z_k^*)\tilde{L}_k (I - Y_k Y_k^*) \tag{3.8}$$

where $\tilde{L}_k$ is a square matrix of order n, and the problem of choice of the preconditioning matrix L_k is reduced to the choice of the matrix $\tilde{L}_k$. In numerical experiments presented in Section 5, for the matrix $\tilde{L}_k$ we take the matrix A_0^{-1} and compute the products by this matrix on the base of exact LU-decomposition (2.4) of the matrix A_0. If the matrix A_0 is extremely large and sparse, then we may use the incomplete LU-decomposition instead of the complete one.

4 Resulting algorithm

Combining the algorithms described in Sections 2 and 3, we get a method for computing the minimal invariant pair of index 1 corresponding to the smallest in magnitude finite eigenvalues of pencil (1.2).

Starting from an arbitrary matrix X_0 having $A_0^*A_0$-orthonormalized columns and from a zero matrix Λ_0, we apply several steps of Algorithm 1 and set $X_0 = X_{\text{out}}$, $\Lambda_0 = \Lambda_{\text{out}}$, $Y_0 = Y_{\text{out}}$, $R_0 = R_{\text{out}}$. After that we apply the block Newton method described in Section 3. It can be formally written in this case as the following algorithm.

Algorithm 3.
Given $\varepsilon > 0$, $\delta > 0$, and the matrices X_0, Λ_0, Y_0, R_0, computed by Algorithm 1.
For $k = 0, 1, \ldots$
1. Compute the Schur decomposition (3.3).
2. Set $X_k := X_kQ_k$, $Y_k := Y_kQ_k$, $R_k := R_kQ_k$.
3. Compute $Z_k = \text{ort}(A_0^*Y_k)$.
4. Solve for Φ_k system (3.4) by the method described in Section 3.
5. Compute $X_{k+1} = X_k - \Phi_k$.
6. Given X_{k+1}, compute the matrices Y_{k+1}, Λ_{k+1} and the residual R_{k+1} in a similar way to Steps 2–4 of Algorithm 1.
7. Test the convergence: if $\|R_{k+1}\|_2 \leqslant \varepsilon$, then set $X_{\text{out}} = X_{k+1}$, $\Lambda_{\text{out}} = \Lambda_{k+1}$ and stop.

To minimize the number of iterations of Algorithm 3, the Schur decomposition computed at Step 1 of Algorithm 3 needs to have the diagonal entries of C_k ordered in a non-increasing order of magnitude (see [3], p. 597).

If generalized Sylvester equation (3.4) is solved according to (3.5), (3.6), (3.7), and (3.8) with the tolerance δ in (3.7) being small enough, then Algorithm 3 demonstrates the quadratic convergence provided that the approximate invariant pair obtained by a few iterations of Algorithm 1 is sufficiently close to the exact one. As δ increases, the solution of the Sylvester equation becomes less accurate and Algorithm 3 will exhibit only linear convergence.

5 Numerical experiments

To illustrate the use of Algorithms 1 and 3, we took a typical quadratic eigenvalue problem appearing in the analysis of spatial stability of hydrodynamic flows. Consider a steady flow of viscous incompressible fluid in the three-dimensional infinite duct $\{-\infty < x < \infty\} \times \Sigma$ of a constant square cross-section

$$\Sigma = \{(y, z) : \ -1 < y < 1, \ -1 < z < 1\}$$

with non-slipping boundary conditions on the duct walls and the constant pressure gradient $(-\tau, 0, 0)$, where τ is a positive constant. This flow is referred to as the Poiseuille's flow [6] and further it will be assumed as the main flow. The normalized velocity vector of the main flow has the form $(U(y, z), 0, 0)$, where the component $U(y, z)$ of the velocity in the streamwise direction x is nonnegative and reaches its maximum value of 1 at the midpoint of the cross-section. The normalized velocity profile $U(y, z)$ can be obtained by solving the Poisson equation

$$\frac{\partial^2 \tilde{U}}{\partial y^2} + \frac{\partial^2 \tilde{U}}{\partial z^2} = -1$$

in the domain Σ with Dirichlet boundary conditions and taking $U(y, z) = \tilde{U}(y, z)/\tilde{U}(0, 0)$.

The analysis of spatial stability of the Poiseuille flow consists in the study of its stability to infinitesimal disturbances of the form

$$\begin{bmatrix} \mathbf{v}'(x, y, z, t) \\ p'(x, y, z, t) \end{bmatrix} = \begin{bmatrix} \mathbf{v}(y, z) \\ p(y, z) \end{bmatrix} e^{\lambda x - i\omega t} \tag{5.1}$$

where $\mathbf{v}$ and p do not depend on x and t, and λ, ω are some numbers. Substituting (5.1) into linearized disturbance equations (see, e.g., [6]), we obtain the following equations:

$$\begin{aligned} \lambda U u - L u + \frac{\partial U}{\partial y} v + \frac{\partial U}{\partial z} w - \frac{\lambda^2}{\mathrm{Re}} u + \lambda p = 0 \\ \lambda U v - L v - \frac{\lambda^2}{\mathrm{Re}} v + \frac{\partial p}{\partial y} = 0 \\ \lambda U w - L w - \frac{\lambda^2}{\mathrm{Re}} w + \frac{\partial p}{\partial z} = 0 \\ \lambda u + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \end{aligned} \tag{5.2}$$

where u, v, and w are components of the velocity vector $\mathbf{v}$ in the directions of x, y, and z, respectively,

$$L = \mathrm{i}\omega + \frac{1}{\mathrm{Re}} \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right)$$

and Re is the Reynolds number.

Studying the spatial stability, one usually fixes some real ω and considers system (5.2) as an eigenvalue problem for the eigenvalue λ and the eigenvector $(u, v, w, p)^T$. For definiteness sake, let $\omega > 0$. In this case the finite eigenvalues of problem (5.2) having physical interest satisfy the inequalities Imag $\lambda > 0$ and Real $\lambda < 0$. In physical sense, these conditions mean that disturbances do not reflect from the boundary $x = \infty$ and there are no sources there. Usually, one needs to obtain several such eigenvalues and eigenvectors. We consider problem (5.2) with $\omega = 0.1$ and $\mathrm{Re} = 3000$. Note that the considered Poiseuille flow is linearly stable for any finite Reynolds number [1, 2, 12, 13].

In the theory of hydrodynamic stability the eigenvalue λ is usually represented as $\mathrm{i}\alpha$, which is more convenient for physical interpretation of disturbances of form (5.1). However, in this paper we do not use this notation because our aim is only to demonstrate the use of the proposed algorithms.

Since the profile $U(y, z)$ of the Poiseuille flow velocity is an even function of y and z, equations (5.2) admit solutions possessing four different symmetries [1, 2, 12, 13]. Solutions of the form

$$(u_{-+}, v_{++}, w_{--}, p_{-+})$$

where, for example, u_{-+} is an odd function with respect to y and an even one with respect to z, are the smallest stable ones. Below we consider solutions with the above symmetry.

By approximating equations (5.2) in y and z with the second order finite differences on staggered grids of type C with m inner nodes in each direction. We obtain a quadratic matrix pencil of the form

$$T(\lambda) = T_1 + \lambda T_2 + \lambda^2 T_3. \tag{5.3}$$

Taking into account the symmetry, we can reduce the order of pencil (5.3) in approximately four times. Thus, the order n of the considered pencil approximately equals m^2.

Using the standard approach, we can transform quadratic pencil (5.3) to the linear pencil

$$\begin{bmatrix} T_1 & 0 \\ 0 & I \end{bmatrix} - \lambda \begin{bmatrix} -T_2 & -T_3 \\ I & 0 \end{bmatrix} \tag{5.4}$$

with matrices of twice order. The eigenvalues of pencils (5.3) and (5.4) coincide, and after computing the eigenvectors of pencil (5.4) we can construct the eigenvectors of pencil (5.3). Therefore, all eigenvalues and eigenvectors of pencil (5.3) can be computed by the QZ-algorithm [8] applied to (5.4). However, this approach leads to large computational cost (we have to perform about $1600n^3$ double precision arithmetic operations and store about $32n^2$ double precision numbers) and, hence, it is applicable only to approximations on a sufficiently coarse grid.

In Figure 1 from the left, the symbols □ denote all finite eigenvalues of pencil (5.3) computed by the QZ-algorithm for $m = 50$. These computations took about 340 seconds. From the right, the same symbols indicate

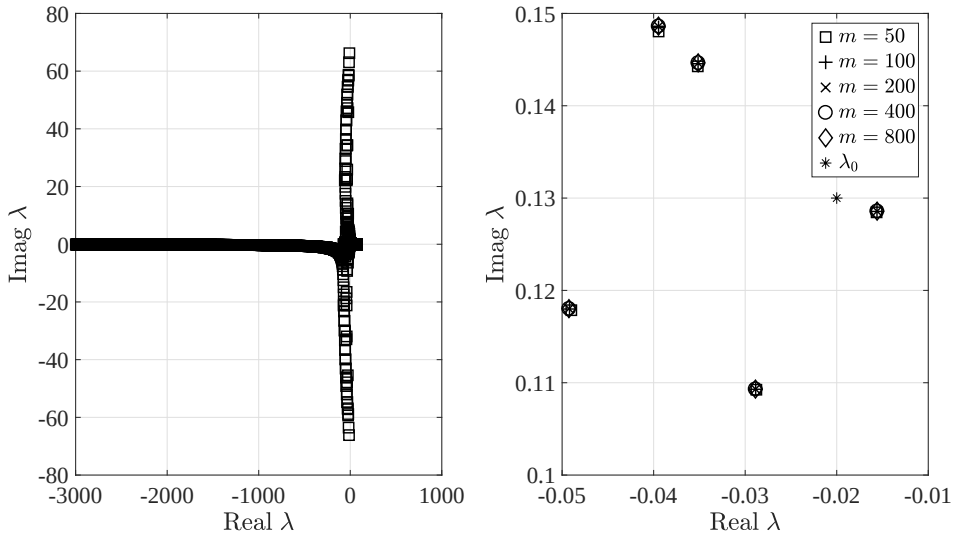

Fig. 1: Left, all finite eigenvalues of pencil (5.3) computed by the QZ-algorithm with $m = 50$ (□); right, five finite eigenvalues closest to the point $\lambda_0 = -0.02 + 0.13\mathrm{i}$ (∗) computed by a combination of Algorithms 1 and 3 for $m = 100$ (+), 200 (×), 400 (◦), and 800 (◇).

$p = 5$ leading eigenvalues from those of physical interest. In addition, the right-hand side of Figure 1 shows the same five eigenvalues computed on finer grids with the use of combination of Algorithms 1 and 3. To do that, we applied the change of variables $\lambda \to \lambda + \lambda_0$ with $\lambda_0 = -0.02 + 0.13\mathrm{i}$ and reduce pencil (5.3) to a pencil of the form

$$A(\lambda) = A_0 + \lambda A_1 + \lambda^2 A_2 \tag{5.5}$$

with the matrices $A_0 = T_0 + \lambda_0 T_1 + \lambda_0^2 T_2$, $A_1 = T_1 + 2\lambda_0 T_2$, and $A_2 = T_2$, and also computed its $p = 5$ minimal in magnitude finite eigenvalues with the use of Algorithms 1 and 3. After that, using the inverse change of variables $\lambda \to \lambda - \lambda_0$, we obtained the required eigenvalues of pencil (5.3).

In accordance with Section 4, Algorithm 1 was used for computing a good initial guess for Algorithm 3. First we applied 10 iterations of Algorithm 1 and computed the initial guess, then we applied 5 iterations of Algorithm 3. At each iteration of Algorithm 1, block systems (2.1) were solved based on complete LU-decomposition (2.4) of the matrix A_0. At each iteration of Algorithm 3, generalized Sylvester equation (3.2) was solved approximately by the algorithm described in Section 3 using GMRES with right preconditioning using the same LU-decomposition of the matrix A_0 for $\tilde{L}_k$. The maximal dimension of Krylov bases in all experiments was set to 50, the parameter δ in (3.7) was equal to 10^{-5}. In the stopping criteria we used $\varepsilon = a10^{-14}$, where a denotes the maximal absolute value of the entries of A_0.

Table 1 illustrates the computational cost of the combination of Algorithms 1 and 3 in computing the approximate invariant pair (X, Λ) corresponding to $p = 5$ finite eigenvalues (see Fig. 1, right) of pencil (5.5), where $\|R\|_2$ is the second norm of the final residual $R = A_0X + A_1X\Lambda + A_2X\Lambda^2$, N_{total} is the total number of multiplications of the matrix A_0^{-1} by a column (based on the LU-decomposition), the solution of all block systems is reduced to such multiplications (these computations formed the main part of total computational costs), T_{total} is the computational time in seconds. Algorithm 2 used at Step 3 of Algorithm 1 and Step 6 of Algorithm 3 always converged in 3–4 iterations for the threshold norm of the residual $\widetilde{\varepsilon} = \varepsilon$.

For $m = 800$ the solid line in Fig. 2 presents the dependence of the second norm of the residual $R_k = A_0X_k + A_1X_k\Lambda_k + A_2X_k\Lambda_k^2$ for the approximate invariant pair (X_k, Λ_k) computed at the kth iteration of the combination of Algorithms 1 and 3 on the number k of that iteration. In this case we have used continuous numbering of iterations. The obtained results show high efficiency of the proposed method including only a slight dependence of the number of particular systems that need to be solved on the order n of matrices of the pencil. It should be noted that for $m = 800$ the norm of the nonlinear term of the final residual was

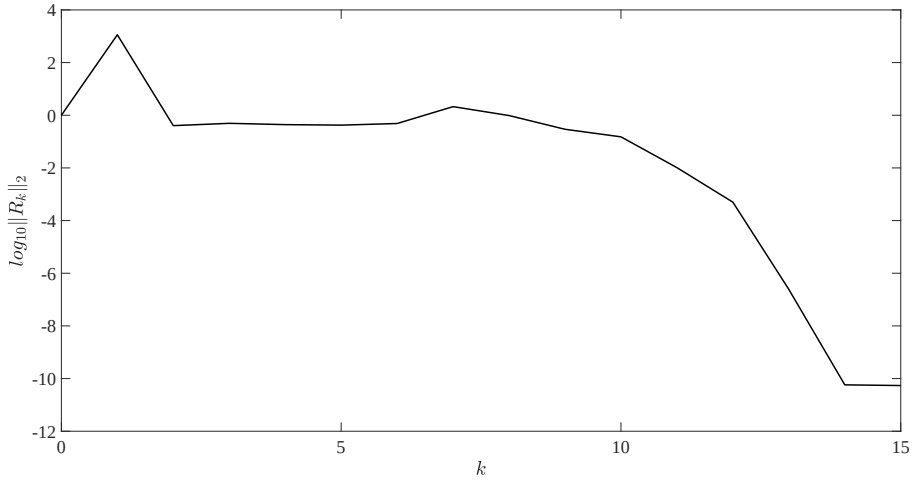

Fig. 2: Convergence of the combination of Algorithms 1 and 3 applied to quadratic pencil (5.5) for $m = 800$.

Tab. 1: Computational cost of the combination of Algorithms 1 and 3.

m	100	200	400	800
n	10^4	4×10^4	1.6×10^5	6.4×10^5
$\|R\|_2$	1.9×10^{-12}	4.7×10^{-12}	1.6×10^{-11}	6.3×10^{-11}
N_{total}	217	223	221	226
T_{total}, s	1	6	29	142

approximately 4.1×10^{-5}. That is, the nonlinear term of pencil (5.5) played a significant role in the considered partial eigenvalue problem.

6 Conclusion

The variant of inverse subspace iteration and block Newton's method proposed previously in [4] for computing the invariant pairs of regular linear matrix pencils are directly generalized in this paper to the case of regular nonlinear matrix pencils. The computation efficiency of these methods is illustrated on the example of a typical quadratic eigenvalue problem appearing in the spatial hydrodynamic stability analysis.

Complete LU-decomposition (2.4) of the matrix A_0 was used in the solution of block systems in Algorithm 1 and in construction of the preconditioner for GMRES in solution of generalized Sylvester equations in Algorithm 3. However, if the order of the matrices of the pencil is sufficiently large, then one should use the incomplete LU-decomposition. In this case, block systems in Algorithm 1 can be solved by GMRES with the use of the right preconditioning and tuning as this was proposed in [4].

The algorithms proposed in this paper are designed to compute the minimal invariant pairs of index $l = 1$. In order to compute minimal invariant pairs of index $l > 1$, these algorithms should be used together with the deflation procedure proposed in [10].

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