Yuri A. Kuznetsov* and Vasily K. Kramarenko

Preconditioners with projectors for mixed hybrid finite element methods

Abstract: We propose and investigate numerically two new preconditioners for the matrices, which arise in the mixed-hybrid finite element methods for diffusion equation in strongly heterogeneous media. Both preconditioners include special projectors on the vector spaces orthogonal to the vectors with constant components. We give general description of the preconditioners and discuss numerical results which demonstrate their efficiency compared to the classical diagonal preconditioner.

Keywords: Preconditioned conjugate gradient method, preconditioners with projectors, mixed-hybrid finite element method, diffusion equation.

MSC 2010: 65F08

DOI: 10.1515/rnam-2017-0004
Received November 24, 2016; accepted November 30, 2016

In the present paper, we propose and investigate numerically two new preconditioners for the condensed matrices, arising in mixed-hybrid finite element approximations of the diffusion equation in strongly heterogeneous media. The construction of preconditioners is based on the ideas originally proposed and investigated in [4]. The preconditioners contain special projectors on the vector spaces orthogonal to vectors with constant components.

The paper is organized as follows. In Section 1, we describe the differential mixed macro-hybrid formulation of the diffusion equation. To simplify the exposition, we consider only the case of polygonal/polyhedral domains and subdomains, and the case of homogeneous Neumann conditions. For the condensed matrix arising in mixed hybrid or mixed macro-hybrid finite element methods we describe the general idea for construction of preconditioners with special projectors [4].

In Section 2, we consider the model diffusion problem in the square domain partitioned in square subdomains with constant diffusion and reaction coefficients in subdomains. Based on the results of Section 1 we propose a variant of a preconditioner with projector which we refer to as a 'coarse mesh preconditioner'. We briefly describe the algorithm of the preconditioner implementation and the respective numerical results.

In Section 3 we consider the domain from Section 2 and assume the checkerboard ordering of square subdomains with large values of the diffusion coefficient. For this particular case we propose a simple block-diagonal preconditioner with projectors. The computational arithmetical cost of this preconditioner is proportional to its dimension. We compare the efficiency of this preconditioner with the classical diagonal (point Jacobi) preconditioner in context of the Preconditioned Conjugate Gradient (PCG) method.

1 Problem formulation

We consider the diffusion problem

$$\begin{aligned} -\nabla\,(K\nabla\, p) + c\, p &= f \quad \text{in} \quad \Omega \\ -\,(K\nabla\, p)\cdot \boldsymbol{n} &= 0 \quad \text{on} \quad \partial\Omega \end{aligned} \tag{1.1}$$

***Corresponding Author: Yuri A. Kuznetsov:** University of Houston, Department of Mathematics, Houston TX, 77204-3476, USA. E-mail: kuz@math.uh.edu
Vasily K. Kramarenko: Institute of Numerical Mathematics, Russian Academy of Sciences, Moscow 119333, Russia

where Ω is a simply connected polygon/polyhedron in $\mathbb{R}^2/\mathbb{R}^3$ with boundary $\partial\Omega$, $\boldsymbol{n}$ is the outward unit normal to $\partial\Omega$, $K = K(\boldsymbol{x})$ is the diffusion tensor, c is a nonnegative coefficient, and $f = f(\boldsymbol{x})$ is a given source function. We partition Ω into $m > 1$ nonoverlapping polygons/polyhedrons E_s, $s = 1, \ldots, m$, and denote by $\Gamma_{s,t}$ the interfaces between E_s and E_t, i.e., $\Gamma_{s,t} = \partial E_s \cap \partial E_t$, $s < t$, $s, t = 1, \ldots, m$. For the sake of simplicity we assume that subdomains E_s are convex, $K \equiv K_s = k_s I_s$ are constant scalar tensors in E_s, where I_s is the identity matrix, and $c \equiv c_s$ are nonnegative constants, $s = 1, \ldots, m$.

The differential mixed macro-hybrid formulation of (1.1) based on the partitioning of Ω into subdomains E_s, $s = 1, \ldots, m$, reads as follows:

$$\begin{aligned} K_s^{-1}\boldsymbol{u}_s + \nabla p_s &= 0 \quad \text{in } E_s \\ \nabla \boldsymbol{u}_s + c_s p_s &= f \quad \text{in } E_s \\ p_s &= p_t \quad \text{on } \Gamma_{s,t},\ s < t \\ \boldsymbol{u}_s \cdot \boldsymbol{n}_s + \boldsymbol{u}_t \cdot \boldsymbol{n}_t &= 0 \quad \text{on } \Gamma_{s,t},\ s < t \\ \boldsymbol{u}_s \cdot \boldsymbol{n} &= 0 \quad \text{on } \partial\Omega \cap \partial E_s \end{aligned} \tag{1.2}$$

where $\boldsymbol{n}_s$ are the outward normals to ∂E_s and ∂E_t, $s, t = 1, \ldots, m$. The variational mixed hybrid formulation of (1.2) is well known [1].

Let Ω_h be a triangular/tetrahedral mesh in Ω, conforming with ∂E_s, i.e., ∂E_s is the union of the sides/faces of mesh cells in Ω_h, $s = 1, \ldots, m$. We approximate (1.2) by the lowest order mixed hybrid Raviart–Thomas finite element method (RT_0 method) on the mesh Ω_h. The method results in the linear algebraic system [1, 2]:

$$\mathcal{A}\begin{pmatrix} \overline{u} \\ \overline{p} \\ \overline{\lambda} \end{pmatrix} = \begin{pmatrix} 0 \\ -\overline{F} \\ 0 \end{pmatrix} \tag{1.3}$$

with the saddle point matrix

$$\mathcal{A} = \begin{pmatrix} M & B^T & C^T \\ B & -\Sigma & 0 \\ C & 0 & 0 \end{pmatrix}. \tag{1.4}$$

Under the appropriate ordering of the flux variables (DOFs) the matrix M is block diagonal with $3\times3/4\times4$ blocks. Eliminating the vector $\overline{u}$ in (1.3) we get the system:

$$S\begin{bmatrix} \overline{p} \\ \overline{\lambda} \end{bmatrix} \equiv \begin{bmatrix} S_{pp} & S_{p\lambda} \\ S_{\lambda p} & S_{\lambda\lambda} \end{bmatrix}\begin{bmatrix} \overline{p} \\ \overline{\lambda} \end{bmatrix} = \begin{bmatrix} \overline{F} \\ 0 \end{bmatrix} \tag{1.5}$$

with the symmetric positive definite matrix S (semidefinite, if $c_s = 0$, $s = 1, \ldots, m$):

$$S = \begin{bmatrix} B \\ C \end{bmatrix} M^{-1}\begin{bmatrix} B^T & C^T \end{bmatrix} + \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}. \tag{1.6}$$

The matrix S in (1.5) can be defined by subassembling matrices

$$S_t = A_t + \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \tag{1.7}$$

associated with subdomains E_t, $t = 1, \ldots, m$, i.e.,

$$S = \sum_{t=1}^{m} N_t S_t N_t^T$$

where N_t are the subassembling matrices, A_t are condensed matrices for subdomains E_t, and Σ_t are the diagonal matrices, $t = 1, \ldots, m$. It is obvious that the matrices A_t are singular, and $\ker(A)$ consists of vectors with constant components, i.e., consists of the vectors, which are proportional to the vector $\overline{e}$ with all components equal to one.

Let M_t be diagonal $m_t \times m_t$ matrices with positive diagonal entries, where m_t is the dimension of A_t. Let us consider the eigenvalue problem

$$A_t \overline{w} = \lambda M_t \overline{w}. \tag{1.8}$$

We denote by $0 = \lambda_{t,1} < \lambda_{t,2} \leqslant \ldots \leqslant \lambda_{t,m_t}$ the eigenvalues in (1.8) and by $\overline{w}_{t,1}, \ldots, \overline{w}_{t,m_t}$ the set of M_t-orthogonal eigenvectors in (1.8), $t = 1, \ldots, m$. Then we get the spectral decompositions

$$A_t = M_t W_t \Lambda_t W_t^T M_t \tag{1.9}$$

where $\Lambda_t = \mathrm{diag}\{\lambda_{t,1}, \lambda_{t,2}, \ldots, \lambda_{t,m_t}\}$ and

$$W_t = \left[\overline{w}_{t,1}, \ldots, \overline{w}_{t,m_t}\right]. \tag{1.10}$$

It is obvious that

$$W_t^T M_t W_t = M_t W_t W_t^T = W_t W_t^T M_t = I_t \tag{1.11}$$

with the $m_t \times m_t$ identity matrix I_t, $t = 1, \ldots, m$. We also remind that

$$\overline{w}_{t,1} = \frac{1}{\|\overline{e}_t\|_{M_t}} \overline{e}_t \tag{1.12}$$

where $\overline{e}_t$ is $m_t \times m_t$ vector with the components equal to one and $\|\overline{e}_t\|_{M_t}$ is the M_t-norm of $\overline{e}_t$, $t = 1, \ldots, m$.

Following [4], we replace the matrices A_t in (1.9) with matrices

$$B_t = \alpha_t M_t W_t \left(I_t - \hat{I}_t\right) W_t M_t = \alpha \left(M_t - M_t \overline{w}_{t,1} \overline{w}_{t,1}^T M_t\right) \tag{1.13}$$

where α_t are some positive numbers and $\hat{I}_t$ are $m_t \times m_t$ matrix with the only nonzero entry in the position (1,1) equals to one, $t = 1, \ldots, m$. It is obvious that $B_t = B_t^T \geqslant 0$, $\ker(B_t) = \ker(A_t)$ and

$$\frac{\lambda_{t,2}}{\alpha_t} B_t \leqslant A_t \leqslant \frac{\lambda_{t,m_t}}{\alpha_t} B_t, \quad t = 1, \ldots, m. \tag{1.14}$$

We define the matrices

$$B = \sum_{t=1}^{m} N_t \left[B_t + \begin{bmatrix} \Sigma_t & 0 \\ 0 & 0 \end{bmatrix}\right] N_t^T. \tag{1.15}$$

Then from [4] we get the inequalities

$$\beta_1 B \leqslant S \leqslant \beta_2 B \tag{1.16}$$

with

$$\beta_1 = \min\left\{\min_{t=1,\ldots,m} \frac{\lambda_{t,2}}{\alpha_t}; 1\right\} \tag{1.17}$$

and

$$\beta_2 = \max\left\{\max_{t=1,\ldots,m} \frac{\lambda_{t,m_t}}{\alpha_t}; 1\right\} \tag{1.18}$$

Remark 1.1. In this section triangular/tetrahedral meshes and RT_0 approximations were chosen to illustrate the algebraic operations (1.9)–(1.15) and estimates (1.16)–(1.18). In fact, we can repeat the same transformations and estimates for many other meshes and approximations. In particular, for numerical results in Sections 2 and 3 we shall use square meshes and macro-hybrid mixed discretizations with piece-wise constant fluxes [5] which practically coincide with the macro-hybridized finite-volume method.

2 Coarse mesh preconditioner

In this section, we consider the diffusion equation in the unit square partitioned into $m = \sqrt{m} \times \sqrt{m}$ subsquares as shown in Fig. 1. We assume that the diffusion coefficients c are constant in each subdomain/subsquare E_t, i.e., $k_t = \mathrm{const} > 0$ and $c \equiv c_t = \mathrm{const} \geqslant 0$ in E_t, $t = 1, \ldots, m$. In $\Omega = [0;1] \times [0;1]$ we

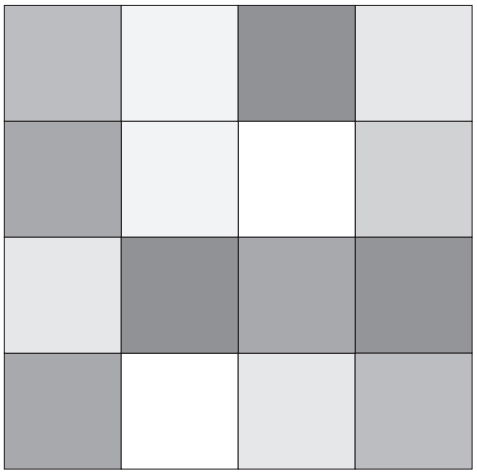

Figure 1. Domain Ω partitioned in subsquares E_s, $s = 1, \ldots, m$, $m = 16$.

define the square mesh Ω_h with the step-size $h = 1/n$, $n \geqslant 2$. We assume that n is a multiple of $\sqrt{m}$. Thus, we defined the macro-mesh (coarse mesh) with the step-size $H = 1/\sqrt{m}$, and the mini-mesh Ω_h.

We approximate the diffusion problem (1.2) by the mixed macro-hybrid finite element method with piecewise constant fluxes [5]. It can be easily shown that the proposed discretization is equivalent to the finite volume discretization of (1.1) on Ω_h hybridized on the interfaces $\Gamma_{s,t}$ between macrocells E_s and E_t, and on the boundary of Ω. This approximation results in system (1.3)–(1.4) with a diagonal matrix M. We observe, in this method DOFs (degrees of freedom) for λ represent the mean values of the solution function p on the boundaries of mesh cells in Ω_h belonging to the boundaries of E_t, $t = 1, \ldots, m$.

To define the matrices B_t, $t = 1, \ldots, m$, in (1.13) we choose M_t to be a diagonal matrix with the diagonal entries equal to h^2 for the components of subvector $\overline{p}$, and equal to h for the components of subvector $\overline{\lambda}$, $t = 1, \ldots, m$. Thus each M_t has $(H/h)^2$ diagonal entries equal to h^2, and $4H\,(H/h)$ diagonal entries equal to h, $t = 1, \ldots, m$.

For numerical experiments we choose

$$\alpha_t = \|M_t^{-1} A_t\| \equiv \alpha = \frac{8k_t}{h^2}, \quad t = 1, \ldots, m. \tag{2.1}$$

Then, in inequalities (1.16) we get

$$\beta_1 = \min_{t=1,\ldots,m} \frac{\lambda_{t,2}}{\alpha_t}, \qquad \beta_2 = 1. \tag{2.2}$$

We observe, that the value of β_1 does not depend on the values k_t and c_t, $t = 1, \ldots, m$.

The numerical results are given in Tables 1–4. Abbreviation PCG stays for the preconditioned conjugate gradient method with preconditioner equal to the diagonal of S, and abbreviation CMP (Coarse Mesh Preconditioner) stays for the Preconditioned Conjugate Gradient method with the above preconditioner B in (1.15). The algorithm for solving a system with the matrix B is described in [4].

In Tables 1 and 2 we give the numbers of iterations of PCG and CMP methods sufficient for minimization of S-norm of the initial error in $\frac{1}{\varepsilon} = 10^6$ times in the case when the reaction coefficient $c \equiv 1$ in Ω. In Tables 3 and 4 we give the number of iterations of the same methods with the same tolerance when the values of c_s are chosen randomly in the interval $[0.01; 100]$.

We can clearly observe that the number of iterations in the PCG method essentially depends on the values of the diffusion coefficients. The number of iterations in CMP method does not depend on the values of coefficients k_t and c_t, $t = 1, \ldots, m$. We can also observe that the number of iterations is proportional to H/h. This statement correlates with theoretical estimates in [4].

Table 1. $H = 1/8$, $m = 8 \times 8 = 64$ subsquares.

$n \times n$	$k_{max} = 1$		$k_{max} = 10^2$		$k_{max} = 10^4$		$k_{max} = 10^6$	
	PCG	CMP	PCG	CMP	PCG	CMP	PCG	CMP
128×128	502	310	927	310	1220	309	1227	307
256×256	910	652	1617	676	2317	657	3741	651
512×512	1705	1373	3419	1398	4767	1377	5615	1381

Table 2. $H = 1/16$, $m = 16 \times 16 = 256$ subsquares.

$n \times n$	$k_{max} = 1$		$k_{max} = 10^2$		$k_{max} = 10^4$		$k_{max} = 10^6$	
	PCG	CMP	PCG	CMP	PCG	CMP	PCG	CMP
128×128	600	144	1267	145	2499	144	5309	142
256×256	944	330	1958	331	4536	330	9139	326
512×512	1678	744	3219	744	8591	741	18044	733

Table 3. $H = 1/8$, $m = 8 \times 8 = 64$ subsquares.

$n \times n$	$k_{max} = 1$		$k_{max} = 10^2$		$k_{max} = 10^4$		$k_{max} = 10^6$	
	PCG	CMP	PCG	CMP	PCG	CMP	PCG	CMP
128×128	276	377	701	317	1278	307	1520	306
256×256	543	736	1324	691	2145	655	2655	650
512×512	970	1497	2217	1421	4508	1392	6680	1303

Table 4. $H = 1/16$, $m = 16 \times 16 = 256$ subsquares.

$n \times n$	$k_{max} = 1$		$k_{max} = 10^2$		$k_{max} = 10^4$		$k_{max} = 10^6$	
	PCG	CMP	PCG	CMP	PCG	CMP	PCG	CMP
128×128	278	199	798	158	2009	147	4726	143
256×256	527	388	1323	339	3524	331	8938	323
512×512	907	794	2277	755	7502	736	15326	733

3 Block diagonal preconditioner

In this section, we partition the macrocells into two groups and assume that any two macrocells E_s and E_t in the first group do not have interfaces, i.e., $|\Gamma_{s,t}| = 0$, $1 \leqslant s < t \leqslant m_0$, $m_0 < m$.

We assume that the diffusion coefficients k_s for the first group of macrocells are much larger than one ($k_s \gg 1$), $s = 1, \ldots, m_0$, and $k_s \approx 1$ for the second group of macrocells, $s = m_0 + 1, \ldots, m$. We represent the matrix S as the sum of two matrices S_1 and S_2, where the matrix S_1 is defined by

$$S_1 = \sum_{t=1}^{m_0} N_t \left[(k_t - 1) A_t + \begin{bmatrix} \Sigma_t & 0 \\ 0 & 0 \end{bmatrix} \right] N_t^T. \tag{3.1}$$

Then, the block diagonal preconditioner B for the matrix S is defined by

$$B = D + \sum_{t=1}^{m_0} N_t \left[(k_t - 1) B_t + \begin{bmatrix} \Sigma_t & 0 \\ 0 & 0 \end{bmatrix} \right] N_t^T \tag{3.2}$$

where the matrix D is a diagonal matrix with positive diagonal entries. The matrix B is a block diagonal matrix

$$B = \sum_{t=1}^{m} N_t B_{tt} N_t^T \tag{3.3}$$

where the blocks

$$B_{tt} = D_{tt} + (k_t - 1) M_t \overline{w}_{t,1} \overline{w}_{t,1}^T M_t \tag{3.4}$$

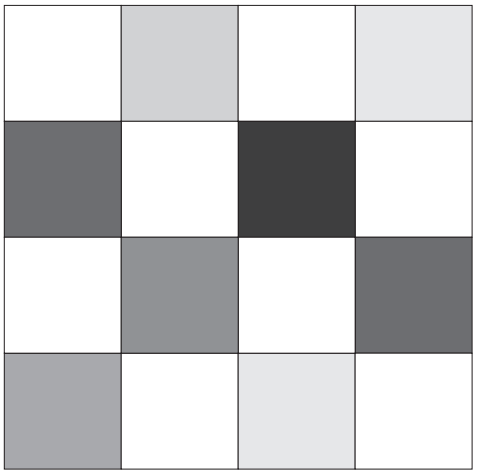

Figure 2. Example of 'checkerboard distribution'.

Table 5. $H = 1/8$, $m = 8 \times 8 = 64$ subsquares.

$n \times n$	$k_{max} = 10^2$			$k_{max} = 10^4$			$k_{max} = 10^6$		
	PCG	CMP	BDP	PCG	CMP	BDP	PCG	CMP	BDP
128×128	1141	306	809	2101	265	747	4804	245	625
256×256	2222	642	1517	4206	558	1347	8337	542	1155
512×512	4455	1375	2826	9111	1219	2487	13796	1151	2188

Table 6. $H = 1/16$, $m = 16 \times 16 = 256$ subsquares.

$n \times n$	$k_{max} = 10^2$			$k_{max} = 10^4$			$k_{max} = 10^6$		
	PCG	CMP	BDP	PCG	CMP	BDP	PCG	CMP	BDP
128×128	1508	143	692	6588	128	625	23093	112	541
256×256	2479	325	1300	11095	276	1153	32797	267	987
512×512	4662	733	2357	16405	639	2080	58545	598	1711

are rank one perturbations of diagonal matrices D_{tt} and the blocks B_{tt}, $t = m_0 + 1, \ldots, m$, are diagonal matrices. Let us assume that

$$\nu_1 D \leqslant S_2 \leqslant \nu_2 D \tag{3.5}$$

with some positive $\nu_1 < \nu_2$. Then

$$\beta_1 B \leqslant S \leqslant \beta_2 B \tag{3.6}$$

where

$$\beta_1 = \min\Big\{ \min_{t=1,\ldots,m} \frac{\lambda_{t,2}}{\alpha_t};\ \nu_1;\ 1 \Big\} \tag{3.7}$$

and

$$\beta_2 = \max\Big\{ \max_{t=1,\ldots,m} \frac{\lambda_{t,m_t}}{\alpha_t};\ \nu_2;\ 1 \Big\}. \tag{3.8}$$

It is obvious that the computational cost of solving a system with matrix B in (3.3)–(3.4) is lower then the computational cost for a residual with the matrix S.

For numerical experiments we choose the domain and partitioning from Section 2 shown in Fig. 1. We partition macrocells into two subgroups based on checkerboard ordering shown in Fig. 2.

The diffusion coefficient for macrocells in the first group is chosen randomly on the segment $[10; k_{\max}]$ and the diffusion coefficient equals to one for the macrocells in the second group. We choose $c \equiv 1$ in Ω and

define the preconditioner B by

$$B = 8I + \sum_{t=m_0+1}^{m} N_t \begin{bmatrix} \Sigma_t & 0 \\ 0 & 0 \end{bmatrix} N_t^T \tag{3.9}$$

where I is the identity matrix of the same size as B in (3.9).

The numerical results are given in Tables 5–6. Abbreviation BDP is used for the Preconditioned Conjugate Gradient method with preconditioner (3.9). An algorithm for inverting a diagonal matrix with rank one perturbation can be found in [3].

The numerical results clearly show that the number of iterations of BDP is $2 \div 2.5$ times larger than the number of iterations for the PCG method in Tables 1–2 for $k_{\max} = 1$, and does not depend on the values of k_s, $s = 1, \ldots, m_0$.

References

[1] D. Boffi, Fr. Brezzi, and M. Fortin, *Mixed Finite Element Methods and Applications*. Springer Series in Computational Mathematics, Vol. 44. Springer, Berlin, 2013.

[2] F.Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*. Springer, New York, 1991.

[3] G. H. Golub and Ch. F. Van Loan, *Matrix Computations*. Johns Hopkins University Press, Baltimore, 1983.

[4] Yu. A. Kuznetsov, Two-level preconditioners with projectors for unstructured grids. *Russ. J. Numer. Anal. Math. Modelling* **15** (2000), No. 3–4. 247–255.

[5] Yu. A. Kuznetsov, Approximations with piece-wise constant fluxes for diffusion equations. *Russ. J. Numer. Anal. Math. Modelling* **19** (2011), No. 4. 309–328.