

Research Article

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A magnetohydrodynamic flow of a water-based hybrid nanofluid past a convectively heated rotating disk surface: A passive control of nanoparticles

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Abstract: One of the basic fluid mechanics problems of fluid flows over a revolving disk has both theoretical and real-world applications. The flow over a rotating disk has been the subject of numerous theoretical studies because it has many real-world applications in areas like rotating machinery, medical equipment, electronic devices, and computer storage. It is also crucial for engineering processes. Therefore, this article deals with a time-independent water-based hybrid nanofluid flow containing copper oxide and silver nanoparticles past a spinning disk. The Newtonian flow is taken into consideration in this analysis. The influence of magnetic field, thermophoresis, nonlinear thermal radiation, Brownian motion, and activation energy has been considered. The present analysis is modeled in a partial differential equation form and is then converted to ordinary differential equations using appropriate variables. A numerical solution using the bvp4c technique is accomplished using MATLAB software. The current results are matched with the previous literature and established a close relationship with previous studies. The purpose of this investigation is to numerically investigate the time-independent hybrid nanofluid flow comprising copper oxide and silver

nanoparticles over a rotating disk surface. The results show that the increased magnetic parameters increase the friction force at the surface, which decreases the radial and azimuthal velocity distribution. At the sheet surface, the radial velocity of the hybrid nanofluid shows dominant performance compared to the nanofluid. On the other hand, the magnetic factor has dominant behavior on the azimuthal velocity component of the nanofluid flow compared to the hybrid nanofluid flow. The higher volume fraction and magnetic factor enhance the skin friction at the disk surface. Furthermore, greater surface drag is found for the hybrid nanofluid flow. The higher solid volume fraction, temperature ratio, and Biot number enhance the rate of heat transmission. Also, a higher rate of heat transmission is observed for the hybrid nanofluid flow.

Keywords: MHD, hybrid nanofluid, nonlinear thermal radiation, activation energy, heat source

Nomenclature

r, ψ, z	coordinate system
u, w, v	velocity components
Ω	angular velocity
B_0	strength of the magnetic field
T_f, T_w, T_∞	reference, surface, and ambient temperatures
C_f, C_w, C_∞	reference, surface, and ambient concentrations
δ	exponential index
p	pressure
μ	dynamic viscosity
σ, k	electrical and thermal conductivities
C_p	specific heat
Q	coefficient of heat source
ν	kinematic viscosity
D_T	thermophoretic coefficient
D_B	Brownian diffusion coefficient

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k_r	chemical reaction coefficient
M	magnetic factor
Pr	Prandtl number
ω	temperature difference factor
Q_e	exponential heat source factor
Bi	thermal Biot number
Rd	radiation factor
Nt	thermophoresis factor
Nb	Brownian motion factor
Sc	Schmidt number
E	activation energy factor
K_r	chemical reaction factor
θ_w	temperature ratio factor
$F(\eta)$	radial velocity component
$G(\eta)$	azimuthal velocity component

Subscripts

f, nf, hnf	base fluid, nanofluid, and hybrid nanofluid
s_1	CuO nanoparticle
s_2	Ag nanoparticle

1 Introduction

In the past, the cooling of various devices in industry and different engineering applications was challenging. Pure fluids like water and engine oil have lower thermal flow characteristics. Therefore, there was a need to enhance such flow characteristics of pure fluids. Scientists and researchers have performed various experiments to attain this goal and have modeled various approaches in this regard. The mixture of tiny-sized particles with pure fluid is one such approach, as introduced by Choi and Eastman [1]. Afterward, a new door opened in the literature to enhance the thermal characteristics of pure fluids in a variety of ways. Shahzad *et al.* [2] examined the convergence of second order for EMHD nanofluid on a rotating disk with the Hall effect and concluded that the volume fraction has a tremendous impact on fluid motion and heat transportation. Shamshuddin *et al.* [3] deliberated on dissipative and radiation mechanisms for nanofluid flow upon a disk where the flow was encountered by the Hall current. Pasha *et al.* [4] investigated statistically the viscous nanofluid flow on an elongating sheet and uncovered that fluid velocity deteriorated and heat flow improved for growth at nanoparticle concentration. Shah *et al.* [5] used nanoparticles of gold in blood to optimize its thermal flow

characteristics subject to thermally radiative effects between two plates. Asghar and Ying [6] studied 2-D radiative and mixed convective nanofluid flow on an exponentially contracting surface with impacts of partial slip constraints and observed that an upsurge in radiation factor supported the growth in thermal characteristics. Khan *et al.* [7] debated the nanofluid flow on a heated moving needle using chemically reactive and viscous dissipative effects. Haider *et al.* [8] discussed computationally a comparative work for a blood-mathematical model with impacts of magnetic field and nanoparticles. Akhtar *et al.* [9] carried out a dynamical analysis with computational evaluation for endoscopy of the left artery and simulated that flow arriving at the slender section of stenosis attained a maximum velocity. Naveed Khan *et al.* [10] studied stagnant point flow for the nanofluid of second grade on an exponentially elongating surface and established that thermal distribution augmented for higher Brownian and thermophoresis factors. Ahsan *et al.* [11] discussed bio-convective Darcy flow on a curved elongating surface with impacts of heterogeneous and homogenous reactions and observed that the velocity of fluid declined with growth in magnetic and porosity factors. Kanwal *et al.* [12] studied the dynamics of nanofluid flow with nonlinear and linear mixed convective thermal radiations and activated energy on a gyrating disk.

Recently, it has been proved that the thermal conductivities of pure fluid can be upgraded more by collaborating two types of nanoparticles in these fluids, which are known as hybrid nanofluids. Waseem *et al.* [13] scrutinized the mixed convective flow of hybrid nanofluid with the combined impression of the magnetic field, homogenous and heterogeneous reactions. Hussain *et al.* [14] used a sintered copper wick pipe to discuss experimentally the thermal performance of hybrid nanoparticles. Guedri *et al.* [15] discussed nanofluid flow over a nonlinear stretched sheet using various effects of the flow of the fluid. Zhang *et al.* [16] examined the magnetohydrodynamics (MHD) nanofluid flow on an elastic surface by mixing nanoparticles of tantalum and nickel and proved that the velocity of fluid increased with growth in tantalum nanoparticles and Darcy number while magnetic effects opposed the fluid motion. Ojjela [17] investigated numerically the heat transference of nanofluid and depicted that the thermal transportation rate increased with alumina nanoparticles compared with silica nanoparticles. Gumber *et al.* [18] debated the thermal characteristics of the hybrid nanoparticles past a plate and concluded that the growth in suction had a positive impact on the fluid velocity. Sriharan *et al.* [19] used the hybrid nanofluid flow through a tube subject to the heat source and concluded that the Nusselt number improved by 28.4% for the Al_2O_3/CuO -DIW hybrid nanofluid. Ullah *et al.* [20] examined thermal and mass flow characteristics for magnetized nanofluid flow

on a cone with convective conditions at the boundary. Zawawi *et al.* [21] used a hybrid nano-lubricant flow with different composition ratios of nanoparticles to augment the thermal behavior of the air-conditioned system. Asghar *et al.* [22] inspected magnetized 3-D gyratory nanofluid flow on contracting and expanding surfaces and noticed that thermal panels augmented with growth in Eckert number and radiation factor. Babar *et al.* [23] inspected the effectiveness of H₂O-based ferric oxide-silica hybrid nanofluid flow to investigate the airfoil-designed pin-fin with a heat sink. They deduced that the Nusselt number enhanced with a thermal power of 75 W. Akbar *et al.* [24] studied magnetized thermal transportation for a dual-phase nanofluid flow model using the impacts of thermal radiations and ANN approach for the solution and noticed that thermal flow augmented with an increase in the radiation factor. Ullah *et al.* [25] evaluated analytically the 3-D problem of a condensed film on a gyrating disk. Raja *et al.* [26] studied nanofluid flow on a gyratory disk with impacts of particle slip and magnetic field effects and matched the efficiency of their designed LMB-NN methodology through a comparative study and performance by utilizing error histograms, regression, and correlation. Ullah *et al.* [27] discussed computationally the magnetized squeezed flow for the Jeffery fluid in a gyrating frame with the mass and heat flux model suggested by Cattaneo-Christov.

MHD, a discipline at the intersection of fluid dynamics and electromagnetism, investigates the intricate interplay between magnetic fields and electrically conductive fluids [28]. In the specific perspective of heat transfer for fluid flow on a disk, MHD manifests profound impacts on the thermal behavior. As a magnetic field permeates the electrically conductive fluid, it induces electric currents, creating a dynamic coupling between the magnetic field and the fluid motion. This interaction significantly alters the flow patterns around the disk, influencing the velocity profiles and modifying the thickness of the boundary layer. MHD normally enhances heat transfer, contingent upon factors such as the orientation and strength of the magnetic field, as well as fluid properties [29,30]. In instances where MHD promotes enhanced heat transfer, it does so by fostering better mixing and circulation of the fluid near the disk's surface [31]. Additionally, the magnetic Reynolds number, indicative of the dominance of magnetic effects, plays a crucial role in determining the overall impact on heat transfer. The complex relationship of these factors is further compounded by the potential development of instabilities and turbulence within the fluid flow induced by MHD, thereby influencing the stability and proficiency of heat transfer phenomena [32]. Solving the complex system of coupled fluid flow problem, heat transfer, and electromagnetic fields is fundamental in comprehending the multifaceted implications of MHD on

heat transfer. Asghar *et al.* [33] evaluated double solutions for rotary convective 3-D nanofluid flow on contracting and expanding surfaces. Rasool *et al.* [34] examined time-based double solutions for hybrid nanofluid flow subject to slip and magnetic constraints and found that flow is retarded in the first solution with an increase in nanoparticles' volumetric fraction of copper. Lund *et al.* [35] discussed Casson magnetized nanofluid flow on a moveable and permeable surface subject to Joule heating and thermally radiative effects.

Nonlinear thermal radiation is the process of heat transfer through electromagnetic waves emitted by a surface that displays a non-linear association between the temperature and radiative heat flux. In traditional linear thermal radiation, the Stefan-Boltzmann law is employed to describe the radiant heat exchange, assuming a linear proportionality between the temperature and emitted radiation. However, in nonlinear thermal radiation, the radiative properties of the material, such as emissivity and absorptivity, are temperature-dependent in a non-linear manner [36]. This nonlinearity introduces complexities in predicting and analyzing heat transfer phenomena. The impacts of nonlinear thermal radiation on heat transfer are profound, influencing several key aspects. First, it alters the radiative heat transfer rates between surfaces, deviating from the simpler linear relationships. This can lead to significant variations in temperature distributions and thermal gradients within a system. Second, nonlinear thermal radiation complicates the design and optimization of heat exchange systems, as traditional linear models may not accurately capture the intricate temperature-dependent radiative properties of the materials. In applications such as high-temperature processes, combustion, or space exploration, where radiative heat transfer plays a critical role, accounting for nonlinear thermal radiation becomes essential for precise and reliable thermal analysis [37]. Additionally, in scenarios involving materials with temperature-sensitive optical properties, nonlinearity can impact the overall energy balance and thermal performance of the system. Accurate modeling of nonlinear thermal radiation is crucial for advancing technologies in fields, such as aerospace, materials science, and energy systems, where a nuanced understanding of heat transfer phenomena is vital for optimizing efficiency, durability, and safety. Shaw *et al.* [38] examined the features of linear and nonlinear radiations on MHD liquid flow on a cylinder and scrutinized the thermal flow characteristics using the values of Prandtl number in the range $10^{-10} \leq Pr \leq 10^4$.

An exponential heat source denotes a heat generation term in the energy equation, which is proportional to an exponential function of temperature. The exponential nature of the heat source implies that as the temperature increases, the heat generation rate diminishes exponentially [39]. The

impact of an exponential heat source on the heat transfer phenomena is significant, introducing nonlinearity and influencing temperature distributions within a system [40]. As the temperature increases, the heat generation decreases rapidly, potentially leading to complex spatial and temporal variations in the temperature field [41]. The impact of an exponential heat source is crucial in applications like electronics cooling, combustion processes, and materials processing, where accurate prediction of temperature profiles is essential for optimizing efficiency, preventing overheating, and ensuring the integrity of the system [42]. Modeling of exponential heat sources in heat transfer simulations is imperative for designing and engineering systems that effectively manage thermal loads and maintain desired temperature conditions in diverse technological contexts [43]. Haq *et al.* [44] explored the impact of exponential heating sources on second-grade MHD fluid flow subject to chemically reactive effects.

A lower quantity of energy, mandatory to start a reaction chemically, is called activation energy. Alqarni *et al.* [45] inspected the motivation of activated energy on a Casson fluid flow over a spinning disk. Rekha *et al.* [46] debated the activation energy influence on the motion of nanoparticles through wedge, cone, and plate and deduced that mass transportation augmented more slowly for the flow of fluid past a cone in comparison to flow on a wedge or plate. Abbasi *et al.* [47] scrutinized the fluid flow at the stagnant point of a horizontal cylinder subject to activation energy and perceived that with augmentation in the chemical reaction and temperature gradient factors, the mass diffusion declined. Ayub *et al.* [48] investigated the activation energy impact upon nanofluid flow on a cylindrical surface using inclined magnetic effects and deduced that an increase in the magnetic factor amplified the energy and weakened the velocity of the fluid flow (Table 1).

One of the basic fluid mechanics problems of fluid flows over a revolving disk has both theoretical and real-world applications. The flow on a gyrating disk has remained the subject of numerous theoretical studies because it has many real-world applications in areas like rotating machinery, medical equipment, electronic devices, and computer storage.

Table 1: Thermophysical features of the water nanofluid, copper oxide, and silver nanoparticles [55]

Base fluid and nanoparticles	H ₂ O	CuO	Ag
σ [$\Omega \cdot \text{m}^{-1}$]	0.05	2.70×10^{-8}	6.30×10^7
k [$\text{W} \cdot \text{m}^{-1} \cdot \text{K}^{-1}$]	0.613	76.5	429
C_p [$\text{J} \cdot \text{kg}^{-1} \cdot \text{K}^{-1}$]	4,179	531.8	235
ρ [$\text{kg} \cdot \text{m}^{-3}$]	997.1	6,320	10,500
Pr	6.2	—	—

It is also crucial to engineering processes. In this attempt, we have considered the time-independent hybrid nanofluid flow over a rotating disk. Newtonian flow is taken into consideration in this analysis. The fluid flow is influenced by the magnetic field, exponential heat source, Brownian motion, chemical reaction, activation energy, thermophoresis, and nonlinear thermal radiation. By incorporating a number of complex phenomena into the modeling of hybrid nanofluid flow, this work improves the field and provides a more comprehensive knowledge of the dynamics at play. Nonlinear thermal radiation and exponential heat sources are included, which is a major improvement over earlier models that frequently omitted these features. We obtain high precision in our numerical solutions using the bvp4c technique to solve the ordinary differential equations (ODEs). This provides a solid foundation for further research into similar systems. The thorough validation procedure guarantees the validity of our results, significantly advancing the current state of fluid dynamics and heat transfer research. A mathematical model is presented in the partial differential equation form and is converted to ODEs using appropriate transformations in Section 2. The numerical solution of the existing model is obtained using the bvp4c technique in Section 3. The model validation is done in Section 4. The results of embedded factors on velocity profiles and detailed discussion are provided in Section 5, while the closing comments are listed in Section 6.

2 Formulation of the problem

Assume a time-independent water-based hybrid nanofluid containing copper oxide and silver nanoparticles on a gyrating disk with angular velocity Ω along the z -direction. The cylindrical coordinates system (r, ψ, z) along the u -, v -, and w -directions is chosen for the existing problem. In the normal direction of the fluid flow, a magnetic field of strength B_0 is employed. The fluid, surface, and ambient temperatures and concentrations are T_f , C_f , T_w , C_w , T_∞ , and C_∞ such that $T_f > T_w$, $C_f > C_w$. A geometrical representation is displayed in Figure 1. Additionally, to understand the flow behavior, the following suppositions are made in this investigation:

- The flow is Newtonian.
- Impacts of the Hall current and induced magnetic field have been ignored.
- The significance of nonlinear linear thermal radiation, Brownian motion, and thermophoresis are accounted for.
- The flow is impacted by the chemical reaction with activation energy.

Thus, the main equations are defined as follows [49–51]:

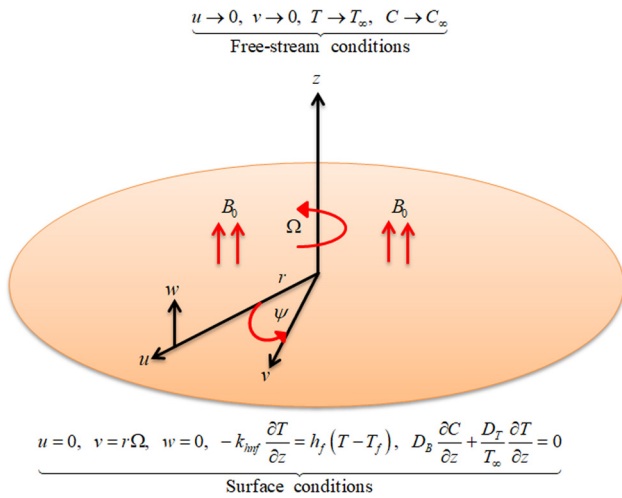


Figure 1: Geometry of flow problem.

$$\frac{\partial u}{\partial r} + \frac{\partial w}{\partial z} + \frac{u}{r} = 0, \quad (1)$$

$$u \frac{\partial u}{\partial r} - \frac{v^2}{r} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho_{\text{hnf}}} \frac{\partial p}{\partial r} + \frac{\mu_{\text{hnf}}}{\rho_{\text{hnf}}} \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2} \right) - \frac{\sigma_{\text{hnf}}}{\rho_{\text{hnf}}} B_0^2 u, \quad (2)$$

$$u \frac{\partial v}{\partial r} + w \frac{\partial v}{\partial z} + \frac{uv}{r} = \frac{\mu_{\text{hnf}}}{\rho_{\text{hnf}}} \left(\frac{\partial^2 v}{\partial r^2} - \frac{v}{r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial z^2} \right) - \frac{\sigma_{\text{hnf}}}{\rho_{\text{hnf}}} B_0^2 v, \quad (3)$$

$$w \frac{\partial w}{\partial z} + u \frac{\partial w}{\partial r} = -\frac{1}{\rho_{\text{hnf}}} \frac{\partial p}{\partial z} + \frac{\mu_{\text{hnf}}}{\rho_{\text{hnf}}} \left(\frac{\partial^2 w}{\partial r^2} + \frac{\partial^2 w}{\partial z^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right), \quad (4)$$

$$u \frac{\partial T}{\partial r} + w \frac{\partial T}{\partial z} = \frac{k_{\text{hnf}}}{(\rho C_p)_{\text{hnf}}} \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} \right) + \frac{1}{(\rho C_p)_{\text{hnf}}} \frac{16\sigma^* T^2}{3k^*} \left(T \frac{\partial^2 T}{\partial z^2} + 3 \left(\frac{\partial T}{\partial z} \right)^2 \right) + \frac{(\rho C_p)_{\text{np}}}{(\rho C_p)_{\text{hnf}}} \left[D_B \left(\frac{\partial T}{\partial r} \frac{\partial C}{\partial r} + \frac{\partial T}{\partial z} \frac{\partial C}{\partial z} \right) + \frac{D_T}{T_\infty} \left(\left(\frac{\partial T}{\partial r} \right)^2 + \left(\frac{\partial T}{\partial z} \right)^2 \right) \right] + \frac{Q}{(\rho C_p)_{\text{hnf}}} (T_f - T_\infty) \exp \left(-\delta \sqrt{\frac{\Omega}{\nu_f}} z \right), \quad (5)$$

$$u \frac{\partial C}{\partial r} + w \frac{\partial C}{\partial z} = D_B \left(\frac{\partial^2 C}{\partial z^2} + \frac{\partial^2 C}{\partial r^2} + \frac{1}{r} \frac{\partial C}{\partial r} \right) + \frac{D_T}{T_\infty} \left(\frac{\partial^2 T}{\partial z^2} + \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right) - k_r^2 (C - C_\infty) \left(\frac{T}{T_\infty} \right)^n \exp \left(-\frac{E_a}{k_b T} \right). \quad (6)$$

The constraints at the boundary are [52]

$$\left\{ \begin{array}{l} u = 0, w = 0, v = \Omega r, -k_{\text{hnf}} \frac{\partial T}{\partial z} = h_f (T_f - T), \\ \frac{D_T}{T_\infty} \frac{\partial T}{\partial z} + D_B \frac{\partial C}{\partial z} = 0 \text{ at } z = 0, \\ u \rightarrow 0, T \rightarrow T_\infty, v \rightarrow 0, C \rightarrow C_\infty \text{ as } z \rightarrow \infty \end{array} \right\}. \quad (7)$$

The thermophysical features of the nanofluid/hybrid nanoparticles are expressed as follows [52–54]:

$$\left\{ \begin{array}{l} \mu = \frac{1}{(1 - \chi)^{2.5}}, (\rho C_p)_{\text{nf}} = (\rho C_p)_f (1 - \chi) + \chi (\rho C_p)_s, \\ \rho_{\text{nf}} = \chi \rho_s + \rho_f (1 - \chi), \\ \sigma_{\text{nf}} = \left[\frac{\sigma_s + 2\sigma_f - 2(\sigma_f - \sigma_s)\chi}{\sigma_s + 2\sigma_f + (\sigma_f - \sigma_s)\chi} \right] \sigma_f, \\ k_{\text{nf}} = \left[\frac{k_s - 2(k_f - k_s)\chi + 2k_f}{k_s + (k_f - k_s)\chi + 2k_f} \right] k_f, \end{array} \right. \quad (8)$$

$$\left\{ \begin{array}{l} k_{\text{hnf}} = \left[\frac{\frac{\chi_1 k_{s1} + \chi_2 k_{s2}}{(\chi_2 + \chi_1)} + 2k_f + 2(\chi_1 k_{s1} + \chi_2 k_{s2}) - 2k_f(\chi_2 + \chi_1)}{\frac{\chi_1 k_{s1} + \chi_2 k_{s2}}{(\chi_2 + \chi_1)} + 2k_f - 2(\chi_1 k_{s1} + \chi_2 k_{s2}) + k_f(\chi_2 + \chi_1)} \right] k_f, \\ \rho_{\text{hnf}} = (1 - \chi_1 - \chi_2) \rho_f + \chi_1 \rho_{s1} + \chi_2 \rho_{s2}, \mu_{\text{hnf}} = \frac{\mu_f}{(1 - \chi_1 - \chi_2)^{2.5}}, \\ (\rho C_p)_{\text{hnf}} = (1 - \chi_1 - \chi_2) (\rho C_p)_f + (\rho C_p)_2 \rho_{s2} + \chi_1 (\rho C_p)_{s1}, \\ \sigma_{\text{hnf}} = \left[\frac{\frac{\chi_1 \sigma_{s1} + \chi_2 \sigma_{s2}}{(\chi_2 + \chi_1)} + 2\sigma_f + 2(\sigma_{s1} \chi_1 + \sigma_{s2} \chi_2) - 2\sigma_f(\chi_2 + \chi_1)}{\frac{\chi_1 \sigma_{s1} + \chi_2 \sigma_{s2}}{(\chi_2 + \chi_1)} + 2\sigma_f - 2(\sigma_{s1} \chi_1 + \sigma_{s2} \chi_2) + \sigma_f(\chi_2 + \chi_1)} \right] \sigma_f. \end{array} \right. \quad (9)$$

A set of appropriate [49,56,57]:

$$\begin{aligned} u &= r\Omega F(\eta), v = r\Omega G(\eta), w = \sqrt{\Omega \nu_f} H(\eta), \\ p - p_\infty &= 2\mu_f \Omega p(\eta) \\ T &= T_\infty (1 + (\theta_w - 1)\theta(\eta)), \phi(\eta) = \frac{C - C_\infty}{C_\infty}, \eta = \sqrt{\frac{\Omega}{\nu_f}} z. \end{aligned} \quad (10)$$

Using Eq. (10) in Eqs. (1)–(6), we have

$$H' + 2F = 0, \quad (11)$$

$$\left(\frac{\mu_{\text{hnf}}/\mu_f}{\rho_{\text{hnf}}/\rho_f} \right) F'' - F^2 + G^2 - HF' - \left(\frac{\sigma_{\text{hnf}}/\sigma_f}{\rho_{\text{hnf}}/\rho_f} \right) MF = 0, \quad (12)$$

$$\left(\frac{\mu_{\text{hnf}}/\mu_f}{\rho_{\text{hnf}}/\rho_f} \right) G'' - 2FG - HG' - \left(\frac{\sigma_{\text{hnf}}/\sigma_f}{\rho_{\text{hnf}}/\rho_f} \right) MG = 0, \quad (13)$$

$$\begin{aligned} &\left(\frac{k_{\text{hnf}}}{k_f} + \text{Rd}(1 + \theta(\theta_w - 1))^3 \right) \theta'' + 4\text{Rd}(\theta_w - 1)(1 + (\theta_w - 1)\theta)^2 \theta'^2 - \text{Pr} \frac{(\rho C_p)_{\text{hnf}}}{(\rho C_p)_f} H\theta' \\ &+ \text{PrNt}\theta'^2 + \text{PrNb}\theta'\phi' + Q_e \exp(-\delta\eta) = 0, \end{aligned} \quad (14)$$

$$\phi'' + \frac{Nt}{Nb}\theta'' - ScH\phi' - KrSc(1 + \omega\theta)^n\phi \exp\left(-\frac{E}{(1 + \omega\theta)}\right) = 0. \quad (15)$$

Transformed constraints at the boundary are as follows:

$$\left\{ \begin{array}{l} F(0) = 0, H(0) = 0, G(0) = 1, \frac{k_{hnf}}{k_f}\theta'(0) \\ = -Bi(1 - \theta(0)), \phi'(0) + \left(\frac{Nt}{Nb}\right)\theta'(0) = 0, \\ F(\infty) = 0, G(\infty) = 0, \theta(\infty) = 0, \phi(\infty) = 0. \end{array} \right. \quad (16)$$

The emerging factors are as follows:

$$\left\{ \begin{array}{l} M = \frac{\sigma_f B_0^2}{\rho_f \Omega}, Pr = \frac{\nu_f(\rho C_p)_f}{k_f}, \omega = \frac{T_f - T_\infty}{T_\infty}, \\ Q_e = \frac{Q}{\Omega(\rho C_p)_f}, Bi = \frac{h_f}{k_f} \sqrt{\frac{\nu_f}{\Omega}}, Rd = \frac{16\sigma^*}{3k^*} \frac{T_\infty^3}{k_f}, \\ Nt = \frac{(\rho C_p)_{np} D_T (T_f - T_\infty)}{(\rho C_p)_f T_\infty \nu_f}, Nb = \frac{(\rho C_p)_{np} D_B C_\infty}{(\rho C_p)_f \nu_f}, \\ Sc = \frac{\nu_f}{D_B}, E = \frac{E_a}{k_b T_\infty}, K_r = \frac{k_r^2}{\Omega}, \theta_w = \frac{T_f}{T_\infty}. \end{array} \right. \quad (17)$$

The skin friction (coefficient) and Nusselt number are described mathematically as

$$C_{fr} = \frac{\sqrt{\tau_{wr}^2 + \tau_{wz}^2}}{\rho_f (\Omega r)^2}, Nu_r = \frac{r q_w}{k_f (T_f - T_\infty)}, \quad (18)$$

$$\mathfrak{R}'(7) = - \frac{\left[Rd(\theta_w - 1)(1 + (\theta_w - 1)\mathfrak{R}(6))^2(\mathfrak{R}(7))^2 - Pr \frac{(\rho C_p)_{hnf}}{(\rho C_p)_f} \mathfrak{R}(1)\mathfrak{R}(7) + \right.}{\left. \frac{k_{hnf}}{k_f} + Rd(1 + (\theta_w - 1)\mathfrak{R}(6))^3 \right]} \frac{Pr Nt (\mathfrak{R}(7))^2 + Pr Nb \mathfrak{R}(7)\mathfrak{R}(9) + Q_e \exp(-\delta\eta)}{\quad}, \quad (25)$$

where

$$\begin{aligned} \tau_{wr} &= \mu_{hnf} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right) \bigg|_{z=0}, \\ \tau_{wz} &= \mu_{hnf} \left(\left(\frac{1}{r} \right) \frac{\partial w}{\partial \psi} + \frac{\partial v}{\partial z} \right) \bigg|_{z=0}. \end{aligned} \quad (19)$$

Thus, the above equation is reduced to

$$\begin{aligned} \sqrt{Re_r} C_f &= \sqrt{G'^2(0) + F'^2(0)}, \\ \frac{1}{\sqrt{Re_r}} Nu &= - \left(\frac{k_{hnf}}{k_f} + Rd(1 + (\theta_w - 1)\theta(0))^3 \right) \theta'(0), \end{aligned} \quad (20)$$

where $Re_r = \frac{\Omega r^2}{\nu_f}$ is the local Reynolds number.

3 Numerical solution

The bvp4c procedure, coded in MATLAB software, is practical and simple to use and is capable of handling problems that are somewhat complex. For the purpose of resolving nonlinear equations, this iterative approach is used [58–60]. The present numerical technique is implemented in Eqs. (11)–(15) subject to Eq. (16). First, we convert the nonlinear equations into a first-order system as follows:

$$\left\{ \begin{array}{l} H = \mathfrak{R}(1), H' = \mathfrak{R}'(1), \\ F = \mathfrak{R}(2), F' = \mathfrak{R}(3), F'' = \mathfrak{R}'(3), \\ G = \mathfrak{R}(4), G' = \mathfrak{R}(5), G'' = \mathfrak{R}'(5), \\ \theta = \mathfrak{R}(6), \theta' = \mathfrak{R}(7), \theta'' = \mathfrak{R}'(7), \\ \phi = \mathfrak{R}(8), \phi' = \mathfrak{R}(9), \phi'' = \mathfrak{R}'(9), \end{array} \right. \quad (21)$$

$$\mathfrak{R}'(1) = -2\mathfrak{R}(2), \quad (22)$$

$$\begin{aligned} \mathfrak{R}'(3) &= - \frac{\left((\mathfrak{R}(4))^2 - (\mathfrak{R}(2))^2 - \mathfrak{R}(1)\mathfrak{R}(3) - \frac{\sigma_{hnf}/\sigma_f}{\rho_{hnf}/\rho_f} M \mathfrak{R}(2) \right)}{\frac{\mu_{hnf}/\mu_f}{\rho_{hnf}/\rho_f}}, \end{aligned} \quad (23)$$

$$\mathfrak{R}'(5) = \frac{\left(2\mathfrak{R}(2)\mathfrak{R}(4) + \mathfrak{R}(1)\mathfrak{R}(5) + \frac{\sigma_{hnf}/\sigma_f}{\rho_{hnf}/\rho_f} M \mathfrak{R}(4) \right)}{\frac{\mu_{hnf}/\mu_f}{\rho_{hnf}/\rho_f}}, \quad (24)$$

$$\begin{aligned} \mathfrak{R}'(9) &= \left(\frac{Nt}{Nb} \mathfrak{R}'(7) - Sc \mathfrak{R}(1)\mathfrak{R}(9) \right. \\ &\quad \left. - Kr Sc(1 + \omega \mathfrak{R}(6))^n \mathfrak{R}(8) \exp\left(-\frac{E}{(1 + \omega \mathfrak{R}(6))}\right) \right). \end{aligned} \quad (26)$$

with the boundary conditions

$$\left\{ \begin{array}{l} \mathfrak{R}_0(1) = 0, \mathfrak{R}_0(2) = 0, \mathfrak{R}_0(4) = 1, \frac{k_{hnf}}{k_f} \mathfrak{R}_0(7) \\ - Bi(1 - \mathfrak{R}_0(6)), \mathfrak{R}_0(9) + \frac{Nt}{Nb} \mathfrak{R}_0(7), \\ \mathfrak{R}_{inf}(2) = 0, \mathfrak{R}_{inf}(6) = 0, \mathfrak{R}_{inf}(8) = 0, \mathfrak{R}_{inf}(4) = 0, \end{array} \right. \quad (27)$$

Table 2: Validation of the current data with the available results of Turkyilmazoglu [62] and Rashidi *et al.* [61] when $\chi_1 = \chi_2 = 0.0$

M	$F(0)$	
	1	4
Rashidi <i>et al.</i> [61]	0.309237	0.165701
Turkyilmazoglu [62]	0.309258	0.165703
Present results	0.309258	0.165703

where $\mathfrak{R}_0$ and $\mathfrak{R}_{\text{inf}}$ are used for the initial and boundary conditions.

4 Present validation

When all other factors are zero, the existing results are compared with the results of Rashidi *et al.* [61] and Turkyilmazoglu [62], who also published their findings. As shown in Table 2, a similar result has been found in our present exploration.

5 Results and discussions

5.1 Results

5.1.1 Discussion of results

This section compacts with the impacts of embedded factors on radial and azimuthal velocity, thermal, and concentration

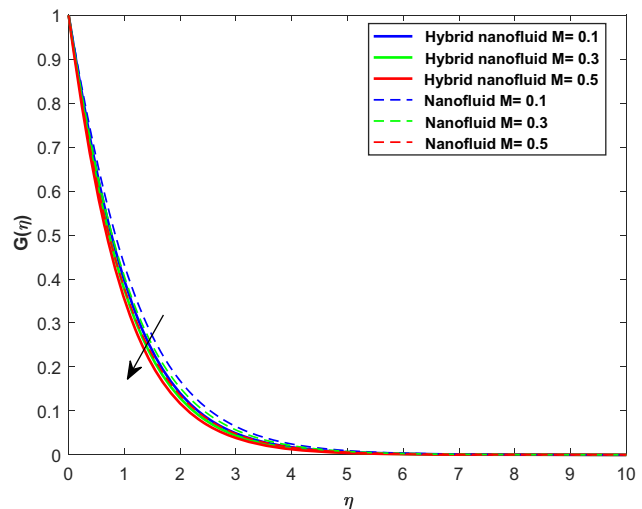


Figure 3: Influence of M on $G(\eta)$.

distributions of the fluid flow past a revolving disk. The numerical solution for the problem is evaluated using the `bvp4c` method. The influence of embedded factors on the flow distributions and quantities of interest is shown in Figures 2–16 and Tables 2–8. Table 2 describes a comparative analysis. Figures 2 and 3 display the influence of magnetic parameter (M) over radial velocity ($F(\eta)$) and azimuthal velocity ($G(\eta)$). From these figures, it can be observed that $F(\eta)$ and $G(\eta)$ reduce due to an increase of M . Actually, growth in M heightens the Lorentz force as created in the flow system, which augments the friction force at the sheet surface and decelerates the flow. Therefore, the magnetic parameter opposes the transport phenomena, which results in the reduction of the velocity profiles, as shown in Figures 2 and 3. Hence, the velocity of the fluid in both radial and azimuthal

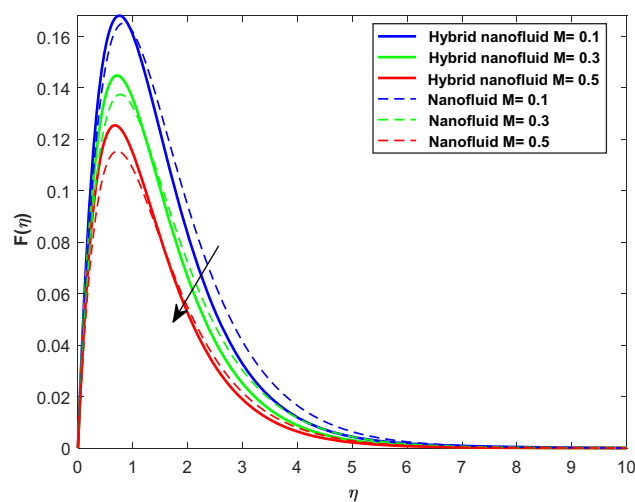


Figure 2: Influence of M on $F(\eta)$.

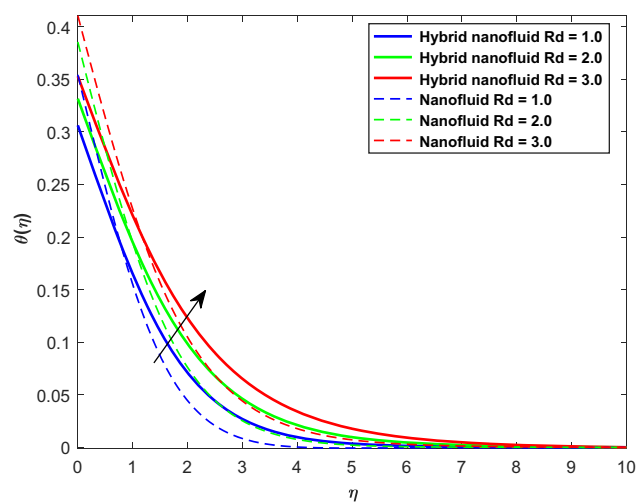
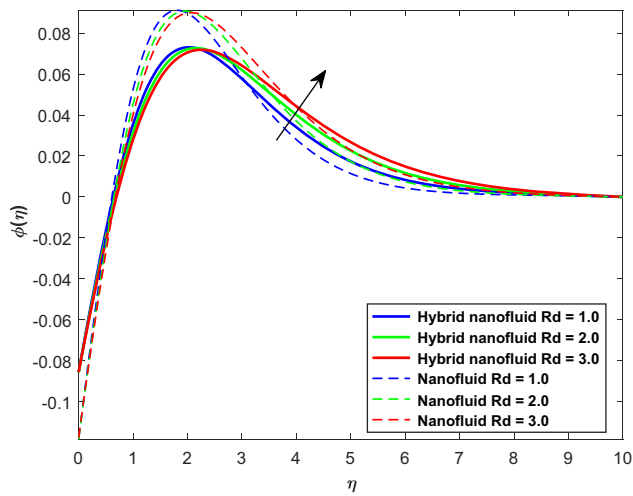
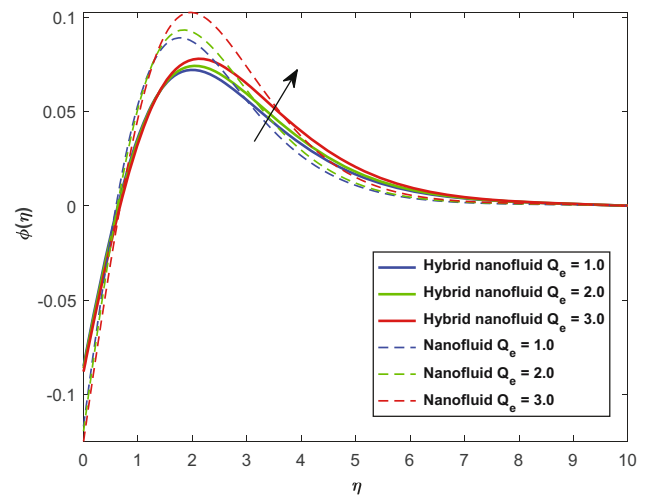
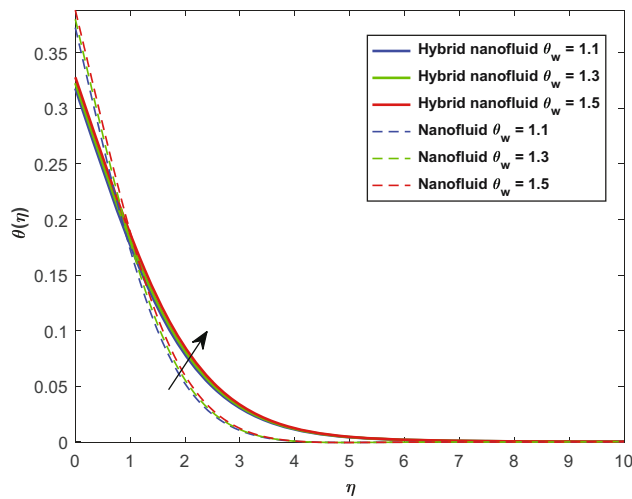
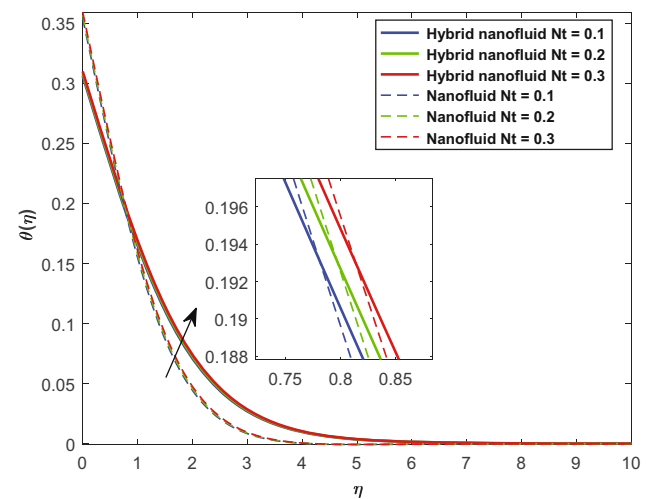
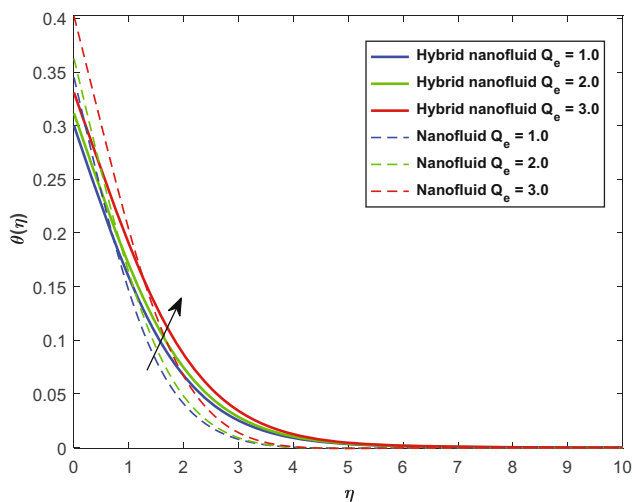
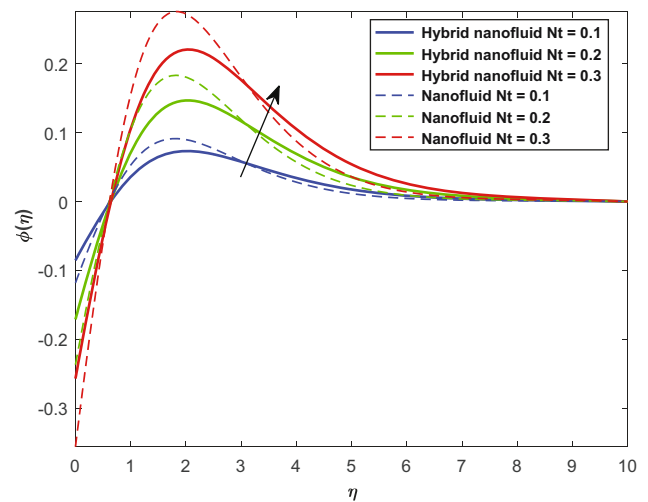


Figure 4: Influence of Rd on $\theta(\eta)$.

Figure 5: Influence of R_d on $\phi(\eta)$.Figure 8: Influence of Q_e on $\phi(\eta)$.Figure 6: Influence of θ_w on $\theta(\eta)$.Figure 9: Influence of Nt on $\theta(\eta)$.Figure 7: Influence of Q_e on $\theta(\eta)$.Figure 10: Influence of Nt on $\phi(\eta)$.

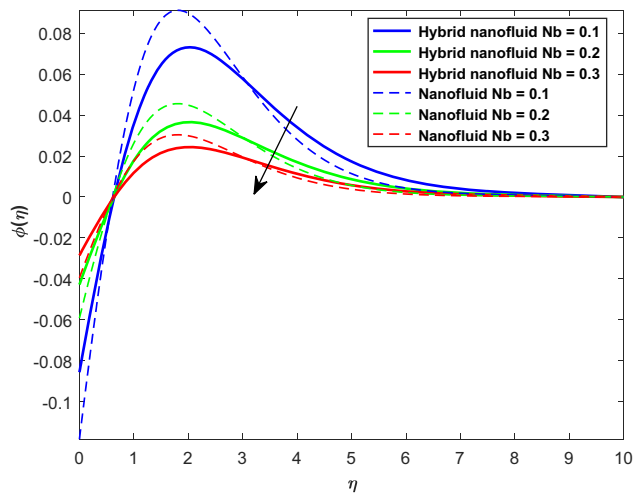
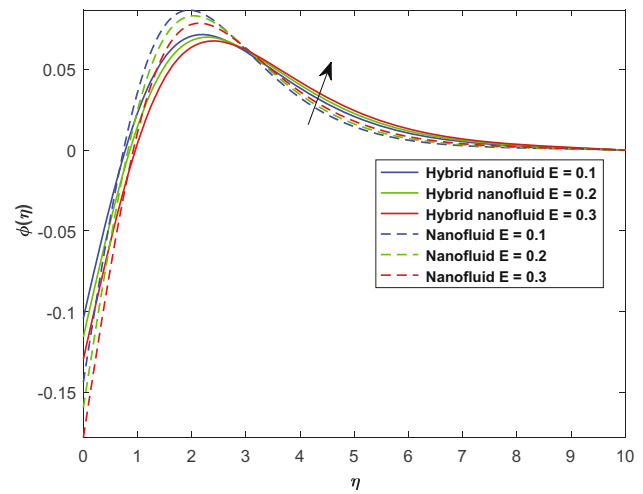
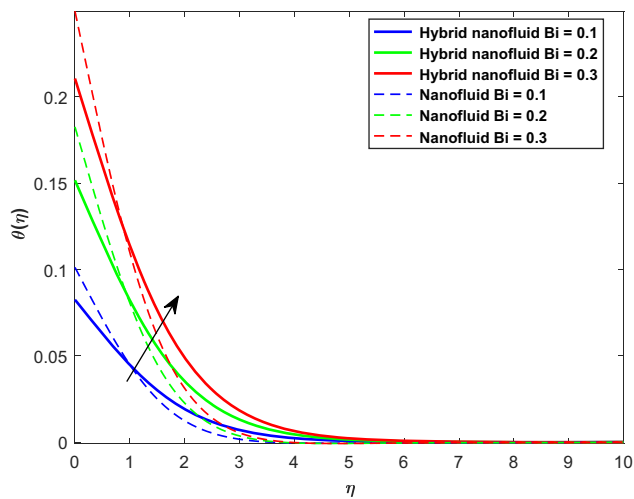
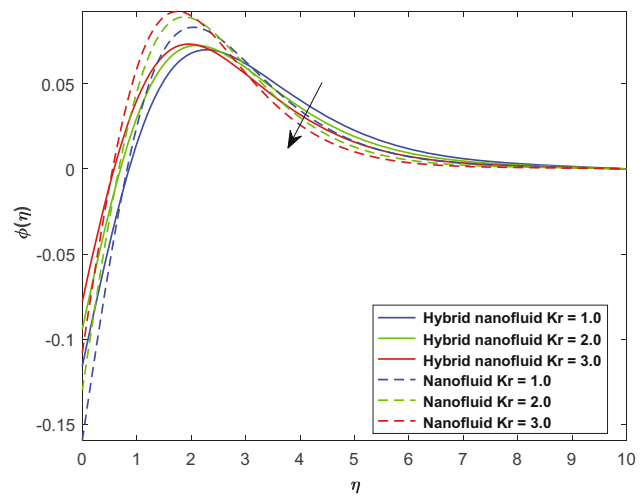
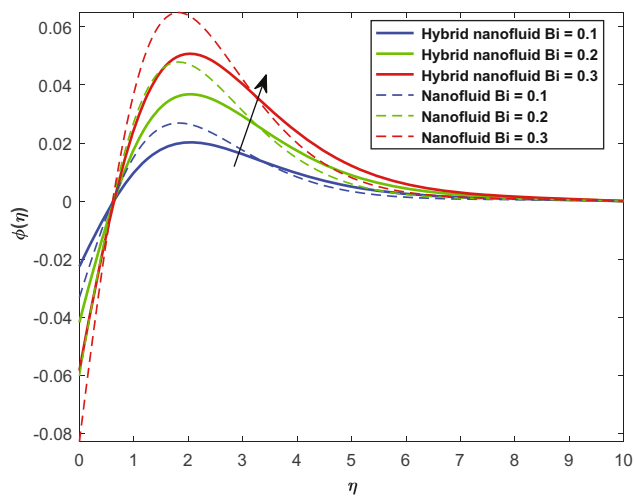
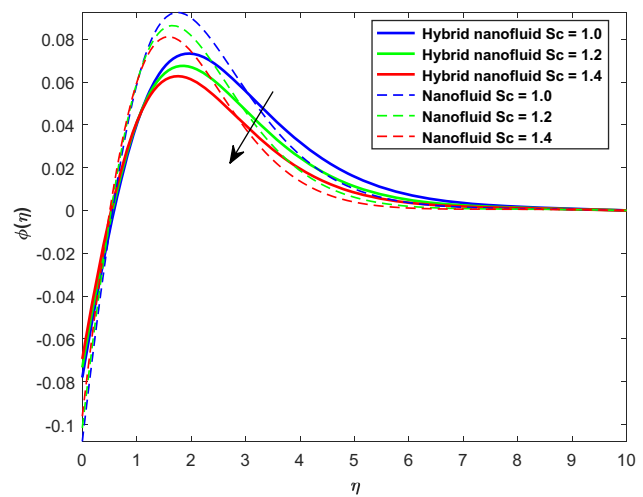
Figure 11: Influence of Nb on $\phi(\eta)$.Figure 14: Influence of E on $\phi(\eta)$.Figure 12: Influence of Bi on $\theta(\eta)$.Figure 15: Influence of K_r on $\phi(\eta)$.Figure 13: Influence of Bi on $\phi(\eta)$.Figure 16: Influence of Sc on $\phi(\eta)$.

Table 3: Impacts of χ_1 and M on $\sqrt{\text{Re}_r} C_f$

χ_1	M	CuO–H ₂ O nanofluid
0.01	0.1	0.467647
0.02		0.476437
0.03		0.487643
0.04		0.498738
	0.3	0.397647
	0.7	0.456784
	1.3	0.549720

Table 4: Impacts of χ_2 and M on $\sqrt{\text{Re}_r} C_f$

χ_2	M	Ag–H ₂ O nanofluid
0.01	0.1	0.470863
0.02		0.480849
0.03		0.495185
0.04		0.515890
	0.3	0.415759
	0.7	0.479058
	1.3	0.568561

directions decreases due to the augmenting magnetic factor. At the sheet surface, the radial velocity of the hybrid nanofluid shows a dominant behavior compared to the nanofluid. However, this effect is reversed above the surface. Figures 4 and 5 demonstrate the impact of the radiation factor (R_d) on $\theta(\eta)$ and $\phi(\eta)$, respectively. From these figures, it can be observed that both $\theta(\eta)$ and $\phi(\eta)$ reduce due to an increase of R_d . Generally, R_d is the source of external energy that provides heat to the working fluid in the electromagnetic waveform, and as a result, the temperature and concentration boundary layer thicknesses increase. Thus, $\theta(\eta)$ and $\phi(\eta)$ increase with an increase in the thermal radiation factor. At the sheet surface, the temperature of the nanofluid shows a dominant behavior compared to the nanofluid. However, this effect is reversed above the surface. Figure 6 shows the influence of the temperature difference factor θ_w on $\theta(\eta)$. It is

Table 5: Impacts of $\chi_1 = \chi_2$ and M on $\sqrt{\text{Re}_r} C_f$

$\chi_1 = \chi_2$	M	CuO–Ag/H ₂ O hybrid nanofluid
0.01	0.1	0.588648
0.02		0.607529
0.03		0.598620
0.04		0.637863
	0.3	0.654264
	0.7	0.670834
	1.3	0.696431

Table 6: Impacts of χ_1 , Bi , and θ_w on $\frac{1}{\sqrt{\text{Re}_r}} \text{Nu}$

χ_1	Bi	θ_w	CuO–H ₂ O nanofluid
0.01	0.1	1.1	0.467647
0.02			0.476437
0.03			0.487643
0.04			0.498738
	0.3	1.1	0.397647
	0.7		0.456784
	1.3		0.549720
		1.2	0.476536
		1.3	0.486525
			0.497533

noticed that the thermal distribution increases with increasing θ_w . The temperature difference factor shows the ratio of T_f with T_∞ (i.e., $\theta_w = T_f/T_\infty$). It is also obvious that θ_w must be greater than 1. According to this definition, as we increase θ_w , the reference heat of fluid flow also increases, which results in a higher thermal distribution of the fluid flow. Thus, an increase in θ_w increases $\theta(\eta)$. At the surface of the sheet, the temperature of the nanofluid shows a dominant behavior compared to the hybrid nanofluid. However, this effect is reversed above the surface. Figure 7 shows the impact of the heat source factor (Q_e) on $\theta(\eta)$. It is noticed that $\theta(\eta)$ increases with increasing Q_e . Actually, Q_e plays an imperative role in the form of extra heat to the flow system. Because of the fundamental principles of thermodynamics, adding a heat source to a system improves its thermal profile. According to the law of conservation of energy, energy cannot be generated or destroyed, but rather transferred or changed from one form to another. As a result, the heat source provides thermal energy to the system, causing a temperature increase throughout its geographical extent. This thermal energy transfer occurs when heat naturally flows from higher temperatures (the heat source) to lower temperatures (the surroundings or the system itself), resulting in a detectable increase in the temperature throughout the system. Thus, the thermal profile reflects how the new heat is redistributed and changes the system's thermal state. Thus, an increase in Q_e increases $\theta(\eta)$. Similarly, $\phi(\eta)$ also increases with an increase in Q_e , as displayed in Figure 8. At the sheet surface, the temperature of the nanofluid shows a dominant behavior compared to the hybrid nanofluid. However, this effect is reversed above the surface. Figures 9 and 10 show the influence of N_t on $\theta(\eta)$ and $\phi(\eta)$. It is observed that both $\theta(\eta)$ and $\phi(\eta)$ increase with increasing N_t . This occurs because the particles near the heated surfaces generate a thermophoretic force that pushes the particles to break down away from the fluid regime, increasing the thicknesses of concentration and temperature boundary layers. At the sheet surface, the temperature and

concentration of the nanofluid show a dominant behavior compared to the hybrid nanofluid. However, this consequence is reversed above the surface. Figure 11 shows the impact of the Brownian factor (Nb) on $\phi(\eta)$. It is found that an increase of Nb reduces $\phi(\eta)$. As Nb values increase, the concentration profile decreases, in turn the concentration layer at the boundary likewise drops. Increasing values of Nb have an impact on the Brownian motion, which also restricts the nanoparticle diffusion in the flow regime and reduces the concentration of the fluid. Thus, an increase in Nb reduces $\phi(\eta)$. At the sheet surface, the concentration of the nanofluid shows dominant behavior compared to the hybrid nanofluid. However, this effect is reversed above the surface. Figures 12 and 13 show the impression of thermal Biot number (Bi) on $\theta(\eta)$ and $\phi(\eta)$. Both $\theta(\eta)$ and $\phi(\eta)$ augment with increasing values of Bi . The reason is that increasing Bi augments the thermal flow, which results in intensification in the heat transfer rate. The increase in the thermal flow rate leads to the enhancement of the thickness of the thermal boundary layer. Thus, $\theta(\eta)$ increases with increasing Bi . Similarly, Bi also augments the concentration boundary layer thickness and thus $\phi(\eta)$ increases. At the sheet surface, the concentration of the nanofluid shows a dominant behavior compared to the hybrid nanofluid. However, this effect is reversed above the surface. Figure 14 shows the influence of E on $\phi(\eta)$. It was found that an increase in E augments $\phi(\eta)$. It was observed that at lower temperatures and higher E , the reaction rate decreases, thereby slowing the chemical process; as a result, the thickness of the concentration boundary layer increases, which increases $\phi(\eta)$. Furthermore, the impact of E is higher for the hybrid nanofluid compared to the nanofluid flow. Figure 15 shows the influence of the reaction factor (K_r) on $\phi(\eta)$. It was found that an increase in K_r reduces $\phi(\eta)$. Actually, the increase in the chemical reaction factor improves $K_r Sc(1 + \omega\theta)^n \phi \exp(-E/(1 + \omega\theta))$. Thus, increasing chemical reaction takes place and reduces the thickness of the concentration boundary layer,

which results in the reduction of the concentration distribution. Thus, an increase in K_r reduces $\phi(\eta)$. Furthermore, the concentration of the nanofluid flow is lower than the flow of the hybrid nanofluid. Figure 16 demonstrates the impact of Sc on $\phi(\eta)$. An increase in Sc reduces the diffusion of mass. Therefore, an increase in Sc diminishes $\phi(\eta)$. Moreover, the concentration of the hybrid nanoparticles flow is greater than the flow of the nanofluid. Table 3 shows the influences of χ_1 and M on $\sqrt{Re_r} C_f$. The increasing χ_1 and M augment $\sqrt{Re_r} C_f$. The reason is that the higher volume fraction of the solid nanoparticles increases the shear stress, which consequently increases the skin friction at the surface. Thus, $\sqrt{Re_r} C_f$ increases with an increase in χ_1 . Also, an increase in M increases $\sqrt{Re_r} C_f$. An increase in M generates Lorentz force, which results in higher friction force at the surface. Thus, an increase in M reduces $\sqrt{Re_r} C_f$. The same findings can be found in Tables 4 and 5. From these tables, it can be found that the friction force for the hybrid nanofluid was found to be the highest. Table 6 shows the impacts of χ_1 , Bi , and θ_w on $\frac{1}{\sqrt{Re_r}} Nu$. An increase in χ_1 , Bi and θ_w augments $\frac{1}{\sqrt{Re_r}} Nu$. Thus, $\frac{1}{\sqrt{Re_r}} Nu$ increases with increasing χ_1 . Similarly, an increase in Bi enhances the coefficient of heat transmission, which consequently augments the rate of thermal flow of the nanofluid. Hence, the higher Bi heightens $\frac{1}{\sqrt{Re_r}} Nu$. The same impact of θ_w is found for $\frac{1}{\sqrt{Re_r}} Nu$. The same findings can be found in Tables 7 and 8. As shown in the tables, the heat transmission rate for the hybrid nanofluid was found to be the highest. The Ag-H₂O nanofluid has a 3% greater impact on $\sqrt{Re_r} C_f$ than the CuO-H₂O nanofluid. Similarly, the CuO-Ag/H₂O hybrid nanofluid has a 28% higher impact on $\sqrt{Re_r} C_f$ than the Ag-H₂O nanofluid. Also, the CuO-Ag/H₂O hybrid nanofluid has a 26% higher impact on $\sqrt{Re_r} C_f$ than the CuO-H₂O nanofluid. The heat transfer rate of the Ag-H₂O nanofluid is 3% greater than the CuO-H₂O

Table 7: Impacts of χ_2 , Bi , and θ_w on $\frac{1}{\sqrt{Re_r}} Nu$

χ_2	Bi	θ_w	Ag-H ₂ O nanofluid
0.01	0.1	1.1	0.470863
0.02			0.480849
0.03			0.495185
0.04			0.515890
	0.3		0.415759
	0.7		0.479058
	1.3		0.568561
		1.1	0.487657
		1.2	0.496426
		1.3	0.509264

Table 8: Impacts of $\chi_1 = \chi_2$, Bi , and θ_w on $\frac{1}{\sqrt{Re_r}} Nu$

$\chi_1 = \chi_2$	Bi	θ_w	CuO-Ag/H ₂ O hybrid nanofluid
0.01	0.1	1.1	0.588648
0.02			0.607529
0.03			0.598620
0.04			0.637863
	0.3		0.654264
	0.7		0.670834
	1.3		0.696431
		1.1	0.523574
		1.2	0.547627
		1.3	0.568973

nanofluid. Also, the heat transfer rate of the CuO–Ag/H₂O hybrid nanofluid is 28% higher than the Ag–H₂O nanofluid.

6 Conclusion

This section presents the concluding remarks that are observed from the present analysis. The current study is done for a time-independent water-based hybrid nanofluid flow over a rotating disk surface. In this investigation, two different nanoparticles, namely copper oxide and silver, are chosen. In the present analysis, the flow is considered to be Newtonian. The influence of the magnetic field, thermophoresis, nonlinear thermal radiation, chemical reaction, Brownian motion, and activation energy is taken into account. A numerical solution using the *bvp4c* technique is accomplished using MATLAB software. The present results are compared with the previous literature and it is found to have a close relationship with previous studies. From this investigation, the following concluding points are found:

- a) A higher magnetic parameter causes the surface to experience more friction, which lowers the radial and azimuthal velocity profiles. When compared to the nanofluid, the hybrid nanofluid's radial velocity at the sheet surface exhibits dominant behavior.
- b) The thermal and concentration profiles are enhanced by the higher thermal radiation factor. In comparison to the nanofluid, the temperature and concentration of the nanofluid exhibit dominant behavior near the sheet surface. Above the surface, though, this impact is the opposite.
- c) The temperature profiles are enhanced when thermophoresis, temperature ratio, and heat source parameters increase. When compared to a hybrid nanofluid, the temperature of the nanofluid exhibits dominant behavior near the sheet surface. Above the surface, though, this impact is the opposite.
- d) The concentration profile is enhanced by the thermophoresis factor and decreased by the Brownian motion factor. When compared to a hybrid nanofluid, the concentration profiles of the former exhibit dominant behavior at the sheet surface. Above the surface, though, this impact is the opposite.
- e) The Biot number increases with the concentration and temperature profiles. Additionally, in comparison to the hybrid nanofluid, the concentration of the nanofluid exhibits dominant behavior near the sheet surface. Above the surface, though, this impact is the opposite.

- f) The skin friction at the disk's surface is increased by the increase in the solid volume fraction and magnetic component. Moreover, the hybrid nanofluid flow is observed to have a greater surface drag.
- g) The rate of heat transmission is accelerated by a greater solid volume fraction, temperature ratio, and Biot number. Furthermore, a greater rate of heat transfer is observed in the case of the hybrid nanofluid flow.

7 Future recommendations

In the future, the present mathematical model can be extended to study ternary hybrid nanofluid flow over a rotating disk surface by including three different metallic and nonmetallic nanoparticles. Furthermore, the addition of the Cattaneo-Christov heat and mass flux model can be studied to analyze heat and mass transport phenomena.

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Conflict of interest: The authors state no conflict of interest.

Data availability statement: All data generated or analyzed during this study are included in this published article.

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