Research Article

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Exact solutions of Einstein's field equations via homothetic symmetries of non-static plane symmetric spacetime

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Abstract: In this paper, we explore some non-static plane symmetric solutions of Einstein's field equations (EFEs) that possess a specific scaling symmetry, known as homothetic symmetry. This symmetry helps in simplifying EFEs and reveals important patterns in gravitational systems. Using a systematic computational method, the Rif tree approach, we find some new exact solutions for expanding universes with plane symmetry. We classify these solutions according to their symmetry properties, finding spacetimes with 4-, 5-, 7-, and 11-dimensional symmetry structures. The physical viability is established through energy-momentum tensors, that reveal solutions describing anisotropic fluids, perfect fluids, and vacuum configurations. By analyzing energy conditions, we have identified which of the derived solutions are physically meaningful. The physical interpretation reveals important connections to some known models, including anisotropic Bianchi type universes, Kasner solutions, Szekeres inhomogeneities, and colliding plane wave geometries.

Keywords: exact solutions; spacetime symmetries; homothetic vector fields; plane symmetric spacetime; Rif tree approach

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1 Introduction

In general relativity, spacetimes symmetries are crucial because they provide deep insights into the physical characteristics of gravitational fields as well as the mathematical structure of Einstein's field equations. Among these symmetries, the isometries, conformal transformations and the homothetic vector fields are particularly important. These symmetries not only help in reducing the complexity of EFEs, enabling the search for their exact solutions, they also provide fundamental conservation laws and a greater understanding of the causal and geometric structure of spacetime.

A Killing vector field (KVF) is defined as a vector field $\eta = (\eta^0, \eta^1, \eta^2, \eta^3)$ satisfying the condition [1]:

$$\mathcal{L}_\eta g_{ab} = 0, \tag{1.1}$$

where g_{ab} denotes the spacetime metric and $\mathcal{L}$ is the Lie derivative operator. This condition is linked with the preservation of the metric of spacetime along the flow of the vector field. Killing vector fields are crucial in describing conserved quantities, such as energy, and linear and angular momentum.

If the metric of spacetime is preserved up to a constant scaling factor along the flow of a vector field $\eta = (\eta^0, \eta^1, \eta^2, \eta^3)$, then such a vector field is defined as a homothetic vector field (HVF). These vector fields satisfy the relation [1]:

$$\mathcal{L}_\eta g_{ab} = 2\alpha g_{ab}, \tag{1.2}$$

where α is some constant. The above condition introduces a self-similarity or scaling symmetry in spacetime. If $\alpha = 0$, a HVF reduces to a KVF. By a proper HVF, we mean a homothetic symmetry defined by Eq. (1.2) with non-zero α. Homothetic symmetries are central to studies involving self-similar solutions, cosmological models, and gravitational collapse, where scale invariance plays a key role.

In the broader context of spacetime symmetries, conformal vector fields (CVFs) preserve the metric up to a local scaling factor that depends on the spacetime coordinates. The CVFs are defined in a similar way by replacing α in Eq. (1.2) by a function of spacetime coordinates, that is:

$$\mathcal{L}_\eta g_{ab} = 2\psi(x^a) g_{ab}, \tag{1.3}$$

where $\psi(x^a)$ is some smooth function. If ψ is a constant function, a CVF becomes a HVF and it reduces to a KVF for $\psi = 0$. A CVF that is neither Killing nor homothetic vector field is known as a proper CVF.

The symmetries of the Ricci and energy-momentum tensors are defined in a similar way by replacing g_{ab} in Eqs. (1.1)-(1.3) by R_{ab} and T_{ab} respectively.

In addition to these spacetime symmetries, there is another important symmetry, called Noether symmetry. For a vector field $V = \eta\partial_s + V^a\partial_{x^a}$, its first prolongation is defined as $V^{[1]} = V + (DV^a - \dot{x}^a D\eta)\partial_{\dot{x}^a}$. A Noether symmetry of the Lagrangian $L(s, x^a, \dot{x}^a)$ is defined by the vector filed V if there exists a gauge function $F(s, x^a)$ such that [2]:

$$V^{[1]}L + (D_s\eta)L = D_s F. \tag{1.4}$$

Here $D_s = \partial_s + \dot{x}^a\partial_{x^a}$, and η and V^a are dependent on the geodesics parameter s. Moreover, the derivative with respect to s is denoted by a dot.

Noether symmetries are found to be valuable in the classification of Lagrangians associated with spacetime metrics and for analyzing differential equations. While solving complicated and nonlinear differential equations, Noether symmetries assist in the reduction of the number of independent variables and the order of higher order differential equations, and in the linearization of nonlinear differential equations. Moreover, Noether symmetries are also crucial from a physical standpoint because they are directly linked with conservation laws via Noether's theorem.

There are some significant relations among Noether symmetries and other spacetime symmetries such as KVFs, HVFs and CVFs. For instance, all KVFs admitted by a spacetime metric are included in the collection of Noether symmetries of the associated Lagrangian. Similarly, if α denotes the homothety constant, then $V + 2\alpha s\partial_s$ is a Noether symmetry of the lagrangian if and only if V is a HVF associated with the corresponding metric [3]. A Noether symmetry that is neither a KVF nor it is linked with a HVF is known as a proper Noether symmetry. For flat Minkowski spacetime, the set of 15 CVFs constitutes a subset of the collection of 17 Noether symmetries of the associated Lagrangian. However, for non-flat spacetimes no such general relationship is identified between CVFs and Noether symmetries.

All the above defined symmetries have been extensively explored in the literature for various spacetimes [4–20]. However, most of this work has focused on simpler or static spacetimes due to the ease of their mathematical treatment and inherent simplicity. As a result, there has been less research done on non-static spacetimes, which are more complicated and harder to study. The difficulties of working with non-static metrics, like their changing nature and the complexity of solving the related equations, have made this area less studied.

In addition to the above mentioned references, the recent literature has highlighted the relevance of symmetry methods in the fields of cosmology and modified theories of gravity. Particularly, Noether symmetries are employed to $f(R)$ and scalar-tensor cosmologies to identify the admissible forms of potentials and couplings, and to derive exact cosmological solutions describing late-time acceleration [21–26]. Similarly, homothetic and conformal symmetries are also investigated in connection with anisotropic and inhomogeneous models, that provide deeper understanding of cosmological evolution and its observational implications [16]. Moreover, CVFs are explored in gravitational wave spacetimes, where related vacuum solutions show self-similar properties [27].

In this research, we investigate proper HVFs of non-static plane symmetric spacetime with the following most general metric [28].

$$ds^2 = -f^2(t,x)\,\mathrm{d}t^2 + g^2(t,x)\,\mathrm{d}x^2 + h^2(t,x)[\mathrm{d}y^2 + \mathrm{d}z^2], \tag{1.5}$$

where $f(t,x) \neq 0$, $g(t,x) \neq 0$ and $h(t,x) \neq 0$. Our goal is to extract all the non-static plane symmetric metrics that admit proper HVFs and to find the associated solutions to the EFEs. With a focus on anisotropic and perfect fluid sources, which are significant in many cosmological models and astrophysical scenarios, we discuss the physical implications of these symmetries.

The motivation behind selecting the metric (1.5) for our study is that it includes several important and well-studied solutions of EFEs. For example, when the metric coefficients f, g, and h depend only on x, the metric (1.5) signifies a static plane symmetric spacetime. Such metrics are essential in various contexts, particularly in deriving the Kasner's and Taub solutions of EFEs. Moreover, when f, g, and h depend only on t, the metric (1.5) becomes the Bianchi type I metric. Bianchi type metrics are significant because they are homogeneous but not necessarily isotropic, offering key cosmological models that solve the EFEs, with different solutions based on the choice of scale factors. Additionally, when $f = g = h = f(t)$, the metric (1.5) simplifies to the Friedmann metric, a fundamental model in cosmology. Therefore, this study naturally covers the classification of all these metrics in terms of their HVFs, as special cases.

In addition to this, the non-static plane symmetric metric (1.5) is a perfect model for studying gravitational waves, anisotropic gravitational fields, and cosmic evolution because of its symmetry in two spatial dimensions. This spacetime provides a simplified framework for investigating the solutions to EFEs since its metric is invariant under translations and rotations in the plane.

In order to explore the symmetries of a spacetime, one always needs to solve a system of determining equations representing these symmetries. The conventional method used to solve these determining equations is known as direct integration technique. In this method, the determining equations are decoupled and integrated directly to find the explicit form of symmetry vector fields. The process usually gives rise to a number of cases depending upon the conditions on the metric functions under which the spacetime under consideration admits the desired symmetries. It is a quite lengthy and cumbersome technique which may result in lack of potential spacetime metrics admitting the required symmetries.

In recent literature, the Rif tree approach has emerged as a powerful computational tool for analyzing systems of partial differential equations that govern the existence of symmetries in spacetimes. This algorithmic method transforms the system of determining equations into an involutive form, systematically solving them and allowing for the classification of vector fields such as HVFs. This method relies on a Maple algorithm (Rif algorithm), which is implemented using the Exterior package in Maple. The process begins by loading the "Exterior" package. Next, the system of differential equations defining HVFs is inserted using the command "sysDEs". The third step involves applying the "symmetry, eq := findsymmetry" command, which analyzes the symmetry equations and identifies the conditions on the metric functions that allow for HVFs. The algorithm then displays these conditions. A graphical representation of these conditions can be viewed using the "caseplot(eq, pivots)" command, resulting in a tree shape, called Rif tree. The branches of the Rif tree illustrate the conditions under which the spacetime may admit HVFs. Finally, the symmetry equations are solved under these branch-specific conditions, yielding the explicit form of the HVFs. The novelty of this approach is that it gives a better classification of the spacetime through its symmetries. Recently, the Rif tree approach has been applied in the classification of HVFs and other symmetries, leading to the discovery of additional spacetime metrics that were previously unidentified using the conventional direct integration tecnique [29–34].

In this paper, we focus on finding the HVFs of non-static plane symmetric spacetimes, using the Rif tree approach. We aim to identify all the non-static plane symmetric metrics for which proper homothetic symmetries exist and to derive the corresponding solutions to the EFEs for anisotropic or perfect fluid sources.

2 Homothetic symmetries

We apply the definition of HVFs given in Eq. (1.2) to the metric (1.5) to derive the following symmetry equations.

$$f_{,t}\eta^0 + f_{,x}\eta^1 + f\eta^0_{,t} = \alpha f, \tag{2.1}$$

$$g_{,t}\eta^0 + g_{,x}\ \eta^1 + g\eta^1_{,x} = \alpha g, \tag{2.2}$$

$$h_{,t}\eta^0 + h_{,x}\ \eta^1 + h\eta^2_{,y} = \alpha h, \tag{2.3}$$

$$h_{,t}\eta^0 + h_{,x}\ \eta^1 + h\eta^3_{,z} = \alpha h, \tag{2.4}$$

$$f^2\eta^0_{,x} - g^2\eta^1_{,t} = 0, \tag{2.5}$$

$$f^2\eta^0_{,y} - h^2\eta^2_{,t} = 0, \tag{2.6}$$

$$f^2\eta^0_{,z} - h^2\eta^3_{,t} = 0, \tag{2.7}$$

$$g^2\eta^1_{,y} + h^2\eta^2_{,x} = 0, \tag{2.8}$$

$$g^2\eta^1_{,z} + h^2\eta^3_{,x} = 0, \tag{2.9}$$

$$\eta^2_{,z} + \eta^3_{,y} = 0, \tag{2.10}$$

In order to find the exact form of the HVF η, we need to solve the above symmetry equations. For this, the metric coefficients $f(t,x)g(t,x)$ and $h(t,x)$ must be subject to some constraints. In order to obtain such constraints, we have analyzed Eqs. (2.1)–(2.10) using the commands of Rif algorithm, that are already explained in the introduction section. As a result, the algorithm generates the Rif tree given in Figure 1 along with its nodes (pivots) as presented in (2.11). Each branch of the Rif tree imposes certain restrictions on the functions $f(t,x)g(t,x)$ and $h(t,x)$. For example, in branch 1 both $p1$ and $p2$ are non-zero, that is $h_{,t} \neq 0$ and $h_{,t}g_{,x} - h_{,x}g_{,t} \neq 0$. We have used these conditions to solve Eqs. (2.1)–(2.10). The same procedure is followed for the constraints of all 25 branches of the Rif tree. Consequently, we have obtained several metrics with 4, 5, 7, and 11-dimensional homothetic algebras. All these homothetic algebras include the set $M_3 = \{\partial_y, \partial_z, y\partial_z - z\partial_y\}$ as a proper subset, which is the set of minimum KVFs admitted by the metric (1.5). Moreover, some cases produce only one proper HVF in addition to the three minimum KVFs, while the other metrics possess a proper HVF and some additional KVFs. The results of all 25 branches of the Rif tree are summarized in Table 1. In every case, the generator V_4 represents the proper homothety.

$$
\begin{aligned}
p1 &= h_{,t}\\
p2 &= h_{,t}g_{,x} - h_{,x}g_{,t}\\
p3 &= h_{,t}f_{,x} - h_{,x}f_{,t}\\
p4 &= h_{,t}h_{,tx} - h_{,tt}h_{,x}\\
p5 &= (h_{,t})^2h_{,xx} - 2h_{,tx}h_{,x} + h_{,tt}(h_{,x})^2\\
p6 &= (h_{,t})^2h_{,ttx} - 2h_{,t}h_{,tt}h_{,tx} - h_{,t}h_{,x}h_{,ttt} + 2(h_{,tt})^2h_{,x}\\
p7 &= (h_{,t})^2h_{,xx} - h_{,tt}(h_{,x})^2\\
p8 &= h_{,x}\\
p9 &= g_{,t}\\
p10 &= f_{,t}\\
p11 &= g_{,t}f_{,x} - g_{,x}f_{,t}\\
p12 &= g_{,t}g_{,tx} - g_{,x}g_{,tt}\\
p13 &= -(g_{,t})^2g_{,xx} + 2g_{,t}g_{,x}g_{,tx} - (g_{,x})^2g_{,tt}\\
p14 &= g_{,ttt}g_{,t}g_{,x} - (g_{,t})^2g_{,ttx} + 2g_{,t}g_{,tt}g_{,tx} - 2g_{,x}(g_{,tt})^2\\
p15 &= (g_{,t})^2g_{,xx} - (g_{,x})^2g_{,tt}\\
p16 &= g_x\\
p17 &= f_{,t}f_{,tx} - f_{,x}f_{,tt}\\
p18 &= (f_{,t})^2f_{,xx} - 2f_{,t}f_{,x}f_{,tx} + (f_{,x})^2f_{,tt}\\
p19 &= (f_{,t})^2f_{,ttx} - f_{,t}f_{,x}f_{,ttt} - 2f_{,t}f_{,tx}f_{,tt} + 2f_{,x}(f_{,tt})^2\\
p20 &= (f_{,t})^2f_{,xx} - (f_{,x})^2f_{,tt}\\
p21 &= f_{,x}
\end{aligned}
\qquad (2.11)
$$

3 Solutions of the field equations

By solving the homothetic symmetry equations for non-static plane symmetric spacetime, we have derived various Lorentzian metrics with 4-, 5-, 7- and 11-dimensional homothetic algebras. Of these Lorentzian metrics, the exact solutions to the EFEs are given by those metrics that satisfy the EFEs and have the energy-momentum tensor associated with some known matter. The EFEs may be used to determine the energy-momentum tensor T_{ab}, corresponding to each of these metrics. Additionally, T_{ab} can be used to verify that these metrics satisfy different energy conditions and to evaluate the physical realism of the metrics. We follow this procedure to determine which of the acquired metrics are

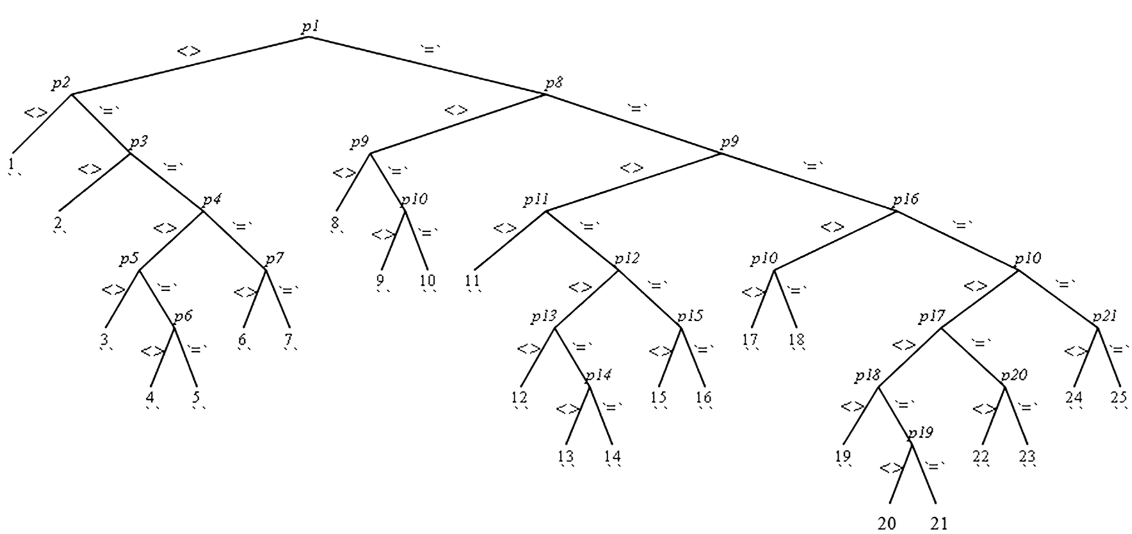

Figure 1: Rif tree.

Table 1: Metrics and their homothetic symmetries.

Metric No./Branch no.	Metric coefficients	Proper HVF
4a	$f=\text{Const.},\ g=a_1t+a_2,\ h=\frac{a_1t+a_2}{a_2}L(x),$	$V_{(4)}=\frac{a_1t+a_2}{a_1}\partial_t$
1	where a_1, a_2 are non-zero constants and $L(x)$ is any function of x.	
4b	$f=f(t),\ g=\text{Const.},\ h=-\frac{(a_1x+2a_2)^{1-\frac{2a_3}{a_1}}}{2a_3(a_1-2a_3)}+(a_1\int f(t)\mathrm{d}t+2a_4)^{1-\frac{2a_3}{a_1}}$	$V_{(4)}=\frac{\int f(t)\mathrm{d}t}{f(t)}\partial_t+x\partial_x,$
2	where $f_{,t}(t)\neq 0$ $a_1\neq 2a_3,\ a_1\neq 0,\ a_3\neq 0$	
4c	$f=(a_1x+2a_2)^{1-\frac{2a_3}{a_1}},\ g=\text{Const.},\ h=(2a_3t+2a_4)^{\frac{a_1-2a_5}{2a_3}}$	$V_{(4)}=x\partial_x$
2	where $a_1\neq 0,\ a_3\neq 0,\ a_1\neq 2a_3\neq 2a_5$	
4d	$f=a_1x+a_2,\ g=\text{Const.},\ h=e^{\frac{(a_3-2a_4)t}{2a_5}}$	$V_{(4)}=\frac{a_1x+a_2}{a_1}\partial_x,$
2	where $a_3\neq 2a_4$, and $a_1\neq 0$	
4e	$f=a_1x+a_2,\ g=\text{Const.},\ h=h(t)\neq e^{\frac{(a_3-2a_4)t}{2a_5}},$	$V_{(4)}=\frac{a_1x+a_2}{a_1}\partial_x+y\partial_y+z\partial_z,$
2	where $h_{,t}(t)\neq 0$ and $a_1\neq 0$	
4f	$f=\sqrt{2a_1x+2a_2},\ g=\text{Const.},\ h=(a_3t+4a_4)^{2-\frac{4a_5}{a_3}},$	$V_{(4)}=\frac{t}{2}\partial_t+\frac{f}{2f_{,x}(x)}\partial_x.,$
2	where $a_3\neq 2a_5$ and $a_1\neq 0,\ a_3\neq 0$	
4g	$f=\text{Const.},\ g=\text{Const.},\ h=\frac{(a_1t+2a_2)^{1-\frac{2a_3}{a_1}}}{a_1-2a_3}+(a_1x+2a_4)^{1-\frac{2a_3}{a_1}},$	$V_{(4)}=t\partial_t+x\partial_x,$
3	where $a_1\neq 2a_3$ and $a_1, a_3\neq 0$	
4h	$f=\text{Const.},\ g=\text{Const.},\ h=\frac{1}{a_1}\ln(\frac{-a_1x+a_2}{a_1t+a_3}),$	$V_{(4)}=\frac{(a_1t+a_3)}{a_1}\partial_t-\frac{(-a_1x+a_2)}{a_1}\partial_x+y\partial_y+z\partial_z$
5	where a_1, a_2, a_3 are non-zero	
4i	$f=\text{Const.},\ g=(a_1t+2a_2)^{1-a_3+\frac{2a_3a_5}{a_1}},\ h=(a_3x+a_4)^{\frac{1}{a_3}},$	$V_{(4)}=t\partial_t+(a_3x+a_5)\partial_x$
8	where $a_1\neq 0,\ a_3\neq 0.$	
4j	$f=\text{const.},\ g=a_1t+2a_2,\ h=h(x)\neq(a_3x+a_4)^{\frac{1}{a_3}},$	$V_{(4)}=t\partial_t+y\partial_y+z\partial_z,$
8	where $a_1\neq 0$ and $h_{,x}(x)\neq 0$	
4k	$f=\text{Const.},\ g=\sqrt{2a_1t+2a_2},\ h=(a_3x+a_4)^{\frac{1}{a_3}},$	$V_{(4)}=\frac{g^2(t)}{2a_1}\partial_t+\frac{h(x)}{2a_3h_{,x}(x)}\partial_x+\frac{2a_3-1}{2a_3}(y\partial_y+z\partial_z),$
8	where $a_1\neq 0,\ a_3\neq 0$	
4l	$f=\text{Const.},\ g=t^{\frac{2a_2}{a_1}},\ h=[(a_1-2a_2)x]^{\frac{a_1-2a_3}{a_1-2a_2}}$	$V_{(4)}=t\partial_t+x\partial_x,$
8	where $a_1\neq 0,\ a_1\neq 2a_2$ and $a_1\neq 2a_3$	
4m	$f=(a_1t+a_2)^{-1+\frac{a_3}{2a_1}}+\frac{(a_3x+2a_4)^{1-\frac{2a_1}{a_3}}}{a_3-2a_1},\ g=\text{Const.},\ h=(a_3x+2a_4)^{1-\frac{2a_5}{a_3}},$	$V_{(4)}=x\partial_x,$
9	where $a_3\neq 2a_1\neq 2a_5,$	
5a	$f=\text{Const.},\ g=a_1t+a_2,\ h=\left(a_3e^{a_1x}+a_4e^{-a_1x}\right)\frac{(a_1t+a_2)}{a_2},$	$V_{(4)}=\frac{h}{a_1}\partial_t,$
1	where $a_i\neq 0$ for $i=1,\dots,4$	$V_{(5)}=-\frac{a_2h_{,x}}{a_1a_4h}\partial_t+\frac{a_2h}{a_4g^2}\partial_x,$
5b	$f=\text{Const.},\ g=g(x),\ h=a_1t+a_2,$	$V_{(4)}=\frac{a_1t+a_2}{a_1}\partial_t+\frac{\int g(x)\mathrm{d}x}{g(x)}\partial_x$
1	where $g_{,x}(x)\neq 0$ and $a_1\neq 0$	$V_{(5)}=\frac{1}{g(x)}\partial_x$
5c	$f=f(t),\ g=\text{const.},\ h=a_1x+a_2\int f(t)\mathrm{d}t,$	$V_{(4)}=\frac{\int f(t)\mathrm{d}t}{f(t)}\partial_t+x\partial_x,$
2	where $f_{,t}(t)\neq 0,\ a_1\neq a_2,\ a_1\neq 0,\ a_2\neq 0$	$V_{(5)}=\frac{1}{f(t)}\partial_t-\frac{a_2}{a_1}\partial_x$
5d	$f=\text{const.},\ g=g(x),\ h=\frac{(a_1t+2a_2)^{1+\frac{a_3}{a_1}}}{a_1+a_3},$	$V_{(4)}=t\partial_t+\frac{\int g(x)\mathrm{d}x}{g(x)}\partial_x,$
1	where $g_{,x}(x)\neq 0,\ a_1\neq 0,\ a_1\neq -a_3$	$V_{(5)}=\frac{1}{g(x)}\partial_x$
5e	$f=\text{const.},\ g=\text{const.},\ h=a_1x+a_2t,$	$V_{(4)}=t\partial_t+x\partial_x$
7	where $a_1\neq a_2$ and $a_1\neq 0,\ a_2\neq 0.$	$V_{(5)}=-\frac{a_2}{a_1}\partial_t+\partial_x$
5f	$f=\text{const.},\ g=\text{const.},\ h=(a_1t+2a_2)^{1-\frac{2a_3}{a_1}},$	$V_{(4)}=t\partial_t+x\partial_x,$
7	where $a_1\neq 0,\ a_1\neq 2a_3$	$V_{(5)}=\partial_x$
5g	$f=\text{const.},\ g=\text{const.},\ h=a_1t+a_2,$	$V_{(4)}=\frac{a_1t+a_2}{a_1}\partial_t+x\partial_x,$
7	where $a_1\neq 0$	$V_{(5)}=\partial_x$

Table 1: (continued).

Metric No./Branch no.	Metric coefficients	Proper HVF
5h	$f=\text{const.},\ g=a_2t,\ h=(a_1t)^{1-\frac{2a_3}{a_1}}$,	$V_{(4)}=t\partial_t$,
7	where $a_1\neq 0$, $a_2\neq 0$ and $a_1\neq 2a_3$	$V_{(5)}=\partial_x$
5i	$f=\text{const.},\ g=(a_1t+2a_2)^{1-\frac{2a_3}{a_1}},\ h=(a_1t+2a_2)^{1-\frac{2a_4}{a_1}}$,	$V_{(4)}=t\partial_t$,
7	where $a_1\neq 0$, $a_1\neq 2a_3$, $a_1\neq 2a_4$	$V_{(5)}=\partial_x$
5j	$f=f(t),\ g=\text{const.},\ h=a_1x+a_2$,	$V_{(4)}=\frac{\int f(t)\mathrm{d}t}{f(t)}\partial_t+\frac{a_1x+a_2}{2a_1}\partial_x$
9	where $f_{,t}(t)\neq 0$ and $a_1\neq 0$	$V_{(5)}=\frac{1}{f(t)}\partial_t$
5k	$f=f(t),\ g=\text{const.},\ h=(a_1x+2a_2)^{1-\frac{2a_3}{a_1}}$,	$V_{(4)}=\frac{\int f(t)\mathrm{d}t}{f(t)}\partial_t+x\partial_x$
9	where $a_1\neq 2a_3$, $f_{,t}(t)\neq 0$,	$V_{(5)}=\frac{1}{f(t)}\partial_t$
5l	$f=\text{const.},\ g=\text{const.},\ h=(a_1x+2a_2)^{1-\frac{2a_3}{a_1}}$,	$V_{(4)}=t\partial_t+x\partial_x$,
10	where $a_1\neq 0$, $a_1\neq 2a_3$	$V_{(5)}=\partial_t$
5m	$f=(a_1x+2a_2)^{1-\frac{2a_3}{a_1}},\ g=\text{const.},\ h=(a_1x+2a_2)^{1-\frac{2a_4}{a_1}}$,	$V_{(4)}=x\partial_x$,
10	where $a_1\neq 0$, $a_1\neq 2a_3$, $a_1\neq 2a_4$	$V_{(5)}=\partial_t$
5n	$f=\text{const.},\ g=\text{const.},\ h=a_1x+a_2$,	$V_{(4)}=t\partial_t+\frac{a_1x+a_2}{a_1}\partial_x$
10	where $a_1\neq 0$	$V_{(5)}=\partial_t$
5o	$f=(a_1\int g(x)\mathrm{d}x+2a_2)^{1-\frac{2a_3}{a_1}},\ g=g(x),\ h=\text{const.}$	$V_{(4)}=\frac{\int g(x)\mathrm{d}x}{g(x)}\partial_x+y\partial_y+z\partial_z$,
18	where $g_{,x}(x)\neq 0$, $a_1\neq 2a_3$ and $a_1\neq 0$.	$V_{(5)}=\partial_t$
5p	$f=(a_1x+2a_2)^{1-\frac{2a_3}{a_1}},\ g=\text{const.},\ h=\text{const.}$,	$V_{(4)}=x\partial_x+y\partial_y+z\partial_z$,
24	where $a_1\neq 2a_3$ and $a_1\neq 0$.	$V_{(5)}=\partial_t$
7a	$f=\text{Const.},\ g=a_1t+a_2,\ h=\frac{(a_1t+a_2)}{a_2}e^{a_3x}$,	$V_{(4)}=\frac{a_1t+a_2}{a_1}\partial_t$
1	where $a_1\neq 0$, $a_2\neq 0$, $a_3\neq 0$	$V_{(5)}=\partial_x-a_3y\partial_y-a_3z\partial_z$
		$V_{(6)}=\frac{y}{a_3}\partial_t+\left(\frac{z^2-y^2}{2}+\frac{a_2^2}{2a_3^2e^{2a_3x}}\right)\partial_y-yz\partial_z$
		$V_{(7)}=-\frac{z}{a_3}\partial_t+yz\partial_y+\left(\frac{z^2-y^2}{2}-\frac{a_2^2}{2a_3^2e^{2a_3x}}\right)\partial_z$
7b	$f=\text{Const.},\ g=a_1t,\ h=\frac{1}{a_2}t$,	$V_{(4)}=t\partial_t$
7	where $a_1\neq 0$, and $a_2\neq 0$.	$V_{(5)}=\partial_x$
		$V_{(6)}=-\frac{y}{a_1^2a_2^2}\partial_x+x\partial_y$
		$V_{(7)}=-\frac{z}{a_1^2a_2^2}\partial_x+x\partial_z$
7c	$f=\text{const.},\ g=h=(a_1t+2a_2)^{1-\frac{2a_3}{a_1}}$,	$V_{(4)}=t\partial_t$,
7	where $a_1\neq 2a_3$, $a_1\neq 0$	$V_{(5)}=-y\partial_x+x\partial_y$
		$V_{(6)}=-z\partial_x+x\partial_z$,
		$V_{(7)}=\partial_x$
7d	$f=h=(a_1x+2a_2)^{1-\frac{2a_3}{a_1}}$,	$V_{(4)}=x\partial_x$,
10	$g=\text{Const.}$,	$V_{(5)}=y\partial_t+t\partial_y$
	where $a_1\neq 2a_3$ and $a_1\neq 0$	$V_{(6)}=z\partial_t+t\partial_z$,
		$V_{(7)}=\partial_t$
7e	$f=a_1x+a_2,\ g=\text{Const.}$,	$V_{(4)}=\frac{a_1x+a_2}{a_1}\partial_x$
10	$h=a_1x+a_2$,	$V_{(5)}=\partial_t$
	where $a_1\neq 0$.	$V_{(6)}=y\partial_t+t\partial_y$
		$V_{(7)}=z\partial_t+t\partial_z$

Table 1: (continued).

Metric No./Branch no.	Metric coefficients	Proper HVF
11a	$f = \text{Const.}$,	$V_{(4)} = \frac{a_1 t + a_2}{a_1} \partial_t$
	$g = a_1 t + a_2$,	$V_{(5)} = \partial_x - a_1 y \partial_y - a_1 z \partial_z$
1	$h = \frac{a_1 t + a_2}{a_2} e^{a_1 x}$,	$V_{(6)} = e^{a_1 x} \partial_t - \frac{e^{a_1 x}}{a_1 t + a_2} \partial_x$
	where $a_1 \neq 0$	$V_{(7)} = \left[\frac{a_1^2}{a_2^2} (y^2 + z^2) e^{a_1 x} + e^{-a_1 x} \right] \partial_t + \left[- \frac{a_1^2}{a_2^2 (a_1 t + a_2)} (y^2 + z^2) e^{a_1 x} + \frac{e^{-a_1 x}}{a_1 t + a_2} \right] \partial_x$
	and $a_2 \neq 0$	$- \frac{2 a_1 y e^{-a_1 x}}{a_1 t + a_2} \partial_y - \frac{2 a_1 z e^{-a_1 x}}{a_1 t + a_2} \partial_z$,
		$V_{(8)} = y e^{a_1 x} \partial_t - \frac{y e^{a_1 x}}{(a_1 t + a_2)} \partial_x - \frac{a_2^2}{a_1 (a_1 t + a_2)} e^{-a_1 x} \partial_y$,
		$V_{(9)} = z e^{a_1 x} \partial_t - \frac{z e^{a_1 x}}{(a_1 t + a_2)} \partial_x - \frac{a_2^2}{a_1 (a_1 t + a_2)} e^{-a_1 x} \partial_z$,
		$V_{(10)} = \frac{y}{a_1} \partial_x + \left(\frac{z^2 - y^2}{2} + \frac{a_2^2}{2 a_1^2} e^{-2 a_1 x} \right) \partial_y - yz \partial_z$
		$V_{(11)} = - \frac{z}{a_1} \partial_x + yz \partial_y + \left(\frac{z^2 - y^2}{2} - \frac{a_2^2}{2 a_1^2} e^{-2 a_1 x} \right) \partial_z$
11b	$f = f(t)$, $g = \text{Const.}$,	$V_{(4)} = \frac{\int f(t) \mathrm{d}t}{f(t)} \partial_t + x \partial_x$
	$h = a_1 x + a_1 \int f(t) \mathrm{d}t$,	$V_{(5)} = - \frac{x}{f(t)} \partial_t - \int f(t) \mathrm{d}t \partial_x + y \partial_y + z \partial_z$
2	where $f_{,t}(t) \neq 0$	$V_{(6)} = \left[\frac{a_1^2 (y^2 + z^2)}{2 f(t)} + \frac{1}{f(t)} \right] \partial_t - a_1^2 \left(\frac{y^2 + z^2}{2} \right) \partial_x - \frac{a_1 y}{h(t,x)} \partial_y - \frac{a_1 z}{h(t,x)} \partial_z$
	and $a_1 \neq 0$	$V_{(7)} = \left(\frac{y^2 + z^2}{2} \right) \frac{a_1^2}{f(t)} \partial_t + \left[\frac{-a_1^2 (y^2 + z^2)}{2} + 1 \right] \partial_x - \frac{a_1 y}{h(t,x)} \partial_y - \frac{a_1 z}{h(t,x)} \partial_z$
		$V_{(8)} = \frac{y}{f(t)} \left[- \int f(t) \mathrm{d}t + \frac{h(t,x)}{a_1} \right] \partial_t + y \int f(t) \mathrm{d}t \partial_x + \left[\frac{z^2 - y^2}{2} + \frac{\int f(t) \mathrm{d}t}{a_1 h(t,x)} \right] \partial_y - yz \partial_z$
		$V_{(9)} = \frac{z}{f(t)} \left[\int f(t) \mathrm{d}t - \frac{h(t,x)}{a_1} \right] \partial_t + z \int f(t) \mathrm{d}t \partial_x + yz \partial_y + \left[\frac{z^2 - y^2}{2} - \frac{\int f(t) \mathrm{d}t}{a_1 h(t,x)} \right] \partial_z$
		$V_{(10)} = \frac{y}{f(t)} \partial_t - y \partial_x - \frac{1}{a_1 h(t,x)} \partial_y$
		$V_{(11)} = \frac{z}{f(t)} \partial_t - z \partial_x - \frac{1}{a_1 h(t,x)} \partial_z$
11c	$f = \text{Const.}$,	$V_{(4)} = \frac{a_1 t + a_2}{a_1} \partial_t + y \partial_y + z \partial_z$
16	$g = a_1 t + a_2$,	$V_{(5)} = y e^{a_1 x} \partial_t - \frac{y e^{a_1 x}}{a_1 t + a_2} \partial_x + \frac{a_1 t + a_2}{a_1} e^{a_1 x} \partial_y$
	$h = \text{Const.}$,	$V_{(6)} = y e^{-a_1 x} \partial_t + \frac{y e^{-a_1 x}}{a_1 t + a_2} \partial_x + \frac{a_1 t + a_2}{a_1} e^{-a_1 x} \partial_y$
	where $a_1 \neq 0$	$V_{(7)} = z e^{a_1 x} \partial_t - \frac{z e^{a_1 x}}{a_1 t + a_2} \partial_x + \frac{a_1 t + a_2}{a_1} e^{a_1 x} \partial_z$
		$V_{(8)} = z e^{-a_1 x} \partial_t + \frac{z e^{-a_1 x}}{a_1 t + a_2} \partial_x + \frac{a_1 t + a_2}{a_1} e^{-a_1 x} \partial_z$
		$V_{(9)} = e^{a_1 x} \partial_t - \frac{e^{a_1 x}}{a_1 t + a_2} \partial_x$
		$V_{(10)} = e^{-a_1 x} \partial_t + \frac{e^{-a_1 x}}{a_1 t + a_2} \partial_x$
		$V_{(11)} = \partial_x$
11d	$f = f(t)$, $g = g(x)$,	$V_{(4)} = \frac{\int f(t) \mathrm{d}t}{f(t)} \partial_t + \int g(x) \mathrm{d}x \, \partial_x + y \partial_y + z \partial_z$
17	$h = \text{const.}$,	$V_{(5)} = \frac{y}{f(t)} \partial_t + \int f(t) \mathrm{d}t \, \partial_y$,
	where $f_{,t}(t) \neq 0$	$V_{(6)} = \frac{z}{f(t)} \partial_t + \int f(t) \mathrm{d}t \, \partial_z$,
	and $g_{,x}(x) \neq 0$	$V_{(7)} = \frac{\int g(x) \mathrm{d}x}{f(t)} \partial_t + \frac{\int f(t) \mathrm{d}t}{g(x)} \partial_x$,
		$V_{(8)} = - \frac{y}{g(x)} \partial_x + \int g(x) \mathrm{d}x \, \partial_y$,
		$V_{(9)} = - \frac{z}{g(x)} \partial_x + \int g(x) \mathrm{d}x \, \partial_z$,
		$V_{(10)} = \frac{1}{f(t)} \partial_t$
		$V_{(11)} = \frac{1}{g(x)} \partial_x$
11e	$f = a_1 \int g(x) \mathrm{d}x$, $g = g(x)$, $h = \text{Const.}$,	$V_{(4)} = \frac{\int g(x) \mathrm{d}x}{g(x)} \partial_x + y \partial_y + z \partial_z$

Table 1: (continued).

Metric No./Branch no.	Metric coefficients	Proper HVF
18	where $a_1 \neq 0$ and $g_{,x}(x) \neq 0$	$V_{(5)} = \frac{a_1 y e^{a_1 t}}{f(x)} \partial_t - \frac{a_1 y e^{a_1 t}}{g(x)} \partial_x + f(x) e^{a_1 t} \partial_y,$
		$V_{(6)} = -\frac{a_1 y e^{-a_1 t}}{f(x)} \partial_t - \frac{a_1 y e^{-a_1 t}}{g(x)} \partial_x + f(x) e^{-a_1 t} \partial_y,$
		$V_{(7)} = \frac{a_1 z e^{a_1 t}}{f(x)} \partial_t - \frac{a_1 z e^{a_1 t}}{g(x)} \partial_x + f(x) e^{a_1 t} \partial_z,$
		$V_{(8)} = -\frac{a_1 z e^{-a_1 t}}{f(x)} \partial_t - \frac{a_1 z e^{-a_1 t}}{g(x)} \partial_x + f(x) e^{-a_1 t} \partial_z,$
		$V_{(9)} = -\frac{e^{a_1 t}}{\int g(x) \mathrm{d}x} \partial_t + \frac{a_1 e^{a_1 t}}{g(x)} \partial_x,$
		$V_{(10)} = -\frac{e^{-a_1 t}}{\int g(x) \mathrm{d}x} \partial_t - \frac{a_1 e^{-a_1 t}}{g(x)} \partial_x,$
		$V_{(11)} = \partial_t$
11f	$f = \text{Const.}$, $g = g(x)$, $h = \text{Const.}$,	$V_{(4)} = t\partial_t + \frac{\int g(x)\mathrm{d}x}{g} \partial_x + y\partial_y + z\partial_z$
18	where $g_{,x}(x) \neq 0$	$V_{(5)} = y\partial_t + t\partial_y,$
		$V_{(6)} = z\partial_t + t\partial_z,$
		$V_{(7)} = \int g(x)\mathrm{d}x \partial_t + \frac{t}{g(x)} \partial_x,$
		$V_{(8)} = -\frac{y}{g(x)} \partial_x + \int g(x)\mathrm{d}x \, \partial_y,$
		$V_{(9)} = -\frac{z}{g(x)} \partial_x + \int g(x)\mathrm{d}x \, \partial_z,$
		$V_{(10)} = \partial_t$
		$V_{(11)} = \frac{1}{g(x)} \partial_x$
11g	$f = f(t)$, $g = \text{Const.}$ $h = \text{Const.}$,	$V_{(4)} = \frac{\int f(t)\mathrm{d}t}{f(t)} \partial_t + x\partial_x + y\partial_y + z\partial_z$
23	where $f_{,t}(t) \neq 0$	$V_{(5)} = \frac{y}{f(t)} \partial_t + \int f(t)\mathrm{d}t \, \partial_y,$
		$V_{(6)} = \frac{z}{f(t)} \partial_t + \int f(t)\mathrm{d}t \, \partial_z,$
		$V_{(7)} = \frac{x}{f(t)} \partial_t + \int f(t)\mathrm{d}t \, \partial_x,$
		$V_{(8)} = -y\partial_x + x\partial_y,$
		$V_{(9)} = -z\partial_x + x\partial_z,$
		$V_{(10)} = \frac{1}{f(t)} \partial_t$
		$V_{(11)} = \partial_x$
11h	$f = a_1 x + a_2$, $g = \text{Const.}$,	$V_{(4)} = \frac{a_1 x + a_2}{a_1} \partial_x + y\partial_y + z\partial_z$
24	$h = \text{Const.}$,	$V_{(5)} = \frac{y e^{a_1 t}}{a_1 x + a_2} \partial_t - y e^{a_1 t} \partial_x + \frac{a_1 x + a_2}{a_1} e^{a_1 t} \partial_y$
	where $a_1 \neq 0$	$V_{(6)} = -\frac{y e^{-a_1 t}}{a_1 x + a_2} \partial_t - y e^{-a_1 t} \partial_x + \frac{a_1 x + a_2}{a_1} e^{-a_1 t} \partial_y$
		$V_{(7)} = \frac{z e^{a_1 t}}{a_1 x + a_2} \partial_t - z e^{a_1 t} \partial_x + \frac{a_1 x + a_2}{a_1} e^{a_1 t} \partial_z$
		$V_{(8)} = -\frac{z e^{-a_1 t}}{a_1 x + a_2} \partial_t - z e^{-a_1 t} \partial_x + \frac{a_1 x + a_2}{a_1} e^{-a_1 t} \partial_z$
		$V_{(9)} = -\frac{e^{a_1 t}}{a_1 x + a_2} \partial_t + e^{a_1 t} \partial_x$
		$V_{(10)} = \frac{e^{-a_1 t}}{a_1 x + a_2} \partial_t + e^{-a_1 t} \partial_x$
		$V_{(11)} = \partial_t$

physically realistic solutions of EFEs. The metric (1.5) has four diagonal and one off-diagonal non-vanishing components of T_{ab}, given by:

$$T_{00} = -\frac{1}{g^3 h^2} \big[2 g f^2 h h_{,xx} - g^3 (h_{,t})^2 - 2 g^2 h g_{,t} h_{,t} + f^2 g (h_{,x})^2 - 2 f^2 g_{,x} h h_{,x} \big],$$

$$T_{01} = -\frac{1}{fgh} \big[2 f g h_{,tx} - f_{,x} g h_{,t} - f g_{,t} h_{,x} \big],$$

$$T_{11} = -\frac{1}{f^3 h^2} \big[2 f g^2 h h_{,tt} + f g^2 (h_{,t})^2 - 2 f_{,t} g^2 h h_{,t} - f^3 (h_{,x})^2 - 2 f^2 f_{,x} h h_{,x} \big],$$

$$T_{22} = T_{33} = \frac{h}{f^3 g^3} \big[f^3 g h_{,xx} - f g^3 h_{,tt} - f g^2 h g_{,tt} + f^2 f_{,xx} g h$$

$$+ f_{,t}g^3h_{,t} - fg^2g_{,t}h_{,t} + f_{,t}g^2g_{,t}h + f^2f_{,x}gh_{,x}$$
$$- f^3g_{,x}h_{,x} - f^2f_{,x}g_{,x}h\big]. \tag{3.1}$$

The energy-momentum tensor T_{ab} has different structures for some known matter sources. For instance, if it is assumed that the matter source for the metric (1.5) is an anisotropic fluid, then the components of T_{ab} are found to be $T_{00} = \rho f^2(t,x)$, $T_{11} = p_{\|} g^2(t,x)$, $T_{22} = T_{33} = p_{\perp} h^2(t,x)$, and $T_{01} = 0$, where ρ is the density and $p_{\|}$ and $p_{\perp}$ are pressures in two directions. A perfect fluid is obtained when $p_{\|} = p_{\perp} = p$. Hence, among the classified metrics, those for which $T_{01} = 0$ suggest anisotropic or perfect fluids. For all such metrics, ρ, $p_{\|}$, and $p_{\perp}$ can be calculated as:

$$\rho = \frac{T_{00}}{f^2(t,x)}, \;\; p_{\|} = \frac{T_{11}}{g^2(t,x)}, \;\; p_{\perp} = \frac{T_{22}}{h^2(t,x)}. \tag{3.2}$$

Out of the metrics obtained during the current study, all the metrics are anisotropic or perfect fluids with $T_{01} = 0$, except the metrics $4c - 4f$ and $4i - 4l$. However, in case of the metrics $4c - 4f$, the condition $T_{00} > 0$ is satisfied, ensuring that these metrics are physically realistic. For all other metrics, one can find the physical terms ρ, $p_{\|}$, and $p_{\perp}$ using Eq. (3.2), which can be subsequently used to check different energy conditions such as strong (SEC), weak (WEC), null (NEC) and dominant (DEC) energy conditions.

All the components of T_{ab} vanish for the metrics possessing 11-dimensional homothetic algebra, labeled by $11a$ – $11h$. Consequently, the terms ρ, $p_{\|}$ and $p_{\perp}$ also vanish for all these metrics and hence all the energy conditions are identically satisfied. Each of these metrics represents a vacuum solution of the EFEs. The metrics $5b$ and $5g$ give anisotropic solutions with $\rho = \frac{a_1^2}{(a_1 t + a_2)^2}$, $p_{\|} = -\rho$ and $p_{\perp} = 0$, that satisfy all energy conditions. The metric $7b$ represents a perfect fluid solution with energy density $\rho = \frac{3}{t^2}$ and pressure $p_{\|} = p_{\perp} = -\frac{1}{t^2}$. Like the previous two metrics, this metric also satisfies all the energy conditions.

Some of the derived metrics satisfy certain energy conditions if we further restrict the parameters in the metric coefficients. For example, the metrics $5c$ and $5e$ are anisotropic solutions with $\rho = -p_{\|} = \frac{a_2^2 - a_1^2}{h^2(t,x)}$ and $p_{\perp} = 0$. These models satisfy all energy conditions provided that $a_2^2 - a_1^2 \geq 0$. Similarly, the metric $7b$ gives a perfect fluid solution with $\rho = \frac{3(a_1^2 - a_3^2)}{(a_1 t + a_2)^2}$ and $p_{\|} = p_{\perp} = -\frac{\rho}{3}$, satisfying all energy conditions if $a_1^2 - a_3^2 > 0$. For metrics $5o$ and $5p$, both ρ and $p_{\|}$ vanish, while $p_{\perp}$ is found to be $p_{\perp} = \frac{2a_3(2a_3 - a_1)}{[a_1 \int g(x) \mathrm{d}x + 2a_2]^2}$ and $p_{\perp} = \frac{2a_3(2a_3 - a_1)}{[a_1 x + 2a_2]^2}$, respectively. If $a_3(2a_3 - a_1) > 0$, all energy conditions are satisfied for these two anisotropic fluid solutions. In Table 2, we give the details of energy conditions for some other metrics.

4 Physical interpretation and connection to known models

The classified metrics of the current study include many physically and geometrically important solutions of the field equations. For example, the metric labeled by $4a$ in Table 1 possesses a separable dependence on t and x in the transverse scale factor $h(t,x)$. This metric resembles with the generalized Bianchi type I spacetimes with spatial inhomogeneity and anisotropic expansion. Similarly, the metric $4c$ admitting a 4-dimensional homothetic algebra exhibits generalized Kasner-like behavior in a non-comoving frame, similar to self-similar anisotropic cosmologies. The metric $4d$, exhibiting an exponential evolution in the transverse directions is consistent with models that describe anisotropic inflation. With a square-root spatial character and a power-law time dependence, the metric $4f$ suggests connections to radiative or wave-like solutions. The moetric $4h$, that involves a logarithmic coupling of space and time is structurally comparable to spacetimes occurring in the scalar field dynamics and colliding plane wave models.

Out of the metrics possessing five homothetic symmetries, the first metric labeled by $5a$, includes a linear time scaling and the exponential spatial modulation, that are the features of the inhomogeneous generalized Szekeres-type models. The metric $5e$ has a linear mixing of time and space in the transverse direction and it captures directional anisotropy and evolving shear. The metric $5f$ involves power-law time dependent scale factor and it resembles the simplified anisotropic Bianchi type models. The metric $5g$ has spatial uniformity and linear time evolution in one direction and it corresponds to the plane symmetric analogues of locally rotationally symmetric Bianchi I spacetimes.

Among the metrics with seven homothetic symmetries, the metric labeled by $7a$ involves an exponential spatial dependence and its homothetic generators involve the transverse coordinates. It resembles generalized Szekeres-type geometries. The metric $7b$ exhibits uniform expansion in all directions, reflecting Milne-like or stiff fluid cosmologies. The metric $7c$ exhibits the same time dependency in both the spatial and transverse parts, it is therefore consistent with Kasner-type anisotropic models under self-similarity. The temporal and transverse scale factors of the metric $7d$ are spatially dependent, making it consistent with colliding plane wave geometries. Finally, the metric $7e$ is entirely a static metric with linear spatial inhomogeneity and it offers a symmetric vacuum configuration suitable for modeling non-evolving systems.

Table 2: Energy conditions.

Metric no.	Physical terms	Energy conditions
5d	$\rho = \frac{(a_1+a_3)^2}{(a_1t+a_2)^2}$ $p_\parallel = -\frac{(a_1+a_3)(a_1+3a_3)}{(a_1t+a_2)^2}$ $p_\perp = -\frac{a_3(a_1+a_3)}{(a_1t+a_2)^2}$	WEC, NEC and SEC are satisfied if $a_3(a_1+a_3) < 0$ and $a_1(a_1+a_3) > 0$ DEC is satisfied if $a_3(a_1+a_3) < 0$, $a_1(a_1+a_3) > 0$ and $(a_1+a_3)(a_1+2a_3) > 0$
5f	$\rho = \frac{(a_1-2a_3)^2}{(a_1t+2a_2)^2}$ $p_\parallel = -\frac{(a_1-2a_3)(a_1-6a_3)}{(a_1t+2a_2)^2}$ $p_\perp = \frac{2a_3(a_1-2a_3)}{(a_1t+2a_2)^2}$	WEC, NEC and SEC are satisfied if $a_3(a_1-2a_3) > 0$ and $a_1(a_1-2a_3) > 0$ DEC is satisfied if $a_3(a_1-2a_3) > 0$, $a_1(a_1-2a_3) > 0$ $(a_1-2a_3)(a_1-4a_3) > 0$
5k, 5l	$\rho = -\frac{(a_1-2a_3)(a_1-6a_3)}{(a_1x+2a_2)^2}$ $p_\parallel = \frac{(a_1-2a_3)^2}{(a_1x+2a_2)^2}$ $p_\perp = -\frac{2a_3(a_1-2a_3)}{(a_1x+2a_2)^2}$	WEC is satisfied if $(a_1-2a_3)(a_1-6a_3) > 0$, $a_3(a_1-2a_3) > 0$ $(a_1-2a_3)(4a_3-a_1) > 0$ NEC and SEC are satisfied if $a_3(a_1-2a_3) > 0$ and $(a_1-2a_3)(4a_3-a_1) > 0$ DEC is satisfied if $a_3(a_1-2a_3) > 0$ $(a_1-2a_3)(4a_3-a_1) > 0$ and $(a_1-2a_3)(8a_3-a_1) > 0$
5j, 5n	$\rho = -\frac{a_1^2}{(a_1x+a_2)^2}$ $p_\parallel = -\rho$ $p_\perp = 0$	An un-physical model with $\rho < 0$ and satisfying none of the energy conditions
7c	$\rho = \frac{3(a_1-2a_3)^2}{(a_1t+2a_2)^2}$ $p_\parallel = p_\perp = -\frac{(a_1-2a_3)(a_1-6a_3)}{(a_1t+2a_2)^2}$	WEC and NEC satisfied if $a_1(a_1-2a_3) > 0$ SEC is satisfied if$a_1(a_1-2a_3) > 0$ and $a_3(a_1-2a_3) > 0$ DEC is satisfied if$a_1(a_1-2a_3) > 0$ and $(a_1-3a_3)(a_1-2a_3) > 0$
7e	$\rho = -\frac{a_1^2}{(a_1x+a_2)^2}$ $p_\parallel = \frac{3a_1^2}{(a_1x+a_2)^2}$ $p_\perp = -\rho$	WEC and DEC are failed, while NEC and SEC are satisfied

Out of the maximally symmetric metrics possessing 11 HVFs, the metric 11a involves exponential spatial dependence and its energy-momentum tensor vanishes. This metric resembles Robinson-Trautman type or plane wave solutions. The metric 11b combines spatially changing transverse scales with integrated time behavior and it is related to scalar field or wave-like cosmologies under self-similarity. The exponential symmetry generators acting on transverse directions for the metric 11c shows that this metric is suitable to model shear-free, plane symmetric vacuum fields. The metric 11g is homogeneous and purely dependent on time, it is consistent with the Milne limit of flat FLRW cosmologies. Lastly, the metric 11h is a spatially homogeneous and static metric, representing a symmetric configuration that is relevant for static interior solutions.

To provide a clearer overview, we provide schematic flow charts in Figures 2 and 3 to illustrate how our derived metrics correspond to vacuum, and perfect and anisotropic fluid configurations and how these metrics connect to some well known models in the literature.

Collectively, our derived results show that the obtained homothetic symmetries enrich the collection of known symmetries for plane symmetric non-static spacetimes and yield new possibilities for finding the exact solutions of EFEs. The derived models may also have applications in many current research areas in relativistic cosmology and gravitation. In

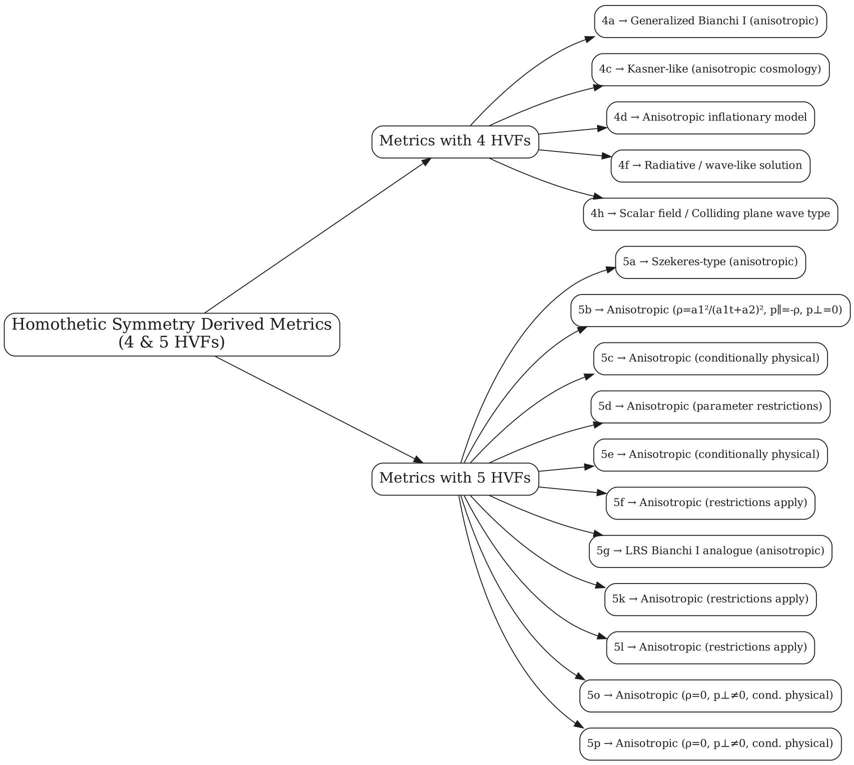

Figure 2: Schematic flow chart for metrics with 4 and 5 HVFs.

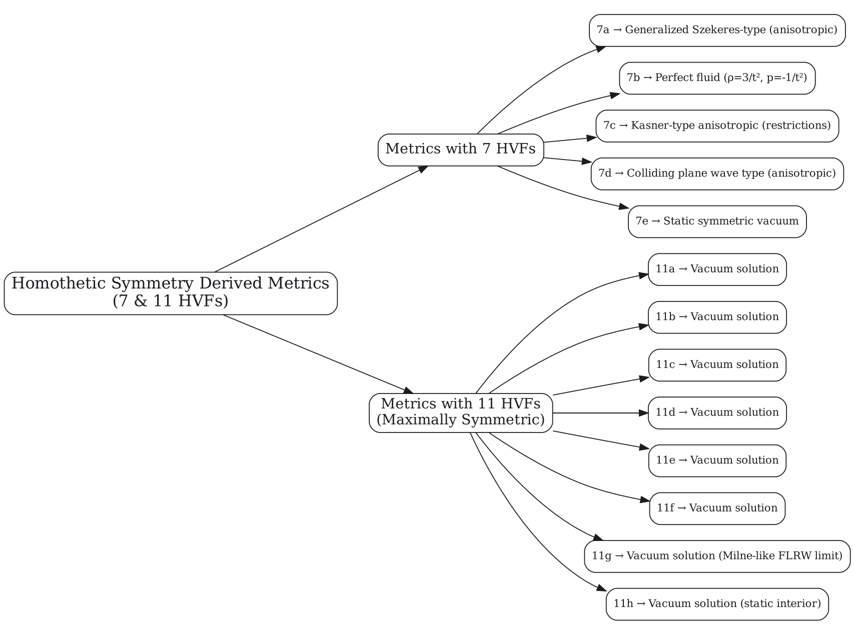

Figure 3: Schematic flow chart for metrics with 7 and 11 HVFs.

anisotropic cosmological models, the spacetimes with planar symmetry provide important geometries for studying anisotropy and inhomogeneity, that play a pivotal role in Bianchi-type universes and in explaining possible deviations from isotropy in the cosmic microwave background. These solutions may further serve as toy models for the early universe, where anisotropic effects could have influenced pre-inflationary or inflationary dynamics. Moreover, non-static plane symmetric models are relevant in the study of gravitational collapse and black hole interiors, where anisotropic pressures and imperfect fluids naturally arise. In this way, the homothetic symmetries obtained in the present study may contribute to understanding self-similar structures that appear both in cosmology and in compact object models.

Anisotropic solutions of EFEs are of significant astrophysical importance. In compact stars, anisotropy can appear due to very high densities, phase transitions, superfluid states, or the presence of electromagnetic fields. Such anisotropic pressures influence the stability, mass-radius relation, and surface redshift of stellar configurations, making them more realistic than isotropic models. Beyond stellar interiors, anisotropic fluids are also relevant in exotic matter scenarios, including dark energy models and non-standard cosmological fluids. Therefore, the anisotropic plane symmetric solutions obtained in this work may provide useful toy models to explore situations where local anisotropy plays a fundamental role in the gravitational dynamics.

5 Conclusions

In this paper, we have explored how symmetries in spacetimes, particularly HVFs, can help us to solve the Einstein's complex field equations more easily. These symmetries act like scaling rules that preserve the structure of spacetime while allowing it to evolve, making them a powerful tool for studying dynamic gravitational systems.

Using an advanced computational method, the Rif tree approach, we have classified a variety of exact solutions for non-static plane-symmetric spacetimes with 4-, 5-, 7-, and 11-dimensional homothetic algebras. Some of the obtained solutions describe universes filled with matter or radiation, while others represent vacuum regions. In addition, we have explored the physical and geometrical significance of many of the derived metrics by comparing their structure with some known solutions of EFEs. Several metrics were identified as generalized versions of Kasner, Szekeres, Bianchi type I, Milne-type, and Robinson-Trautman spacetimes. These results contribute new exact solutions and deepen the understanding of homothetic symmetry in modeling anisotropic, inhomogeneous, and self-similar cosmological geometries.

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