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# Analytical Solution of Pantograph Equation with Incommensurate Delay

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Pantograph equation is a delay differential equation (DDE) arising in electrodynamics. This paper studies the pantograph equation with two delays. The existence, uniqueness, stability and convergence results for DDEs are presented. The series solution of the proposed equation is obtained by using Daftardar-Gejji and Jafari method and given in terms of a special function. This new special function has several properties and relations with other functions. Further, we generalize the proposed equation to fractional-order case and obtain its solution.

**Keywords:** pantograph equation, Daftardar-Gejji and Jafari method, proportional delay

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## 1 Introduction

Ordinary differential equations (ODE) are widely used by researchers to model various phenomena arising in Science and Technology. However, it is observed that such equations cannot model the actual behavior of the system. Since the ordinary derivative is a local operator, it cannot model the memory and hereditary properties in the real-life models. Such phenomena can be modeled in a more accurate way by introducing some nonlocal component, e.g. delay in it.

The delay differential equations (DDEs) are the equations where the rate of change of certain quantity depends on the value of that quantity at previous time [1]. The DDE models are proved very useful while modeling natural systems [2, 3]. The analysis of DDE is more difficult than that of ODE. The characteristic equation corresponding to DDE is a transcendental equation unlike a polynomial in case of ODE. Some first-order nonlinear DDEs may exhibit chaotic oscillations [4].

The DDE

$$y'(x) = ay(x) + by(qx), \quad (1)$$

is a famous equation called pantograph equation arising in electrodynamics. The pantograph [5, 6] is a device used in electric trains to collect electric current from the overload lines. The equation was modeled by Ockendon and Tayler [7] in 1971. The analytical solution of pantograph eq. (1) and its asymptotic properties are discussed by Kato [8]. Liu [9] used trapezium rule to find numerical solution of eq. (1). The DJM is used by Bhalekar and Patade [10] to find analytical series solution of eq. (1). The convergence of the series solution is discussed and the properties of the novel special function defined in terms of the series are also discussed by these authors. Iserles [11] presented generalization of pantograph equation namely

$$y'(t) = Ay(t) + By(qt) + Cy'(qt) \quad (2)$$

Bellen, Guglielmi and Torelli [12] studied the stability properties of  $\theta$ -methods for eq. (2). The eq. (2) in complex plane is described in [13]. Koto [14] discussed the stability of Runge-Kutta methods for the eq. (2).

Yet another generalization namely multi-pantograph equation

$$y'(t) = \lambda y(t) + \sum_{i=1}^l \mu_i y(q_i t) \quad (3)$$

is proposed by Liu and Li [15]. Adomian decomposition method and variational iteration method are used to solve some particular cases of eq. (3) in [16] and [17] respectively. Runge-Kutta methods are used to solve

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this equation numerically by Li and Liu [18]. The approximate solution of eq. (3) with variable coefficients is presented in [19] in terms of Taylor polynomials. However, the solution of eq. (3) is not given in the literature in terms of a special function.

Our aim in this paper is to analyze the pantograph equation

$$y'(x) = ay(x) + by(px) + cy(qx) \tag{4}$$

with incommensurate delays. We use Daftardar-Gejji and Jafari method (DJM) to obtain series solution of eq. (4). We define the new special function using this series and analyze its properties. The paper is organized as below: Basic definition and results given in Section 2.1. The iterative scheme DJM is discussed in Section 2.2. Existence, uniqueness and convergence results are described in Section 2.3. Section 3 deals with stability analysis. The solution of pantograph equation is given in Section 4. Analysis of special function generated from this solution is given in Section 5. The equation is generalized to fractional-order case in Section 6. The conclusions are summarized in Section 7.

## 2 Preliminaries

### 2.1 Basic definitions and results

We recall some basic definitions and results from [20], [21] and [22] which will be used in this paper.

**Definition 1.**

The upper and lower incomplete gamma functions are defined as

$$\Gamma(n, x) = \int_x^\infty t^{n-1}e^{-t}dt \quad \text{and}$$

$$\gamma(n, x) = \int_0^x t^{n-1}e^{-t}dt \quad \text{respectively.}$$

Various properties of incomplete gamma functions are discussed in [20].

**Definition 2.**

Kummer's confluent hyper-geometric functions  ${}_1F_1(a; c; x)$  and  $U(a; c; x)$  are defined as below

$${}_1F_1(a; c; x) = \sum_{n=0}^\infty \frac{(a)_n x^n}{(c)_n n!}, \quad c \neq 0, -1, -2, \dots \text{ and}$$

$$U(a; c; x) = \frac{\pi}{\sin(\pi c)} \left( \frac{{}_1F_1(a; c; x)}{\Gamma(c)\Gamma(1+a-c)} - x^{1-c} \frac{{}_1F_1(1+a-c; 2-c; x)}{\Gamma(a)\Gamma(2-c)} \right),$$

$$-\pi < \arg x \leq \pi.$$

**Definition 3.**

The generalized Laguerre polynomials are defined as

$$L_n^{(\alpha)}(x) = \sum_{m=0}^n (-1)^m \binom{n+\alpha}{n-m} \frac{x^m}{m!}$$

$$= \binom{n+\alpha}{n} {}_1F_1(-n; \alpha+1; x).$$

**Definition 4.**

A real function  $f(x), x > 0$ , is said to be in space  $C_\alpha, \alpha \in \mathbb{R}$ , if there exists a real number  $p(> \alpha)$ , such that  $f(x) = x^p f_1(x)$  where  $f_1(x) \in C[0, \infty)$ .

**Definition 5.**

A real function  $f(x), x > 0$ , is said to be in space  $C_\alpha^m, m \in \mathbb{N} \cup \{0\}$ , if  $f^{(m)} \in C_\alpha$ .

**Definition 6.**

Let  $f \in C_\alpha$  and  $\alpha \geq -1$ , then the (left-sided) Riemann-Liouville integral of order  $\mu$ ,  $\mu > 0$  is given by

$$I^\mu f(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t - \tau)^{\mu-1} f(\tau) d\tau, \quad t > 0.$$

**Definition 7.**

The (left sided) Caputo fractional derivative of  $f, f \in C_{-1}^m, m \in \mathbb{N} \cup \{0\}$ , is defined as:

$$\begin{aligned} D^\mu f(t) &= \frac{d^m}{dt^m} f(t), \quad \mu = m \\ &= I^{m-\mu} \frac{d^m}{dt^m} f(t), \quad m-1 < \mu < m, \quad m \in \mathbb{N}. \end{aligned}$$

Note that for  $0 \leq m-1 < \alpha \leq m$  and  $\beta > -1$

$$\begin{aligned} I^\alpha (x-b)^\beta &= \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} (x-b)^{\beta+\alpha}, \\ (I^\alpha D^\alpha f)(t) &= f(t) - \sum_{k=0}^{m-1} f^{(k)}(0) \frac{t^k}{k!}. \end{aligned}$$

**Theorem 8.**

Rudin [23] Suppose  $\{f_n\}$  is a sequence of functions defined on  $E$ , and suppose that there exists  $M_n \in \mathbb{R}$  such that

$$|f_n| \leq M_n, \quad \text{on } E, \quad n = 1, 2, 3 \dots.$$

Then  $\sum f_n$  converges uniformly on  $E$  if  $\sum M_n$  converges.

**2.2 Daftardar-Gejji and Jafari method**

A new iterative method (DJM) was introduced by Daftardar-Gejji and Jafari [24] in 2006 for solving nonlinear functional equations. The DJM has been used to solve a variety of equations such as fractional differential equations [25], partial differential equations [26], boundary value problems [27, 28], evolution equations [29] and system of nonlinear functional equations [30]. The method is successfully employed to solve Newell-Whitehead-Segel equation, Fishers equation [31, 32], fractional-order logistic equation [33] and some nonlinear dynamical systems [34] also. Recently DJM has been used to generate new numerical methods [35–38] for solving differential equations. In this section we describe DJM which is very useful for solving the equations of the form

$$u = f + L(u) + N(u), \quad (5)$$

where  $f$  is a given function,  $L$  and  $N$  are linear and nonlinear operators respectively. DJM provides the solution to eq. (5) in the form of series

$$u = \sum_{i=0}^{\infty} u_i. \quad (6)$$

Since  $L$  is linear

$$L\left(\sum_{i=0}^{\infty} u_i\right) = \sum_{i=0}^{\infty} L(u_i). \quad (7)$$

The nonlinear operator  $N$  in eq. (5) is decomposed by Daftardar-Gejji and Jafari as below:

$$\begin{aligned} N\left(\sum_{i=0}^{\infty} u_i\right) &= N(u_0) + \sum_{i=1}^{\infty} \left\{ N\left(\sum_{j=0}^i u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right) \right\} \\ &= \sum_{i=0}^{\infty} G_i, \end{aligned} \quad (8)$$

where

$$G_0 = N(u_0) \text{ and}$$

$$G_i = \left\{ N \left( \sum_{j=0}^i u_j \right) - N \left( \sum_{j=0}^{i-1} u_j \right) \right\}, \quad i \geq 1.$$

Using eqs (6), (7) and (8) in eq. (5), we get

$$\sum_{i=0}^{\infty} u_i = f + \sum_{i=0}^{\infty} L(u_i) + \sum_{i=0}^{\infty} G_i. \tag{9}$$

From eq. (9), the DJM series terms are generated as below:

$$u_0 = f,$$

$$u_{m+1} = L(u_m) + G_m, \quad m = 0, 1, 2, \dots$$

The  $k$ -term approximate solution is given by

$$u = \sum_{i=0}^{k-1} u_i,$$

for suitable integer  $k$ .

Convergence of DJM is given in following results.

**Theorem 9.**

*Bhalekar and Daftardar-Gejji [39] If  $N$  is  $C^{(\infty)}$  in a neighborhood of  $y_0$  and  $\|N^{(n)}(y_0)\| \leq L$ , for any  $n$  and for some real  $L > 0$  and  $\|y_i\| \leq M < \frac{1}{e}$ ,  $i = 1, 2, \dots$ , then the series  $\sum_{n=0}^{\infty} G_n$  is absolutely convergent to  $N$  and moreover,*

$$\|G_n\| \leq LM^n e^{n-1} (e - 1), \quad n = 1, 2, \dots$$

**Theorem 10.**

*Bhalekar and Daftardar-Gejji [39] If  $N$  is  $C^{(\infty)}$  and  $\|N^{(n)}(y_0)\| \leq M \leq e^{-1}$ ,  $\forall n$ , then the series  $\sum_{n=0}^{\infty} G_n$  is absolutely convergent to  $N$ .*

**2.3 Existence, uniqueness and convergence**

In this section we generalize theorems described in [40]. The equation

$$y'(x) = f(x, y(x), y(px), y(qx)),$$

is a particular case of time-dependent DDE

$$y'(x) = f(x, y(x), y(x - \tau_1(x)), y(x - \tau_2(x))) \quad \text{with} \quad \tau_1(x) = (1 - p)x, \quad \tau_2(x) = (1 - q)x.$$

**Theorem 11.**

*(Local existence) Consider the equation*

$$y'(x) = f(x, y(x), y(x - \tau_1(x)), y(x - \tau_2(x))), \quad x_0 \leq x < x_f,$$

$$y(x_0) = y_0, \tag{10}$$

*and assume that the function  $f(x, u, v, w)$  is continuous on  $A \subseteq [x_0, x_f] \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m$  and locally Lipschitz continuous with respect to  $u, v$  and  $w$ . Moreover, assume that the delay function  $\tau_1(x) \geq 0, \tau_2(x) \geq 0$  is continuous in  $[x_0, x_f]$ ,  $\tau_1(x_0) = 0, \tau_2(x_0) = 0$  and, for some  $\xi > 0, x - \tau_1(x) > x_0, x - \tau_2(x) > x_0$  in the interval  $(x_0, x_0 + \xi]$ . Then the problem (10) has a unique solution in  $[x_0, x_0 + \delta)$  for some  $\delta > 0$  and this solution depends continuously on the initial data.*

**Theorem 12.**

(Global existence) Under the hypothesis of Theorem Theorem 11, if the unique maximal solution of (10) is bounded, then it exist on the entire interval  $[x_0, x_f)$ .

Now, we present convergence result motivated from Bhalekar and Patade [41] for DJM solution.

**Theorem 13.**

Let  $f$  be a continuous function defined on a four-dimensional rectangle  $R = \{(x, y_1, y_2, y_3) | 0 \leq x \leq b, -\delta \leq y_1 \leq \delta, -\mu \leq y_2 \leq \mu, -\eta \leq y_3 \leq \eta\}$  and  $|f(x, y_1, y_2, y_3)| \leq M, \forall (x, y_1, y_2, y_3) \in R$ . Suppose that  $f$  satisfies Lipschitz type condition  $|f(x, y_1, y_2, y_3) - f(x, u_1, u_2, u_3)| \leq L_1 |y_1 - u_1| + L_2 |y_2 - u_2| + L_3 |y_3 - u_3|$ . Then the DJM series solution of the initial value problem (IVP),

$$y'(x) = f(x, y(x), y(px), y(qx)), \quad y(0) = 1, \quad 0 < p < 1, 0 < q < 1, \quad (11)$$

converges uniformly in the interval  $[0, b]$ .  $\square$

**proof.**

The equivalent integral equation of eq. (11) is

$$y(x) = 1 + \int_0^x f(t, y(t), y(pt), y(qt)) dt.$$

Using DJM, we get

$$y_0(x) = 1,$$

$$y_1(x) = \int_0^x f(t, y_0(t), y_0(pt), y_0(qt)) dt.$$

$$\Rightarrow |y_1(x)| \leq Mx.$$

$$\text{Since } p, q \in (0, 1), \frac{b}{p} > b \text{ and } \frac{b}{q} > b.$$

$$\Rightarrow |y_1(px)| \leq Mpx, \quad \forall x \in [0, b].$$

Further,

$$y_2(x) = \int_0^x (f(t, y_1(t) + y_0(t), y_1(pt) + y_0(pt), y_1(qt) + y_0(qt)) - f(t, y_0(t), y_0(pt), y_0(qt))) dt.$$

$$\Rightarrow |y_2(x)| \leq \int_0^x (L_1 |y_1(x)| + L_2 |y_1(pt)| + L_3 |y_1(qt)|) dt$$

$$\leq M(L_1 + L_2p + L_3q) \frac{x^2}{2!}$$

$$\leq M(L_1 + L_2 + L_3) \frac{x^2}{2!}.$$

$$\Rightarrow |y_2(px)| \leq Mp^2(L_1 + L_2p + L_3q) \frac{x^2}{2!}, \quad x \in [0, b]$$

$$\leq M(L_1 + L_2 + L_3) \frac{x^2}{2!}$$

$$\text{and } |y_2(qx)| \leq Mq^2(L_1 + L_2p + L_3q) \frac{x^2}{2!}, \quad x \in [0, b]$$

$$\leq M(L_1 + L_2 + L_3) \frac{x^2}{2!}.$$

Similarly,

$$|y_3(x)| \leq M(L_1 + L_2p + L_3q)(L_1 + L_2p^2 + L_3q^2) \frac{x^3}{3!}$$

$$\leq M(L_1 + L_2 + L_3)^2 \frac{x^3}{3!}.$$

In general

$$\begin{aligned} |y_n(x)| &\leq M \prod_{j=1}^{n-1} (L_1 + L_2 p^j + L_3 q^j) \frac{x^n}{n!} \\ &\leq M (L_1 + L_2 + L_3)^{n-1} \frac{x^n}{n!}, \quad n = 1, 2, 3 \dots \end{aligned}$$

Taking summation overn, we get

$$\begin{aligned} \left| \sum_{n=0}^{\infty} y_n \right| &\leq \frac{M}{(L_1 + L_2 + L_3)} e^{(L_1 + L_2 + L_3)x} + \left( 1 - \frac{M}{(L_1 + L_2 + L_3)} \right), \quad x \in [0, b], \\ &\leq \frac{M}{(L_1 + L_2 + L_3)} e^{(L_1 + L_2 + L_3)b} + \left( 1 - \frac{M}{(L_1 + L_2 + L_3)} \right). \end{aligned}$$

By using Theorem **Theorem 8**, we can conclude that the DJM series solution of eq. (11) converges uniformly in the interval  $[0, b]$ .

### 3 Stability analysis

Now, we consider a particular case of nonlinear eq. (11) namely pantograph equation

$$y'(x) = ay(x) + by(px) + cy(qx). \tag{12}$$

The following result gives sufficient condition for asymptotic stability of zero solution of eq. (12) by using technique of upper bounds.

**Theorem 14.**

If  $(a + b + c) < 0$  then zero solution of eq. (12) is asymptotically stable.

**Proof**

Define

$$z(x) = \max_{0 \leq t \leq x} y^2(x).$$

Now,

$$\begin{aligned} \frac{1}{2} z'(x) &= \frac{1}{2} \frac{d}{dx} (y^2(x)) \\ &= y(x)y'(x) \\ &= y(x)(ay(x) + by(px) + cy(qx)) \\ &= ay^2(x) + by(x)y(px) + cy(x)y(qx) \\ &\leq (a + b + c)z(x) \\ \Rightarrow z(x) &\leq z(0)e^{2(a+b+c)x} \\ \therefore \lim_{t \rightarrow \infty} y(x) &= 0, \quad \text{if } (a + b + c) < 0. \end{aligned}$$

This concludes the proof.  $\square$

**Definition 15.**

Consider the time-dependent DDE,

$$y'(x) = g(y(x), y(x - \tau_1(x)), y(x - \tau_2(x))), \tag{13}$$

where  $g : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ . The flow  $\phi_x(x_0)$  is a solution  $y(x)$  of eq. (13) with initial condition  $y(x) = x_0, x \leq 0$ . The point  $y^*$  is called equilibrium solution of eq. (13) if  $g(y^*, y^*, y^*) = 0$ . **(a)** If, for any  $\epsilon > 0$ , there exist  $\delta > 0$  such that  $|x_0 - y^*| < \delta \Rightarrow |\phi_x(x_0) - y^*| < \epsilon$ , then the system (13) is stable (in the Lyapunov sense) at the equilibrium  $y^*$ . **(b)** If the system (13) is stable at  $y^*$  and moreover,  $\lim_{x \rightarrow \infty} |\phi_x(x_0) - y^*| = 0$  then the system (13) is said to be asymptotically stable at  $y^*$ .

The following results are analogous to the results in [42].

**Theorem 16.**

Assume that the equilibrium solution  $y^*$  of the equation

$$y' = g(y(x), y(x - \tau_1^*), y(x - \tau_2^*)), \quad \tau_1^* = \tau_1(x_0), \tau_2^* = \tau_2(x_0)$$

is stable and

$$\|g(y(x), y(x - \tau_1(x)), y(x - \tau_2(x))) - g(y(x), y(x - \tau_1(x_1)), y(x - \tau_2(x_2)))\| < \epsilon_1|x - x_1| + \epsilon_2|x - x_2|,$$

for some  $\epsilon_1, \epsilon_2 > 0$  and  $x, x_1, x_2 \in [x_0, x_0 + c]$ ,  $c$  is a positive constant, then there exists  $\bar{x} > 0$  such that the equilibrium solution  $y^*$  of eq. (13) is stable on finite time interval  $[x_0, \bar{x}]$ .

**Corollary 17.**

If the real parts of all roots of  $\lambda - a - be^{-\lambda\tau_1^*} - ce^{-\lambda\tau_2^*} = 0$  are negative, where  $a = \partial_1 f, b = \partial_2 f, c = \partial_3 f$  evaluated at equilibrium. Then there exist  $\epsilon_c, \bar{x} (> x_0)$ , such that when  $\epsilon_1 < \epsilon_c, \epsilon_2 < \epsilon_c$ , the solution  $y^* = 0$  of eq. (13) is stable on finite time interval  $[x_0, \bar{x}]$ .

## 4 The pantograph equation and its solution

Consider the pantograph equation involving two delays,

$$y'(x) = ay(x) + by(px) + cy(qx), \quad y(0) = 1, \quad (14)$$

where  $0 < 1, 0 < q < 1, a \in \mathbb{R}, b \in \mathbb{R}$  and  $c \in \mathbb{R}$ .

Integrating eq. (14), we get

$$y(x) = 1 + \int_0^x (ay(t) + by(pt) + cy(qt)) dt$$

which is of the form eq. (5). Applying DJM, we obtain

$$\begin{aligned} y_0(x) &= 1, \\ y_1(x) &= \int_0^x (ay_0(t) + by_0(pt) + cy_0(qt)) dt \\ &= (a + b + c) \frac{x}{1!}, \\ y_2(x) &= \int_0^x (ay_1(t) + by_1(pt) + cy_1(qt)) dt \\ &= \int_0^x (a(a + b + c)t + b(a + b + c)tp + c(a + b + c)tq) dt \\ &= (a + b + c)(a + bp + cq) \frac{x^2}{2!}, \\ y_3(x) &= \int_0^x (ay_2(t) + by_2(pt) + cy_2(qt)) dt \\ &= (a + b + c)(a + bq + cq)(a + bp^2 + cq^2) \frac{x^3}{3!}, \\ &\vdots \\ y_n(x) &= \frac{x^n}{n!} \prod_{j=0}^{n-1} (a + bp^j + cq^j), \quad n = 1, 2, 3 \dots \end{aligned}$$

$\therefore$  The DJM solution of eq. (14) is

$$\begin{aligned}
 y(x) &= y_0(x) + y_1(x) + y_2(x) + \dots \\
 &= 1 + (a + b + c) \frac{x}{1!} + (a + b + c)(a + bp + cq) \frac{x^2}{2!} + \dots \\
 &= 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} \prod_{j=0}^{n-1} (a + bp^j + cq^j).
 \end{aligned} \tag{15}$$

From this solution eq. (15) of eq. (14) we propose a novel special function

$$\mathcal{R}(a, b, c, p, q, x) = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} \prod_{j=0}^{n-1} (a + bp^j + cq^j). \tag{16}$$

## 5 Analysis

### Theorem 18.

If  $0 < p < 1, 0 < q < 1$ , then the power series

$$\mathcal{R}(a, b, c, p, q, x) = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} \prod_{j=0}^{n-1} (a + bp^j + cq^j) \tag{17}$$

has infinite radius of convergence.

### proof.

Suppose

$$a_n = \frac{1}{n!} \prod_{j=0}^{n-1} (a + bp^j + cq^j), \quad n = 1, 2, \dots$$

If  $R$  is radius of convergence of eq. (17) then by using ratio test [43]

$$\begin{aligned}
 \frac{1}{R} &= \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{1}{(n+1)!} \prod_{j=0}^n (a + bp^j + cq^j)}{\frac{1}{n!} \prod_{j=0}^{n-1} (a + bp^j + cq^j)} \right| \\
 &= \lim_{n \rightarrow \infty} \left| \frac{a}{(n+1)} + \frac{bp^n}{(n+1)} + \frac{cq^n}{(n+1)} \right| \\
 &\Rightarrow \frac{1}{R} = 0 \quad (\because 0 < p < 1, 0 < q < 1).
 \end{aligned}$$

Thus the series has infinite radius of convergence.  $\square$

### Corollary 19.

The power series eq. (16) is absolutely convergent for all  $x$ , if  $0 < p < 1, 0 < q < 1$  and hence it is uniformly convergent on any compact interval on  $\mathbb{R}$ .

Proof of the following theorem is trivial.

### Theorem 20.

For  $0 < p < 1, 0 < q < 1, a \in \mathbb{R}, b \in \mathbb{R}, c \in \mathbb{R}$  and  $m \in \mathbb{N} \cup \{0\}$ , we have

$$\begin{aligned}
 \text{(a)} \quad \frac{d}{dx} \mathcal{R}(a, b, c, p, q, r^m x) &= r^m (a \mathcal{R}(a, b, c, p, q, r^m x) + b \mathcal{R}(a, b, c, p, q, r^m px) \\
 &\quad + c \mathcal{R}(a, b, c, p, q, r^m qx)) \\
 \text{and (b)} \quad \frac{d^m}{dx^m} \mathcal{R}(a, b, c, p, q, x) &= \sum_{n=m}^{\infty} \frac{x^{n-m}}{(n-m)!} \prod_{j=0}^{n-1} (a + bp^j + cq^j).
 \end{aligned}$$

**Theorem 21.**

(Addition Theorem) For  $0 < p < 1, 0 < q < 1, a \in \mathbb{R}, b \in \mathbb{R}, c \in \mathbb{R}$  and  $r \in \mathbb{N} \cup \{0\}$ , we have

$$\mathcal{R}(a, b, c, p, q, x + y) = \sum_{r=0}^{\infty} \frac{x^r}{r!} \mathcal{R}^{(r)}(a, b, c, p, q, y)$$

**proof.**

We have

$$\begin{aligned} \mathcal{R}(a, b, c, p, q, x + y) &= 1 + \sum_{n=1}^{\infty} \frac{(x + y)^n}{n!} \prod_{j=0}^{n-1} (a + bp^j + cq^j) \\ &= 1 + \sum_{n=1}^{\infty} \sum_{r=0}^n \frac{x^r}{r!} \frac{y^{n-r}}{(n-r)!} \prod_{j=0}^{n-1} (a + bp^j + cq^j). \end{aligned}$$

Define

$$\prod_{j=0}^{n-1} (a + bp^j + cq^j) = 1 \quad \text{for } n = 0.$$

$$\begin{aligned} \text{to } \quad & \therefore \mathcal{R}(a, b, c, p, q, x + y) = \sum_{r=0}^{\infty} \sum_{n=r}^{\infty} \frac{x^r}{r!} \frac{y^{n-r}}{(n-r)!} \prod_{j=0}^{n-1} (a + bp^j + cq^j) \\ = \sum_{r=0}^{\infty} \frac{x^r}{r!} \sum_{n=r}^{\infty} \frac{y^{n-r}}{(n-r)!} \prod_{j=0}^{n-1} (a + bp^j + cq^j). \end{aligned}$$

to

Using Theorem 20(b), we have

$$\mathcal{R}(a, b, c, p, q, x + y) = \sum_{r=0}^{\infty} \frac{x^r}{r!} \mathcal{R}^{(r)}(a, b, c, p, q, y).$$

□

**Theorem 22.**

For  $0 < p < 1, 0 < q < 1, a \geq 0, b \geq 0$  and  $c \geq 0$  the function  $\mathcal{R}(a, b, c, p, q, x)$  satisfies the following inequality

$$e^{ax} \leq \mathcal{R}(a, b, c, p, q, x) \leq e^{(a+b+c)x}, \quad 0 \leq x < \infty.$$

**proof.**

Since  $0 < p < 1, 0 < q < 1, a \geq 0, b \geq 0$  and  $c \geq 0$ , we have

$$\begin{aligned} \prod_{j=0}^{n-1} (a + bp^j + cq^j) &\leq (a + b + c)^n \\ \Rightarrow \frac{x^n}{n!} \prod_{j=0}^{n-1} (a + bp^j + cq^j) &\leq \frac{x^n (a + b + c)^n}{n!}. \end{aligned}$$

Taking summation over  $n$ , we get

$$\mathcal{R}(a, b, c, p, q, x) \leq e^{(a+b+c)x}, \quad 0 \leq x < \infty. \quad (18)$$

Similarly, we have

$$\begin{aligned} a^n &\leq \prod_{j=0}^{n-1} (a + bp^j + cq^j) \\ \Rightarrow e^{ax} &\leq \mathcal{R}(a, b, c, p, q, x), \quad 0 \leq x < \infty. \end{aligned} \quad (19)$$

From eqs (18) and (19), we get

$$e^{ax} \leq \mathcal{R}(a, b, c, p, q, x) \leq e^{(a+b+c)x}, \quad 0 \leq x < \infty.$$

Result is illustrated in Figure 1. for the values  $a = 2, b = 3, c = 4, p = \frac{1}{2}$  and  $q = \frac{1}{3}$ .

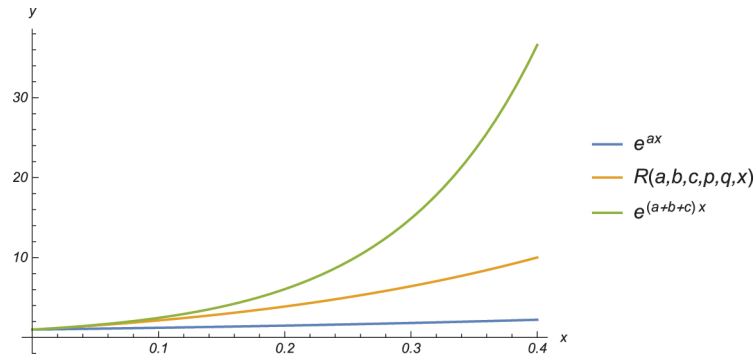


Figure 1 Bounds on  $R(a,b,c,p,q,x)$  for the values  $a=2,b=3,c=4,p=1/2,a=2,b=3,c=4,p=1/2$  and  $q=1/3,q=1/3$ .

**Theorem 23.**

$$\int_0^\infty e^{-t} \mathcal{R}(a,b,c,p,q,e^{-t}t) dt = 1 + \sum_{n=1}^\infty \frac{1}{(1+n)^n} \prod_{j=0}^{n-1} (a + bp^j + cq^j).$$

**proof.**

Consider

$$\begin{aligned} \int_0^\infty e^{-t} \mathcal{R}(a,b,c,p,q,e^{-t}t) dt &= \int_0^\infty e^{-t} \left( 1 + \sum_{n=1}^\infty \frac{t^n}{n!} \prod_{j=0}^{n-1} (a + bp^j + cq^j) \right) dt \\ &= \int_0^\infty e^{-t} dt + \sum_{n=1}^\infty \frac{1}{n!} \int_0^\infty e^{-t} t^n dt \prod_{j=0}^{n-1} (a + bp^j + cq^j) \\ &= 1 + \sum_{n=1}^\infty \frac{1}{n!} \frac{\Gamma(n+1)}{(n+1)^n} \prod_{j=0}^{n-1} (a + bp^j + cq^j) \\ &= 1 + \sum_{n=1}^\infty \frac{1}{(1+n)^n} \prod_{j=0}^{n-1} (a + bp^j + cq^j). \end{aligned}$$

□

**Theorem 24.**

$$\int_x^\infty e^{-t} \mathcal{R}(a,b,c,p,q,t) dt = \Gamma(1,x) \left( 1 + \sum_{n=1}^\infty \sum_{k=0}^n \frac{x^k}{k!} \prod_{j=0}^{n-1} (a + bp^j + cq^j) \right).$$

**proof.**

Consider

$$\begin{aligned} \int_x^\infty e^{-t} \mathcal{R}(a,b,c,p,q,t) dt &= \int_x^\infty e^{-t} + \sum_{n=1}^\infty \prod_{j=0}^{n-1} \frac{(a + bp^j + cq^j)}{n!} \int_x^\infty e^{-t} t^n dt \\ &= e^{-x} + \sum_{n=1}^\infty \sum_{k=0}^n \prod_{j=0}^{n-1} \frac{(a + bp^j + cq^j)}{n!} \Gamma(n+1,x) \\ &= e^{-x} + \sum_{n=1}^\infty \prod_{j=0}^{n-1} \frac{(a + bp^j + cq^j)}{n!} n! e^{-x} \sum_{k=0}^n \frac{x^k}{k!} \\ &= \Gamma(1,x) \left( 1 + \sum_{n=1}^\infty \sum_{k=0}^n \frac{x^k}{k!} \prod_{j=0}^{n-1} (a + bp^j + cq^j) \right). \end{aligned}$$

□

**Theorem 25.**

$$\int_0^x e^{-t} \mathcal{R}(a, b, c, p, q, t) dt = 1 + \sum_{n=1}^{\infty} \prod_{j=0}^{n-1} (a + bp^j + cq^j) \\ - \Gamma(1, x) \left( 1 + \sum_{n=1}^{\infty} \sum_{k=0}^n \frac{x^k}{k!} \prod_{j=0}^{n-1} (a + bp^j + cq^j) \right).$$

**proof.**

Consider

$$\int_0^x e^{-t} \mathcal{R}(a, b, c, p, q, t) dt = \int_0^x e^{-t} dt + \sum_{n=1}^{\infty} \prod_{j=0}^{n-1} \frac{(a + bp^j + cq^j)}{n!} \int_0^x e^{-t} t^n dt \\ = 1 - e^{-x} + \sum_{n=1}^{\infty} \prod_{j=0}^{n-1} \frac{(a + bp^j + cq^j)}{n!} \gamma(n+1, x) \\ = 1 - e^{-x} + \sum_{n=1}^{\infty} \prod_{j=0}^{n-1} \frac{(a + bp^j + cq^j)}{n!} n! \left( 1 - e^{-x} \sum_{k=0}^n \frac{x^k}{k!} \right) \\ = 1 + \sum_{n=1}^{\infty} \prod_{j=0}^{n-1} (a + bp^j + cq^j) \\ - \Gamma(1, x) \left( 1 + \sum_{n=1}^{\infty} \sum_{k=0}^n \frac{x^k}{k!} \prod_{j=0}^{n-1} (a + bp^j + cq^j) \right).$$

**5.1 The relation between  $\mathcal{R}(a, b, c, p, q, x)$  and Kummer's confluent hypergeometric function****Theorem 26.**

$$\int_0^x e^{-t} \mathcal{R}(a, b, c, p, q, t) dt = 1 + \Gamma(1, x) \left( \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{x^{(n+m)}}{n! (n+1)_m} \prod_{j=0}^{n-1} (a + bp^j + cq^j) \right) \\ - \mathcal{R}(a, b, c, p, q, x)$$

**proof.**

Consider

$$\int_0^x e^{-t} \mathcal{R}(a, b, c, p, q, t) dt = \int_0^x e^{-t} dt + \sum_{n=1}^{\infty} \frac{\int_0^x e^{-t} t^n dt}{n!} \prod_{j=0}^{n-1} (a + bp^j + cq^j) \\ = 1 - e^{-x} + \sum_{n=1}^{\infty} \frac{\gamma(n+1, x)}{n!} \prod_{j=0}^{n-1} (a + bp^j + cq^j) \\ = 1 - e^{-x} + \sum_{n=1}^{\infty} \frac{(n\gamma(n, x) - x^n e^{-x})}{n!} \prod_{j=0}^{n-1} (a + bp^j + cq^j)$$

$$\begin{aligned}
 &= 1 - e^{-x} + \sum_{n=1}^{\infty} \frac{(\gamma(n, x))}{(n-1)!} \prod_{j=0}^{n-1} (a + bp^j + cq^j) \\
 &\quad - e^{-x} \sum_{n=1}^{\infty} \frac{x^n}{n!} \prod_{j=0}^{n-1} (a + bp^j + cq^j) \\
 &= 1 + \sum_{n=1}^{\infty} \frac{(n^{-1}x^n e^{-x} {}_1F_1(1; n+1; x))}{(n-1)!} \prod_{j=0}^{n-1} (a + bp^j + cq^j) \\
 &\quad - e^{-x} \mathcal{R}(a, b, c, p, q, x) \\
 &= 1 + \Gamma(1, x) \left( \sum_{n=1}^{\infty} \frac{x^n}{n!} {}_1F_1(1; n+1; x) \prod_{j=0}^{n-1} (a + bp^j + cq^j) \right) \\
 &\quad - \mathcal{R}(a, b, c, p, q, x) \\
 &= 1 + \Gamma(1, x) \left( \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{x^{(n+m)}}{n! (n+1)_m} \prod_{j=0}^{n-1} (a + bp^j + cq^j) \right) \\
 &\quad - \mathcal{R}(a, b, c, p, q, x).
 \end{aligned}$$

Using properties of incomplete gamma functions described in [20], we have following Corollaries.□

**Corollary 27.**

$$\begin{aligned}
 \int_0^x e^{-t} \mathcal{R}(a, b, c, p, q, t) dt &= 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} {}_1F_1(n; n+1; -x) \prod_{j=0}^{n-1} (a + bp^j + cq^j) \\
 &\quad - \Gamma(1, x) \mathcal{R}(a, b, c, p, q, x).
 \end{aligned}$$

**Corollary 28.**

$$\begin{aligned}
 \int_0^x e^{-t} \mathcal{R}(a, b, c, p, q, t) dt &= 1 + \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m (n)_m x^{(n+m)}}{n! m! (n+1)_m} \prod_{j=0}^{n-1} (a + bp^j + cq^j) \\
 &\quad - \Gamma(1, x) \mathcal{R}(a, b, c, p, q, x).
 \end{aligned}$$

**Theorem 29.**

$$\begin{aligned}
 \int_x^{\infty} e^{-t} \mathcal{R}(a, b, c, p, q, t) dt &= \Gamma(1, x) + \pi x \Gamma(1, x) \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)! \sin((1+n)\pi)} \\
 &\quad \left( \frac{{}_1F_1(1; 1+n; x)}{\Gamma(1+n)\Gamma(1-n)} - x^{-n} \frac{{}_1F_1(1-n; 1-n; x)}{\Gamma(1-n)} \right) \\
 &\quad \prod_{j=0}^{n-1} (a + bp^j + cq^j) + e^x (\mathcal{R}(a, b, c, p, q, x) - 1).
 \end{aligned}$$

**proof.**

Consider

$$\begin{aligned}
 \int_x^\infty e^{-t} \mathcal{R}(a, b, c, p, q, t) dt &= \int_x^\infty e^{-t} + \sum_{n=1}^\infty \frac{\int_x^\infty e^{-t} t^n dt}{n!} \prod_{j=0}^{n-1} (a + bp^j + cq^j) \\
 &= e^{-x} + \sum_{n=1}^\infty \frac{\Gamma(n+1, x)}{n!} \prod_{j=0}^{n-1} (a + bp^j + cq^j) \\
 &= \Gamma(1, x) + \sum_{n=1}^\infty \frac{(n\Gamma(n, x) + x^n e^{-x})}{n!} \prod_{j=0}^{n-1} (a + bp^j + cq^j) \\
 &= \Gamma(1, x) + \sum_{n=1}^\infty \frac{\Gamma(n, x)}{(n-1)!} \prod_{j=0}^{n-1} (a + bp^j + cq^j) \\
 &\quad + e^x \sum_{n=1}^\infty \frac{x^n}{n!} \prod_{j=0}^{n-1} (a + bp^j + cq^j) \\
 &= \Gamma(1, x) + \sum_{n=1}^\infty \frac{(x^n e^{-x} U(1; 1+n; x))}{(n-1)!} \prod_{j=0}^{n-1} (a + bp^j + cq^j) \\
 &\quad + e^x (\mathcal{R}(a, b, c, p, q, x) - 1) \\
 &= \Gamma(1, x) + x\Gamma(1, x) \sum_{n=1}^\infty \frac{x^{n-1}}{(n-1)!} U(1; 1+n; x) \\
 &\quad \prod_{j=0}^{n-1} (a + bp^j + cq^j) + e^x (\mathcal{R}(a, b, c, p, q, x) - 1) \\
 &= \Gamma(1, x) + \pi x \Gamma(1, x) \sum_{n=1}^\infty \frac{x^{n-1}}{(n-1)! \sin((1+n)\pi)} \\
 &\quad \left( \frac{{}_1F_1(1; 1+n; x)}{\Gamma(1+n)\Gamma(1-n)} - x^{-n} \frac{{}_1F_1(1-n; 1-n; x)}{\Gamma(1-n)} \right) \\
 &\quad \prod_{j=0}^{n-1} (a + bp^j + cq^j) + e^x (\mathcal{R}(a, b, c, p, q, x) - 1).
 \end{aligned}$$

□

Proof of following Corollaries are immediate from the properties of incomplete gamma functions [20].

**Corollary 30.**

$$\begin{aligned}
 \int_x^\infty e^{-t} \mathcal{R}(a, b, c, p, q, t) dt &= \Gamma(1, x) + \Gamma(1, x) \sum_{n=1}^\infty \frac{U(1-n; 1-n; x)}{(n-1)!} \prod_{j=0}^{n-1} (a + bp^j + cq^j) \\
 &\quad + e^x (\mathcal{R}(a, b, c, p, q, x) - 1).
 \end{aligned}$$

**Corollary 31.**

$$\begin{aligned}
 \int_x^\infty e^{-t} \mathcal{R}(a, b, c, p, q, t) dt &= \Gamma(1, x) + \pi \Gamma(1, x) \sum_{n=1}^\infty \frac{1}{(n-1)! \sin((1+n)\pi)} \\
 &\quad \left( \frac{{}_1F_1(1-n; 1-n; x)}{\Gamma(1-n)} - x^n \frac{{}_1F_1(1; 1+n; x)}{\Gamma(1-n)\Gamma(1+n)} \right) \\
 &\quad \prod_{j=0}^{n-1} (a + bp^j + cq^j) + e^x (\mathcal{R}(a, b, c, p, q, x) - 1).
 \end{aligned}$$

**Corollary 32.**

Using properties of incomplete gamma functions and generalized Laguerre polynomial, we have following expression for  $\int_x^\infty e^{-t} \mathcal{R}(a, b, c, p, q, t) dt$  :

$$\begin{aligned}
 (i) \quad & \gamma(1, x) + x\Gamma(1, x) \sum_{n=1}^\infty \sum_{m=0}^\infty \prod_{j=0}^{n-1} (a + bp^j + cq^j) \frac{x^{n-1}}{(n-1)!} \frac{L_m^{(n)}(x)}{(m+1)!} \\
 & + e^x (\mathcal{R}(a, b, c, p, q, x) - 1) \\
 (ii) \quad & \gamma(1, x) + x\Gamma(1, x) \sum_{n=1}^\infty \sum_{m=0}^\infty \sum_{k=0}^m \prod_{j=0}^{n-1} (-1)^k \binom{m+n}{m-k} \\
 & (a + bp^j + cq^j) \frac{x^{n+k-1}}{k!(n-1)!} + e^x (\mathcal{R}(a, b, c, p, q, x) - 1) \\
 (iii) \quad & \gamma(1, x) + x\Gamma(1, x) \sum_{n=1}^\infty \sum_{m=0}^\infty \prod_{j=0}^{n-1} (-1)^k \binom{m+n}{m} (a + bp^j + cq^j) \\
 & {}_1F_1(-m; n+1; x) \frac{x^{n-1}}{(n-1)!} + e^x (\mathcal{R}(a, b, c, p, q, x) - 1),
 \end{aligned}$$

where  $L_m^{(n)}$  is generalized Laguerre polynomial.

**Theorem 33.**

$$\int_0^x e^{-t} \mathcal{R}(a, b, c, p, q, \lambda t) dt = \gamma(1, x) + \lambda \sum_{n=1}^\infty \sum_{m=0}^{n+1} \prod_{j=0}^{n-1} (a + bp^j + cq^j) \frac{\lambda^n}{n!} \frac{\gamma(n+m+1, x)}{m!} (1-\lambda)^m.$$

**proof.**

We have

$$\begin{aligned}
 \int_0^x e^{-t} \mathcal{R}(a, b, c, p, q, \lambda t) dt &= 1 - e^{-x} + \sum_{n=1}^\infty \prod_{j=0}^{n-1} \frac{(a + bp^j + cq^j)}{n!} \int_0^x e^{-t} (\lambda t)^n dt \\
 &= \gamma(1, x) + \sum_{n=1}^\infty \prod_{j=0}^{n-1} \frac{(a + bp^j + cq^j)}{n!} \gamma(n+1, \lambda x).
 \end{aligned}$$

By using the property Gautschi et al. [44]  $\gamma(n, \lambda x) = \lambda^n \sum_{m=0}^n \frac{\gamma(n+m, x)}{m!} (1-\lambda)^m$ , we get

$$\int_0^x e^{-t} \mathcal{R}(a, b, c, p, q, \lambda t) dt = \gamma(1, x) + \lambda \sum_{n=1}^\infty \sum_{m=0}^{n+1} \prod_{j=0}^{n-1} (a + bp^j + cq^j) \frac{\lambda^n}{n!} \frac{\gamma(n+m+1, x)}{m!} (1-\lambda)^m.$$

□

**Theorem 34.**

$$\int_x^\infty e^{-t} \mathcal{R}(a, b, c, p, q, \lambda t) dt = \Gamma(1, x) + \lambda \sum_{n=1}^\infty \sum_{m=0}^{n+1} \prod_{j=0}^{n-1} (a + bp^j + cq^j) \frac{\lambda^n}{n!} \frac{\Gamma(n+m+1, x)}{m!} (1-\lambda)^m.$$

**proof.**

$$\begin{aligned}
 \int_x^\infty e^{-t} \mathcal{R}(a, b, c, p, q, \lambda t) dt &= e^{-x} + \sum_{n=1}^\infty \prod_{j=0}^{n-1} \frac{(a + bp^j + cq^j)}{n!} \int_x^\infty e^{-t} (\lambda t)^n dt \\
 &= \Gamma(1, x) + \sum_{n=1}^\infty \prod_{j=0}^{n-1} \frac{(a + bp^j + cq^j)}{n!} \Gamma(n+1, \lambda x).
 \end{aligned}$$

By using the property Gautschi et al. [44]  $\Gamma(n, \lambda x) = \lambda^n \sum_{m=0}^n \frac{\Gamma(n+m, x)}{m!} (1-\lambda)^m$ , we obtain

$$\int_x^\infty e^{-t} \mathcal{R}(a, b, c, p, q, \lambda t) dt = \Gamma(1, x) + \lambda \sum_{n=1}^\infty \sum_{m=0}^{n+1} \prod_{j=0}^{n-1} (a + bp^j + cq^j) \frac{\lambda^n}{n!} \frac{\Gamma(n+m+1, x)}{m!} (1-\lambda)^m.$$

□

## 6 Generalizations to fractional-order DDE

Consider the fractional DDE with proportional delay

$$D_0^\alpha y(x) = ay(x) + by(px) + cy(qx), \quad y(0) = 1, \quad (20)$$

where  $0 < \alpha \leq 1$ ,  $0 < p < 1$ ,  $0 < q < 1$ ,  $a \in \mathbb{R}$ ,  $b \in \mathbb{R}$  and  $c \in \mathbb{R}$ . Equivalently

$$y(x) = 1 + I^\alpha (ay(x) + by(px) + cy(qx)).$$

The DJM solution of eq. (20) is

$$y(x) = 1 + \sum_{n=1}^{\infty} \frac{x^{\alpha n}}{\Gamma(\alpha n + 1)} \prod_{j=0}^{n-1} (a + bp^{\alpha j} + cq^{\alpha j}). \quad (21)$$

We denote the series in eq. (21) by

$$\mathcal{R}_\alpha(a, b, c, p, q, x) = 1 + \sum_{n=1}^{\infty} \frac{x^{\alpha n}}{\Gamma(\alpha n + 1)} \prod_{j=0}^{n-1} (a + bp^{\alpha j} + cq^{\alpha j}).$$

### Theorem 35.

If  $0 < q < 1$ , then the power series

$$\mathcal{R}_\alpha(a, b, c, p, q, x) = 1 + \sum_{n=1}^{\infty} \frac{x^{\alpha n}}{\Gamma(\alpha n + 1)} \prod_{j=0}^{n-1} (a + bp^{\alpha j} + cq^{\alpha j}),$$

is convergent for all finite values of  $x$ .

### proof.

Result follows immediately by ratio test.  $\square$

## 7 Conclusions

In this paper, we have obtained a new special function arising from pantograph equation with two delays. The solution is obtained by applying the iterative scheme namely DJM. The existence, uniqueness, stability and convergence results for the time-dependent DDE are presented in this paper. The new special function exhibits different properties and relations with other functions. The generalization to fractional-order case is also presented. We hope that the researchers will get motivated from this work and work on more properties of this new special function.

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The DOI of this article has been used for another publication by mistake. If you intended to access the other publication, please use this link: <https://doi.org/10.1515/psr-2016-0103>

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