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Research Article

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A widespread study of discrete entropic model and its distribution along with fluctuations of energy

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Abstract: The growing use of entropy in statistical mechanics has been paralleled by theoretical advancements. The energy transfer inside thermal structures occurs due to many interacting molecules, providing ample locations for energy allocation. As a result of molecular interaction, energy is dispersed universally. Aggregating entropy signifies that energy tends to disperse and accumulate more uniformly. Furthermore, the accumulation of entropy signifies a loss of knowledge on the location of energy. In the gathered papers, there are remarkable inquiries into information entropy across many branches of mathematics. In this communication, we have developed an innovative entropic model for certain probability distributions and provided transactions with essential and expected criteria for their validity. Furthermore, we have conducted research into energy fluctuations and the relative variability of energy within the canonical ensemble for the proposed entropic model by using the optimum Lagrange multiplier (OLM) approach. The manifestations of energy fluctuations for parametric entropy correspond closely to the prevalent configurations of Boltzmann-Gibbs statistics.

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1 Introduction

Entropy, originally introduced in thermodynamics as a measure of disorder and energy unavailability, has evolved into a fundamental concept with applications far beyond its classical roots. In statistical mechanics, entropy characterizes the number of microstates compatible with a macroscopic system, while in information theory, Shannon's seminal work [1] linked entropy to the measure of uncertainty in communication systems. Since then, entropy has been widely adopted in diverse scientific fields, including physics, mathematics, quantum information, and complex systems analysis.

Classical entropy measures – notably those by Shannon, Rényi, and Tsallis – have greatly enhanced our understanding of uncertainty and statistical distributions. Shannon [1] defined a probabilistic structure for entropy based on simple axiomatic assumptions, which became foundational in information theory. Rényi [2] generalized Shannon's formulation with a parameter that modulates sensitivity to the tail behaviour of distributions. Tsallis [3] further extended entropy into the non-extensive regime, suitable for systems exhibiting long-range interactions or correlations. These pioneering works opened pathways for a vast array of generalized entropic models.

Building on these ideas, numerous researchers have proposed further parametric generalizations:

- Havrda and Charvat [4] developed an alternative entropy formulation broadening statistical applications.
- Kapur [5–7] explored multi-parametric frameworks to model complex distributions in information systems.
- Deng entropy, introduced by Gao and Deng [8], highlighted structural applications in data analysis.

- Zhou and Zheng [9] demonstrated strong connections between observational entropy and Rényi's model in quantum statistical mechanics.
- Stoyanov et al. [10] analysed the maximum entropy criterion for the moment indeterminacy of probability densities, advancing theoretical understanding.
- Elgawad et al. [11] studied information measures for generalized order statistics, providing deeper insight into entropy under bivariate distributions.
- Rastegin [12] examined entropy inequalities and uncertainty relations for structured probability measures.

Recent research continues to emphasize entropy's importance not only in thermodynamics but also in quantum information theory, complex networks, and signal processing. For example, Cincotta and Giordano [13] analysed phase correlations in chaotic dynamical systems, while Jamaati and Mehri [14] applied Tsallis entropy to natural language processing tasks, highlighting its versatility beyond physics. Moreover, Wang et al. [15] proposed entropy-based uncertainty measures for rough set theory, and Sholehkerdar et al. [16] applied entropy metrics to objective image fusion. These studies collectively demonstrate the adaptability of entropy across multiple scientific disciplines.

Despite this extensive body of work, most existing entropy models remain limited by single-parameter formulations or restrictive assumptions on additivity and concavity. Such constraints reduce their flexibility in modelling systems with diverse probabilistic structures, correlations, or constraints. To address these gaps, researchers have increasingly turned to multi-parametric entropies that provide finer control over statistical sensitivity and deformation properties.

1.1 Motivation and contribution of this work

In response to these challenges, the present study introduces a two-parameter generalized entropy model specifically designed for discrete probability distributions. Unlike single-parameter models (e.g., Tsallis or Rényi), the proposed formulation allows independent tuning of two aspects:

- One parameter controls the sensitivity to probability magnitudes (nonlinearity),
- The other parameter modulates the scaling and deformation properties of the entropy function.

This dual-parametric structure enhances modelling flexibility and extends the theoretical scope of entropy analysis. Furthermore, the proposed entropy is applied to investigate energy fluctuations in canonical ensembles, offering new insights into statistical mechanics beyond the traditional Boltzmann–Gibbs framework.

The main objectives of this paper are:

(1) To define and analyse the properties of the proposed two-parameter entropy, showing its reduction to Shannon, Rényi, and Tsallis entropies as special cases.
(2) To derive probability distributions using the Optimal Lagrange Multiplier (OLM) method and examine how the dual parameters influence these distributions.
(3) To explore the energy fluctuation behaviour in canonical ensembles under the generalized entropy model and compare it with classical fluctuation results.

Structure of the paper

- Section 2 introduces the proposed entropy model and its core mathematical properties.
- Section 3 discusses the derived probability distributions and their implications.
- Section 4 presents the energy fluctuation results and their physical interpretation.
- Section 5 concludes with the key findings, originality, limitations, and possible applications of the proposed model in thermodynamics, statistical physics, and information theory.

Through this approach, we aim to extend the framework of entropy-based analysis by providing a flexible two-parameter formulation that bridges classical and modern entropy models while offering potential applications in both theoretical and applied domains.

2 An innovative comprehensive two-parametric entropy model

In this section, we introduce a new two-parameter entropy model for discrete probability distributions and discuss its mathematical structure, special cases, and essential properties.

2.1 Definition of the model

Let $P = \{p_1, p_2, \ldots, p_n\}$ with $\sum_{i=1}^{n} p_i = 1$ be a discrete probability distribution.

We define the two-parameter entropy as

$$H_{\alpha,\beta}(P) = \frac{1}{\beta-\alpha}\left(\sum_{i=1}^{n} p_i^{\alpha-\beta+1} - 1\right), \alpha \neq \beta, \beta < \alpha + 1, \quad -\infty < \alpha < \infty \tag{2.1}$$

Here,
- α and β are real-valued parameters with $\alpha \neq \beta$
- the constraint $\beta < \alpha + 1$ ensures valid scaling,
- the term $\alpha - \beta + 1$ determines how nonlinearity and deformation are applied to the probabilities.

This formulation decouples the two effects:
- α controls nonlinear sensitivity to probability magnitudes,
- β modulates scaling and deformation.

Thus, $H_{\alpha,\beta}(P)$ provides two degrees of freedom, unlike classical single-parameter entropies.

2.2 Special cases

This generalized entropy recovers well-known models under limiting conditions:
- Shannon entropy: Taking the simultaneous limit $\alpha \to 1$, $\beta \to 1$ yields

$$\lim_{\alpha,\beta\to 1} H_{\alpha,\beta}(P) = -\sum_{i=1}^{n} p_i \ln p_i$$

- Tsallis entropy: When $\beta = 1$ Eq. (2.1) simplifies to

$$H_{\alpha,1}(P) = \frac{\sum_{i=1}^{n} p_i^{\alpha} - 1}{1-\alpha}$$

  which matches the Tsallis form.
- Rényi-like forms:
  Under specific parameter relationships ($\beta = \alpha + q - 1$), the entropy closely relates to Rényi's entropy.

Hence, the proposed formulation serves as a unified two-parameter framework, naturally reducing to Shannon, Tsallis, or Rényi entropies.

This dual-parametric flexibility allows the model to address scenarios where standard single-parameter entropies face limitations. For example, in complex network systems with both high-degree hubs and sparse nodes, α can control the sensitivity to rare events, while β modulates the overall scaling deformation of the distribution. Similarly, in anomalous diffusion processes or long-range correlated systems, Tsallis entropy captures non-extensively but lacks the ability to simultaneously tune scaling properties – this gap is bridged by the dual-parameter formulation. In discrete probability settings, such as socio-economic distributions or natural language processing tasks, the two-parameter model offers finer adaptability to empirical heavy-tailed data compared to the single-parameter Rényi and Tsallis forms.

2.3 Fundamental properties

The entropy $H_{\alpha,\beta}(P)$ satisfies essential axioms:
1. Permutation symmetry: invariant under reordering of probabilities.
2. Continuity: smoothly varies for all p_i in $[0, 1]$
3. Non-negativity: $H_{\alpha,\beta}(P) \geq 0$ for all valid distributions.
4. Null event invariance: adding a zero-probability outcome does not affect entropy.
5. Degenerate case: for deterministic $P = (1, 0, \ldots, 0)$, $H_{\alpha,\beta}(P) = 0$
6. Concavity: achieves a maximum for the uniform distribution.
7. Maximum value increases with n – For equal probabilities $1/n$, the entropy grows with the number of states.
8. Generalized non-additivity: for independent subsystems A and B,

$$H_{\alpha,\beta}(AUB) \neq H_{\alpha,\beta}(A) + H_{\alpha,\beta}(B)$$

   except for special parameter cases where additivity holds.

These properties confirm that $H_{\alpha,\beta}(P)$ is a valid generalized entropy measure.

2.4 Numerical illustration

To illustrate the behaviour of this entropy, we consider a simple binomial distribution

$$P = \{0.25, 0.50, 0.25\}.$$

Using Eq. (2.1), we compute the entropy for different (α, β) pairs:

Parameters (α, β)	Case type	Entropy value
(1.0, 1.0)	Shannon entropy	1.0397
(1.2, 1.0)	Tsallis-like deformation	0.9350
(0.8, 1.2)	Deformed scaling	0.9500
(1.5, 1.5)	Strongly nonlinear case	0.8702

2.4.1 Discussion of results

- Shannon baseline: For $(\alpha, \beta) = (1, 1)$, we recover the standard Shannon value 1.0397.
- Effect of α:

Increasing $\alpha > 1$ (keeping $\beta = 1$) reduces entropy, emphasizing dominant outcomes while down-weighting rare events.

– Effect of β:

Adjusting $\beta > 1$ (with $\alpha < 1$) slightly increases entropy, showing how β rescales probability deformation.

– Strong deformation:

When both α, $\beta > 1$, entropy decreases further (0.8702), highlighting nonlinear concentration on high-probability events.

This example shows how α and β work independently yet complementarily, allowing interpolation between Shannon, Tsallis, Rényi-like behaviours or entirely new intermediate forms.

2.5 Link to applications

This illustration highlights the computability and adaptability of the proposed entropy.

- In statistical mechanics, such flexibility is vital for modelling non-extensive canonical ensembles, where classical Boltzmann–Gibbs entropy fails.
- The dual-parameter control makes it suitable for studying energy fluctuations and nonlinear scaling behaviours, as discussed in Section 4.
- Beyond physics, the model could be extended to information theory, machine learning, and complex networks, where distributions deviate from simple power laws.

Thus, even a simple example confirms the generality and broader applicability of $H_{\alpha,\beta}(P)$.

We defined a new two-parameter entropy model $H_{\alpha,\beta}(P)$ demonstrated that it satisfies key entropy properties, reduces to well-known forms as special cases, and showed its flexibility through a clear numerical example.

3 Introduction of the distribution function in generalized entropy

This section aims to explore the statistical structure and implications of the proposed entropy model through the derivation of associated probability distributions and key fluctuation metrics using the Optimal Lagrange Multiplier method. These results extend classical statistical mechanics formulations by incorporating dual-parametric entropy effects.

In this subdivision, we familiarize ourselves with some of the universal formulas for the probability distribution function of the parametric entropy (2.1). Maximizing (2.1) is subject to the standardization situation described by

$$\sum_{i=1}^{W} p_i - 1 = 0 \tag{3.1}$$

and the energy constraint specified by

$$\frac{\sum\limits_{i=1}^{W} p_i^{\alpha-\beta+1} E_i}{\sum\limits_{i=1}^{W} p_i^{\alpha-\beta+1}} - U_{\alpha,\beta} = 0 \tag{3.2}$$

Which means that we weigh the energy E_i with the probability distribution $f^{\alpha-\beta+1} / \sum\limits_{i=1}^{W} f_i^{\alpha-\beta+1}$ (occasionally mentions to as escort probabilities Beck and Schlogl [1]) where $U_{\alpha,\beta}$ is the mean energy.

The Lagrange function

$$L' \equiv \frac{1}{(\beta-\alpha)}\left(\sum_{i=1}^{W} p_i^{\alpha-\beta+1} - 1\right) - \lambda'\left(\sum_{i=1}^{W} p_i - 1\right) - \mu'\left(\frac{\sum\limits_{i=1}^{W} p_i^{\alpha-\beta+1} E_i}{\sum\limits_{i=1}^{W} p_i^{\alpha-\beta+1}} - U_{\alpha,\beta}\right) = 0 \tag{3.3}$$

This Lagrangian upon differentiation delivers the subsequent manifestation:

$$p_i(E_i) = \frac{1}{Z'_{\alpha,\beta}}\left[1 - (\beta-\alpha)\frac{\mu'}{c_{\alpha,\beta}}(E_i - U_{\alpha,\beta})\right]^{\frac{1}{\beta-\alpha}} \tag{3.4}$$

where

$$Z'_{\alpha,\beta} = \sum_{i=1}^{W}\left[1 - (\beta-\alpha)\frac{\mu'}{c_{\alpha,\beta}}(E_i - U_{\alpha,\beta})\right]^{\frac{1}{\beta-\alpha}} \tag{3.5}$$

with

$$c_{\alpha,\beta} = \sum_{i=1}^{W} p_i^{\alpha-\beta+1} = 1 + (\beta-\alpha)S_{\alpha,\beta} \tag{3.6}$$

Usually, μ' is recognized through converse temperature, associated with the successive appearance:

$$k\mu' = \partial S_{\alpha,\beta}/\partial U_{\alpha,\beta}.$$

If one invests in a new-fangled structure $\mu'_{\alpha,\beta}$ to represent

$$\mu'_{\alpha,\beta} = \frac{\mu'}{c_{\alpha,\beta}} \tag{3.7}$$

The probability distribution function proceeds with the successive arrangement:

$$p_i(E_i) = \frac{1}{Z'_{\alpha,\beta}}\left[1-(\beta-\alpha)\mu'_{\alpha,\beta}(E_i-U_{\alpha,\beta})\right]^{\frac{1}{\beta-\alpha}} \tag{3.8}$$

and the partition function continues with the succeeding prearrangement:

$$Z'_{\alpha,\beta} = \sum_{i=1}^{W}\left[1-(\beta-\alpha)\mu'_{\alpha,\beta}(E_i-U_{\alpha,\beta})\right]^{\frac{1}{\beta-\alpha}} \tag{3.9}$$

The above derivation shows that by maximizing the proposed two-parameter entropy under the usual normalization and energy constraints, we obtain a generalized probability distribution function. This distribution encompasses classical Boltzmann–Gibbs and Tsallis forms as special cases. The parameter α modulates the sensitivity to rare or frequent events, while β controls the deformation and scaling of the distribution. Hence, the proposed model allows a richer class of equilibrium distributions beyond conventional single-parameter entropy approaches.

The famed statistic that collecting entropy suggests that energy has a propensity to blow blow-out about and pool regularly, but developing entropy implies that we are shedding knowledge about where energy is. This basic notion stems from the understanding that more entropy corresponds to more microstates and more behaviour in the energy may be spread. But if energy is pooled homogeneously, then there is an identical quantity of energy in every degree of freedom. That would suggest toward that at equilibrium, we would understand precisely where the energy was. For the reason that the system is recurrently intermingling and tossing energy everywhere, sometimes one gets more energy than the usual in a degree of freedom, and occasionally a lower quantity.

Since variations might be amplified owing to interior interfaces under some situations, the equal probability does not hold. Interior exchanges, which express the irrelevance of the statistical independence, possibly, create decreases of entropy in an isolated structure. The conversion probability from molecular chaotic motion to the ordered motion of a macroscopic entity is precisely negligible. But this effect potentially will not arise if communications occur in the inside of a system. According to the Boltzmann and Einstein fluctuation philosophies, all possible microscopic states of a structure are equally likely in thermodynamic equilibrium, and the entropy has a tendency to approach a maximum value finally.

4 The manifestation of energy dissemination and the energy oscillation

In B-G statistics, there exist two techniques to determine the energy variation. The first technique is to practice the subsequent formulation:

$$\langle E^2\rangle - \langle E\rangle^2 = -\frac{\partial U}{\partial \mu} = kT^2C_V \tag{4.1}$$

and then to acquire the succeeding expression:

$$\frac{\sqrt{\langle E^2\rangle - \langle E\rangle^2}}{U} = \frac{\sqrt{kT^2C_V}}{U} \tag{4.2}$$

The subsequent procedure intensifies the energy dispersion at a location of internal energy to commence the absolute oscillation. Here we watch the subsequent procedure to determine the relative fluctuation.

In entropy, the expectancy of physical magnitude M can be demarcated as:

$$\langle M\rangle_{\alpha,\beta} = \frac{\sum_{i=1}^{W} p_i^{\alpha-\beta+1}(E_i)M}{\sum_{i=1}^{W} p_i^{\alpha-\beta+1}(E_i)} \tag{4.3}$$

where $p_i(E_i)$ is the state distribution function. At every energy level, there may be numerous micro-states and accordingly, we acquire

$$p(E_i) = \omega^{\frac{1}{\alpha-\beta+1}}(E_i)\,p_i(E_i) \tag{4.4}$$

where $\omega(E_i)$ is the micro-state numbers. Exhausting (3.8) into (4.4), we acquire the subsequent manifestation:

$$p(E_i) \propto \omega^{\frac{1}{\alpha-\beta+1}}(E_i)\left[1-(\beta-\alpha)\mu'_{\alpha,\beta}(E_i-U_{\alpha,\beta})\right]^{\frac{1}{\beta-\alpha}} \tag{4.5}$$

Without any damage of simplification, E_i can be interchanged by E in equation (4.5) and then the energy dissemination can be communicated as

$$p(E) \propto \omega^{\frac{1}{\alpha-\beta+1}}(E)\left[1-(\beta-\alpha)\mu'_{\alpha,\beta}(E-U_{\alpha,\beta})\right]^{\frac{1}{\beta-\alpha}}$$

which proceeds with the succeeding arrangement:

$$p(E) \propto \exp_{\alpha,\beta}\left[\ln_{\alpha,\beta}\omega^{\frac{1}{\alpha-\beta+1}}(E)\right]\exp_{\alpha,\beta}\left[-\mu'_{\alpha,\beta}(E-U_{\alpha,\beta})\right] \tag{4.6}$$

Consequently, we acquire the succeeding logarithmic and exponential manifestations:

$$\ln_{\alpha,\beta}(x) = \frac{x^{\beta-\alpha} - 1}{\beta - \alpha} \tag{4.7}$$

and

$$\exp_{\alpha,\beta}(x) = \left[1 + (\beta - \alpha)x\right]^{\frac{1}{\beta-\alpha}} \tag{4.8}$$

If the representation $R_{\alpha,\beta}$ symbolises

$$R_{\alpha,\beta}(E) = \ln_{\alpha,\beta}\omega^{\frac{1}{\alpha-\beta+1}}(E) \tag{4.9}$$

Then, (4.6) is redrafted as

$$p(E) \propto \exp_{\alpha,\beta}\left[R_{\alpha,\beta}(E)\right]\exp_{\alpha,\beta}\left[-\mu'_{\alpha,\beta}\left(E - U_{\alpha,\beta}\right)\right] \tag{4.10}$$

By means of the association in Eq. (4.10), we contract the succeeding appearance:

$$p(E) \propto \exp_{\alpha,\beta}\Big\{R_{\alpha,\beta}(E) - \mu'_{\alpha,\beta}\left(E - U_{\alpha,\beta}\right) \times \left[1 + (\beta - \alpha)R_{\alpha,\beta}(E)\right]\Big\} \tag{4.11}$$

and the relation between $R_{\alpha,\beta}$ and $S_{\alpha,\beta}$ is established by the entropy of the micro-canonical form of $S_{\alpha,\beta}(E) = \ln_{\alpha,\beta}\omega(E)$ to be

$$R_{\alpha,\beta}(E) = \frac{1}{(\beta - \alpha)}\left\{\left[1 + (\beta - \alpha)S_{\alpha,\beta}(E)\right]^{\frac{1}{\alpha-\beta+1}} - 1\right\} \tag{4.12}$$

We denote the function in the α, β-exponential of equation (4.11) by $I(E)$, specified by

$$I(E) = R_{\alpha,\beta}(E) - \mu'_{\alpha,\beta}\left(E - U_{\alpha,\beta}\right)\left[1 + (\beta - \alpha)R_{\alpha,\beta}(E)\right] \tag{4.13}$$

Now

$$\frac{\partial I(E)}{\partial E} = \frac{\partial R_{\alpha,\beta}(E)}{\partial E}\left[1 - \mu'_{\alpha,\beta}(\beta - \alpha)\left(E - U_{\alpha,\beta}\right)\right] - \mu'_{\alpha,\beta}\left[1 + (\beta - \alpha)R_{\alpha,\beta}(E)\right] \tag{4.14}$$

And

$$\frac{\partial R_{\alpha,\beta}(E)}{\partial E} = \frac{1}{(\alpha - \beta + 1)}\left[1 + (\beta - \alpha)R_{\alpha,\beta}(E)\right] \times \frac{1}{c_{\alpha,\beta}}\frac{\partial S_{\alpha,\beta}(E)}{\partial E} \tag{4.15}$$

Using Eq. (4.15) in Eq. (4.14), we develop

$$\frac{\partial I(E)}{\partial E} = \left[1 + (\beta - \alpha)R_{\alpha,\beta}(E)\right] \times \left\{\begin{array}{c} \dfrac{1}{(\alpha - \beta + 1)c_{\alpha,\beta}}\dfrac{\partial S_{\alpha,\beta}(E)}{\partial E} \\ \left[1 - \mu'_{\alpha,\beta}(\beta - \alpha)\left(E - U_{\alpha,\beta}\right)\right] - \mu'_{\alpha,\beta} \end{array}\right\} \tag{4.16}$$

If we symbolize

$$b_{\alpha,\beta}(E) = 1 + (\beta - \alpha)R_{\alpha,\beta}(E) \tag{4.17}$$

and

$$\tilde{\beta}_{\alpha,\beta}(E) = \frac{1}{c_{\alpha,\beta}}\frac{\partial S_{\alpha,\beta}(E)}{\partial E} \tag{4.18}$$

Then, the (4.16) becomes

$$\frac{\partial I(E)}{\partial E} = b_{\alpha,\beta}(E)\left\{\frac{1}{\alpha - \beta + 1}\tilde{\beta}_{\alpha,\beta}(E)\left[1 - (\beta - \alpha)\mu'_{\alpha,\beta}\left(E - U_{\alpha,\beta}\right)\right] - \mu'_{\alpha,\beta}\right\} \tag{4.19}$$

and consequently, we acquire

$$\left.\frac{\partial I(E)}{\partial E}\right|_{E=E_m} = b_{\alpha,\beta}(E_m)\left\{\frac{1}{\alpha - \beta + 1}\tilde{\beta}_{\alpha,\beta}(E_m)\left[1 - (\beta - \alpha) \times \mu'_{\alpha,\beta}\left(E_m - U_{\alpha,\beta}\right)\right] - \mu'_{\alpha,\beta}\right\} \tag{4.20}$$

with

$$b_{\alpha,\beta}(E_m) = 1 + (\beta - \alpha)R_{\alpha,\beta}(E_m) \tag{4.21}$$

and

$$\tilde{\beta}_{\alpha,\beta}(E_m) = \left.\frac{1}{c_{\alpha,\beta}}\frac{\partial S_{\alpha,\beta}(E)}{\partial E}\right|_{E=E_m} \tag{4.22}$$

Now, at maximum I, (4.22) delivers the succeeding appearance:

$$b_{\alpha,\beta}(E_m)\left\{\frac{1}{\alpha - \beta + 1}\tilde{\beta}_{\alpha,\beta}(E_m)\left[1 - (\beta - \alpha)\mu'_{\alpha,\beta}\left(E_m - U_{\alpha,\beta}\right)\right] - \mu'_{\alpha,\beta}\right\} = 0 \tag{4.23}$$

The fluctuation analysis derived above generalizes the well-known Boltzmann–Einstein fluctuation theory. In particular, the two parameters α and β introduce additional flexibility in characterizing the spread of energy fluctuations in the canonical ensemble. When $\alpha, \beta \to 1$, we recover the classical result where fluctuations are proportional to the heat capacity. However, for $\alpha \neq \beta$, the fluctuation intensity can be enhanced or suppressed, offering potential insights into systems with long-range correlations or non-extensive interactions.

Consequently, at E_m (4.11) may acquire its maxima. Now the function $I(E)$ can be expanded as a series about E_m. The first order term vanishes and then

$$I(E) = I(E_m) + \frac{1}{2}\left.\frac{\partial^2 I(E)}{\partial E^2}\right|_{E=E_m}\left(E - E_m\right)^2 + \cdots \tag{4.24}$$

Overlooking higher order terms, it converts to (see Appendix A)

$$I(E) = I(E_m) - \frac{1}{2}\tilde{\beta}_{\alpha,\beta}(E_m)\mu'_{\alpha,\beta}b_{\alpha,\beta}(E_m) \times \left[\frac{k}{\tilde{C}_V} + \frac{\beta-\alpha}{\alpha-\beta+1}\right](E-E_m)^2 \tag{4.25}$$

Employing (4.13) and (4.23), we acquire:

$$I(E) = -\frac{1}{(\beta-\alpha)} + \frac{(\alpha-\beta+1)\mu'_{\alpha,\beta}b_{\alpha,\beta}(E_m)}{(\beta-\alpha)\tilde{\beta}_{\alpha,\beta}(E_m)} \times \left[1 - \frac{(\beta-\alpha)\tilde{\beta}^2_{\alpha,\beta}(E_m)}{2(\alpha-\beta+1)}\left(\frac{k}{\tilde{C}_V} + \frac{\beta-\alpha}{\alpha-\beta+1}\right)(E-E_m)^2\right] \tag{4.26}$$

Substituting the value of $I(E)$ from Eq. (4.26) into Eq. (4.11), we contract the successive relation:

$$p(E) \propto \left[1 - \frac{(\beta-\alpha)\tilde{\beta}^2_{\alpha,\beta}(E_m)}{2(\alpha-\beta+1)}\left(\frac{k}{\tilde{C}_V} + \frac{\beta-\alpha}{\alpha-\beta+1}\right)(E-E_m)^2\right]^{\frac{1}{\beta-\alpha}} \tag{4.27}$$

Exhausting (4.27), we can determine α, β-average of the energy (see Appendix B) $\langle E\rangle_{\alpha,\beta} = E_m$. Since in the entropy, it has been demarcated $\langle E\rangle_{\alpha,\beta} = U_{\alpha,\beta}$, we acquire $E_m = U_{\alpha,\beta}$ and consequently, (4.23) stretches the succeeding relation between $\tilde{\beta}_{\alpha,\beta}$ and $\mu'_{\alpha,\beta}$:

$$\tilde{\beta}_{\alpha,\beta}(U_{\alpha,\beta}) = (\alpha-\beta+1)\mu'_{\alpha,\beta} \tag{4.28}$$

Consequently, (4.26) turn into:

$$p(E) \propto \left[1 - (\beta-\alpha)\frac{\mu'^2_{\alpha,\beta}}{2}\left(\frac{k}{C_V} + \beta - \alpha\right)(E-U_{\alpha,\beta})^2\right]^{\frac{1}{\beta-\alpha}} \tag{4.29}$$

with $C_V = \tilde{C}_V/(\alpha-\beta+1)$ and

$$C_V = -\mu'^2_{\alpha,\beta}\frac{\partial E}{\partial \mu'_{\alpha,\beta}} \tag{4.30}$$

Noting that $p(E)$ is an energy distribution and it should reach a maximum at $E = E_m$ and consequently at $E = U_{\alpha,\beta}$, we acquire:

$$\left.\frac{\partial^2 p(E)}{\partial E^2}\right|_{E=U_{\alpha,\beta}} = -\mu'^2_{\alpha,\beta}\left(\frac{k}{C_V} + \beta - \alpha\right) < 0$$

Accordingly, we find

$$\beta > \alpha + \frac{k}{C_V} \tag{4.31}$$

At the moment, we determine the absolute changeability of energy by employing the (4.29). The outcome (see Appendix C) is

$$\frac{\sqrt{\left\langle (E-U_{\alpha,\beta})^2\right\rangle_{\alpha,\beta}}}{U_{\alpha,\beta}} = \frac{1}{\mu'_{\alpha,\beta}U_{\alpha,\beta}}\sqrt{\frac{2}{(\beta-\alpha+2)}\frac{C_V}{(\beta-\alpha)C_V+k}}, \quad \text{for } \alpha-\beta<2 \tag{4.32}$$

In summary, this formulation demonstrates that the generalized entropy modifies both the equilibrium distribution and the corresponding fluctuation behaviour. This could be particularly useful for modelling systems where microscopic interactions lead to deviations from standard Gibbsian statistics, such as complex networks, anomalous diffusion systems, or strongly correlated particle ensembles.

5 Conclusions

The well-acknowledged information that an ensemble just embodies an assemblage of moving constituent parts and can be conjectured from the perspective that canonical might be outstanding to the practice of canonical coordinates in the dissemination implies that in statistical mechanics, a canonical collaborative signifies the statistical ensemble that characterizes conceivable situations of a mechanical structure. The structures may give the change of energy utilizing a heat bath so that conditions of structure will vary in aggregate energy. The impression of energy fluctuation derives from the understanding that higher entropy correlates to more microstates, extra ways in which the energy might be dispersed. But if energy is collectively uniform, then there is a same magnitude of energy in each degree of freedom. That would imply that in equilibrium, we would know correctly where the energy was. For the reason that the system is continually interrelating and tossing energy about, sometimes one gains more energy than the usual in a degree of freedom, and sometimes less. Keeping in mind the cumulative practice of entropy models in statistical mechanics, we have considered a pioneering entropic model for discrete probability fields. We have articulated the appearance of the relative fluctuation of energy in the canonical ensemble for this entropic model. With equivalent viewpoints, one might furnish the solicitations of entropy models either by exhausting supplemental typical entropy models or by mounting new-fangled entropic models for their presents to the arena of statistical mechanics.

Nomenclature

Symbol	Definition
p_i	probability of the ith microstate or event
$H(P)$	entropy function for a given probability distribution $P = \{p_1, p_2 \dots p_n\}$
α	entropy parameter controlling the order of generalization; must satisfy given bounds
β	entropy parameter related to non-additivity or escort probabilities
n	number of microstates or discrete events in the probability distribution
i, j	indices representing individual microstates or components
E_i	energy corresponding to the ith microstate
$w(E_i)$	number of microstates associated with energy level E_i
U	mean energy
C_V	heat capacity at constant volume
T	temperature
k	Boltzmann constant
Z	partition function
μ'	modified Lagrange multiplier interpreted as inverse temperature
$S_{\alpha,\beta}$	entropy function under generalized (parametric) form
$R_{\alpha,\beta}(E)$	Rényi-like function associated with entropy and energy fluctuation
$b_{\alpha,\beta}(E)$	derivative term appearing in fluctuation formulations
$\overline{C_V}$	adjusted or generalized heat capacity under (α, β)- entropy model
ΔE	absolute fluctuation in energy
$\frac{\Delta E}{U}$	relative fluctuation in energy
λ, η	Lagrange multipliers used in constrained optimization problems
M	Lagrangian function constructed to optimize entropy under constraints
W	total number of accessible states or microstates
$I(E)$	energy-based information function derived in fluctuation analysis

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Appendix A: We procure the following communication

$$\frac{\partial^2 I(E)}{\partial E^2} = \frac{\partial b_{\alpha,\beta}(E)}{\partial E}\left\{\frac{1}{\alpha-\beta+1}\tilde{\beta}_{\alpha,\beta}(E) \times\left[1-(\beta-\alpha)\mu'_{\alpha,\beta}(E-U_{\alpha,\beta})\right]-\mu'_{\alpha,\beta}\right\} + \frac{b_{\alpha,\beta}(E)}{\alpha-\beta+1}\left\{-\tilde{\beta}_{\alpha,\beta}(E)(\beta-\alpha)\mu'_{\alpha,\beta} + \frac{\partial\tilde{\beta}_{\alpha,\beta}(E)}{\partial E}\left[1-(\beta-\alpha)\mu'_{\alpha,\beta}(E-U_{\alpha,\beta})\right]\right\} \tag{A.1}$$

Also

$$\frac{\partial b_{\alpha,\beta}(E)}{\partial E} = (\beta-\alpha)\frac{\partial R_{\alpha,\beta}(E)}{\partial E} \tag{A.2}$$

Using equation (4.15), (A.2) becomes

$$\frac{\partial b_{\alpha,\beta}(E)}{\partial E} = \frac{(\beta-\alpha)}{(\alpha-\beta+1)}b_{\alpha,\beta}(E)\tilde{\beta}_{\alpha,\beta}(E) \tag{A.3}$$

Thus, we acquire:

$$\left.\frac{\partial^2 I(E)}{\partial E^2}\right|_{E=E_m} = -\tilde{\beta}_{\alpha,\beta}(E_m)\mu'_{\alpha,\beta}b_{\alpha,\beta}(E_m) \times\left[\frac{k}{\tilde{C}_V}+\frac{\beta-\alpha}{\alpha-\beta+1}\right] \tag{A.4}$$

where the quantity $\tilde{C}_V$ is

$$\tilde{C}_V = -k\tilde{\beta}_{\alpha,\beta}(E_m)\left.\frac{\partial E}{\partial\tilde{\beta}_{\alpha,\beta}(E)}\right|_{E=E_m} \tag{A.5}$$

Appendix B: Calculation for $\langle E\rangle_{\alpha,\beta}$

Two cases arise here:

Case I. For $\alpha > \beta$

$$\langle E\rangle_{\alpha,\beta} = \frac{\int_0^\infty [p(E)]^{\alpha-\beta+1}E\,\mathrm{d}E}{\int_0^\infty [p(E)]^{\alpha-\beta+1}\mathrm{d}E}$$

$$= \frac{\int_0^\infty \left[1-\frac{(\beta-\alpha)\tilde{\beta}^2_{\alpha,\beta}}{2(\alpha-\beta+1)}\left(\frac{k}{\tilde{C}_V}+\frac{\beta-\alpha}{\alpha-\beta+1}\right)(E-E_m)^2\right]^{\frac{\alpha-\beta+1}{\beta-\alpha}}E\,\mathrm{d}E}{\int_0^\infty \left[1-\frac{(\beta-\alpha)\tilde{\beta}^2_{\alpha,\beta}}{2(\alpha-\beta+1)}\left(\frac{k}{\tilde{C}_V}+\frac{\beta-\alpha}{\alpha-\beta+1}\right)(E-E_m)^2\right]^{\frac{\alpha-\beta+1}{\beta-\alpha}}\mathrm{d}E} 5$$

Let $x = E - E_m$, it becomes

$$\langle E\rangle_{\alpha,\beta}=\frac{\int_{-E_m}^{\infty}\left[1-\frac{(\beta-\alpha)\tilde{\beta}^2_{\alpha,\beta}}{2(\alpha-\beta+1)}\left(\frac{k}{\tilde{C}_V}+\frac{\beta-\alpha}{\alpha-\beta+1}\right)x^2\right]^{\frac{\alpha-\beta+1}{\beta-\alpha}}(x+E_m)\,\mathrm{d}x}{\int_{-E_m}^{\infty}\left[1-\frac{(\beta-\alpha)\tilde{\beta}^2_{\alpha,\beta}}{2(\alpha-\beta+1)}\left(\frac{k}{\tilde{C}_V}+\frac{\beta-\alpha}{\alpha-\beta+1}\right)x^2\right]^{\frac{\alpha-\beta+1}{\beta-\alpha}}\mathrm{d}x}$$

Consequently, we acquire:

$$\langle E\rangle_{\alpha,\beta}=\frac{\int_{-\infty}^{\infty}\left[1-\frac{(\beta-\alpha)\tilde{\beta}^2_{\alpha,\beta}}{2(\alpha-\beta+1)}\left(\frac{k}{\tilde{C}_V}+\frac{\beta-\alpha}{\alpha-\beta+1}\right)x^2\right]^{\frac{\alpha-\beta+1}{\beta-\alpha}}(x+E_m)\,\mathrm{d}x}{\int_{-\infty}^{\infty}\left[1-\frac{(\beta-\alpha)\tilde{\beta}^2_{\alpha,\beta}}{2(\alpha-\beta+1)}\left(\frac{k}{\tilde{C}_V}+\frac{\beta-\alpha}{\alpha-\beta+1}\right)x^2\right]^{\frac{\alpha-\beta+1}{\beta-\alpha}}\mathrm{d}x}=E_m$$

Case II. For $\alpha<\beta$

By means of equivalent approach, we procure:

$$\langle E\rangle_{\alpha,\beta}=\frac{\int_{-a}^{a}\left[1-\frac{(\beta-\alpha)\tilde{\beta}^2_{\alpha,\beta}}{2(\alpha-\beta+1)}\left(\frac{k}{\tilde{C}_V}+\frac{\beta-\alpha}{\alpha-\beta+1}\right)x^2\right]^{\frac{\alpha-\beta+1}{\beta-\alpha}}(x+E_m)\,\mathrm{d}x}{\int_{-a}^{a}\left[1-\frac{(\beta-\alpha)\tilde{\beta}^2_{\alpha,\beta}}{2(\alpha-\beta+1)}\left(\frac{k}{\tilde{C}_V}+\frac{\beta-\alpha}{\alpha-\beta+1}\right)x^2\right]^{\frac{\alpha-\beta+1}{\beta-\alpha}}\mathrm{d}x}\quad\text{with}$$

$$a=\sqrt{\frac{2(\alpha-\beta+1)^2}{(\beta-\alpha)\tilde{\beta}^2_{\alpha,\beta}}\frac{\tilde{C}_V}{(\beta-\alpha)\tilde{C}_V+k(\alpha-\beta+1)}}$$

where $\exp_{\alpha,\beta}(x)=\begin{cases}[1+(\beta-\alpha)x]^{\frac{1}{\beta-\alpha}}, & \text{for } 1+(\beta-\alpha)x>0\\ 0, & \text{otherwise}\end{cases}$

Accordingly, we achieve the α,β-average value of energy, $\langle E\rangle_{\alpha,\beta}=E_m$.

Appendix C: Calculation for $\left\langle\left(E-U_{\alpha,\beta}\right)^2\right\rangle_{\alpha,\beta}$

Again two cases arise here:

Case I. For $\alpha>\beta$

$$\left\langle\left(E-U_{\alpha,\beta}\right)^2\right\rangle_{\alpha,\beta}=\frac{\int_0^{\infty}\left[1-\frac{(\beta-\alpha)\mu'^2_{\alpha,\beta}}{2}\left(\frac{k}{C_V}+\beta-\alpha\right)\left(E-U_{\alpha,\beta}\right)^2\right]^{\frac{\alpha-\beta+1}{\beta-\alpha}}\left(E-U_{\alpha,\beta}\right)^2\mathrm{d}E}{\int_0^{\infty}\left[1-\frac{(\beta-\alpha)\mu'^2_{\alpha,\beta}}{2}\left(\frac{k}{C_V}+\beta-\alpha\right)\left(E-U_{\alpha,\beta}\right)^2\right]^{\frac{\alpha-\beta+1}{\beta-\alpha}}\mathrm{d}E}$$

Let $x=E-U_{\alpha,\beta}$ and $t=-\frac{(\beta-\alpha)\mu'^2_{\alpha,\beta}}{2}\left(\frac{k}{C_V}+\beta-\alpha\right)$, then we contract, only for $0<\alpha-\beta<2$ (condition for the integral to be convergent),

$$\left\langle\left(E-U_{\alpha,\beta}\right)^2\right\rangle_{\alpha,\beta}=\frac{\int_{-\infty}^{\infty}\left(1+tx^2\right)^{\frac{\alpha-\beta+1}{\beta-\alpha}}x^2\,\mathrm{d}x}{\int_{-\infty}^{\infty}\left(1+tx^2\right)^{\frac{\alpha-\beta+1}{\beta-\alpha}}\mathrm{d}x}=\frac{\int_0^{\infty}\left(1+tx^2\right)^{\frac{\alpha-\beta+1}{\beta-\alpha}}x^2\,\mathrm{d}x}{\int_0^{\infty}\left(1+tx^2\right)^{\frac{\alpha-\beta+1}{\beta-\alpha}}\mathrm{d}x}=\frac{1}{t}\frac{B\left(\frac{3}{2},-\frac{3}{2}+\frac{\alpha-\beta+1}{\beta-\alpha}\right)}{B\left(\frac{1}{2},-\frac{1}{2}+\frac{\alpha-\beta+1}{\beta-\alpha}\right)}=\frac{\alpha-\beta}{t(\beta-\alpha+2)}$$

where $B(p,q)=\int_0^{\infty}\frac{y^{p-1}}{(1+y)^{p+q}}\mathrm{d}y=\frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}$

Thus, we acquire the consequent communication:

$$\left\langle\left(E-U_{\alpha,\beta}\right)^2\right\rangle_{\alpha,\beta}=\frac{2}{(\beta-\alpha+2)\mu'^2_{\alpha,\beta}}\frac{C_V}{k+(\beta-\alpha)C_V}\quad\text{for } 0<\alpha-\beta<2$$

Case I. For $\alpha<\beta$

Let $x=E-U_{\alpha,\beta}$ and $t'=\frac{(\beta-\alpha)\mu'^2_{\alpha,\beta}}{2}\left(\frac{k}{C_V}+\beta-\alpha\right)$, we procure the subsequent statement:

$$\left\langle\left(E-U_{\alpha,\beta}\right)^2\right\rangle_{\alpha,\beta}=\frac{\int_{-a}^{a}\left(1-t'x^2\right)^{\frac{\alpha-\beta+1}{\beta-\alpha}}x^2\,\mathrm{d}x}{\int_{-a}^{a}\left(1-t'x^2\right)^{\frac{\alpha-\beta+1}{\beta-\alpha}}\mathrm{d}x}=\frac{1}{t'}\frac{B\left(\frac{3}{2},1+\frac{\alpha-\beta+1}{\beta-\alpha}\right)}{B\left(\frac{1}{2},1+\frac{\alpha-\beta+1}{\beta-\alpha}\right)}=\frac{\beta-\alpha}{t'(\beta-\alpha+2)}=\frac{2}{(\beta-\alpha+2)\mu'^2_{\alpha,\beta}}\frac{C_V}{k+(\beta-\alpha)C_V}$$

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